On the supersingular reduction of $K3$ surfaces with complex multiplication

Kazuhiro Ito
(Department of Mathematics, Kyoto University)

Abstract. We study the good reduction modulo p of $K3$ surfaces with complex multiplication (CM). We determine when the good reduction is supersingular. Moreover, for almost all p, we calculate its Artin invariant. Our results generalize Shimada's results on complex projective $K3$ surfaces with Picard number 20.

1. Introduction

Let $X_{\mathbb{C}}$ be a projective $K3$ surface over $\mathbb{C}$. Let

$$T(X_{\mathbb{C}}) := \mathrm{Pic}(X_{\mathbb{C}})^{\perp} \subset H^2(X_{\mathbb{C}}, \mathbb{Z}(1))$$

be the transcendental lattice. Let E be a *CM field* with maximal totally real subfield F. Assume that $X_{\mathbb{C}}$ has *complex multiplication* (CM) by E, i.e.

$$E \simeq \mathrm{End}_{\mathrm{Hdg}}(T(X_{\mathbb{C}})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and

$$\dim_E(T(X_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1.$$

Pjateckiĭ-Šapiro and Šafarevič showed that $X_{\mathbb{C}}$ has a model X_K over a number field $K \subset \mathbb{C}$ which contains E.

Let v be a finite place of K whose residue characteristic is p. We assume that the $K3$ surface X_K has *good reduction* at v.

- We say the geometric special fiber $\mathscr{X}_{\overline{v}}$ is (Shioda-)*supersingular* if
  $\rho(\mathscr{X}_{\overline{v}}) = \mathrm{rank}_{\mathbb{Z}} \mathrm{Pic}(\mathscr{X}_{\overline{v}}) = 22.$
- The *Artin invariant* a of $\mathscr{X}_{\overline{v}}$ is defined by
  $\mathrm{disc}\, \mathrm{Pic}(\mathscr{X}_{\overline{v}}) = -p^{2a} \quad (1 \leq a \leq 10).$

2. Main Theorem

Theorem 2.1 ([**1**]). *Recall $F \subset E \subset K \subset \mathbb{C}$. Let $\mathfrak{q}$ be the finite place of F below v. For almost all finite places v, the following hold.*

(1) *The $K3$ surface $\mathscr{X}_{\overline{v}}$ is supersingular if and only if $\mathfrak{q}$ does not split in E.*
(2) *If $\mathfrak{q}$ does not split in E, the Artin invariant of $\mathscr{X}_{\overline{v}}$ is equal to $[k(\mathfrak{q}) : \mathbb{F}_p]$. Here $k(\mathfrak{q})$ is the residue field of $\mathfrak{q}$.*

Our results generalize Shimada's results [**5**] for $K3$ surfaces with $\rho(X_{\mathbb{C}}) = 20$. See below.

3. Some examples of $K3$ surfaces with CM

$K3$ surfaces with Picard number 20. Projective $K3$ surfaces $X_{\mathbb{C}}$ over $\mathbb{C}$ with $\rho(X_{\mathbb{C}}) = 20$ are classified by Shioda-Inose. They have CM by imaginary quadratic fields.

Kummer surfaces. Let $A_{\mathbb{C}}$ be a simple abelian surface over $\mathbb{C}$ with CM, i.e. $\mathrm{End}(A_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field of degree 4. Then the *Kummer surface* $\mathrm{Km}(\mathrm{A}_{\mathbb{C}})$ has CM by a CM field of degree 4.

$K3$ surfaces with automorphisms. A projective $K3$ surface $X_{\mathbb{C}}$ over $\mathbb{C}$ has CM by the cyclotomic field $\mathbb{Q}(\zeta_N)$ if it has an automorphism $f \in \mathrm{Aut}(X_{\mathbb{C}})$ such that the order of $f^* \in \mathrm{Aut}(T(X_{\mathbb{C}}))$ is N with

$$\phi(N) = \mathrm{rank}_{\mathbb{Z}} T(X_{\mathbb{C}}),$$

where ϕ is the Euler's totient function. For each

$$N \in \{3, 5, 7, 9, 11, 12, 13, 17, 19, 25, 27, 28, 36, 42, 44, 66\},$$

Kondō proved that there exists a projective $K3$ surface $X_{\mathbb{C}}$ over $\mathbb{C}$ with CM by $\mathbb{Q}(\zeta_N)$ [**3**].

4. Sketch of the proof

- First, we use the main theorem of CM for $K3$ surfaces (Rizov [**4**]) to calculate the Frobenius action on the Galois module
  $H^2_{\text{ét}}(X_{\overline{K_v}}, \mathbb{Q}_\ell) \quad (\ell \neq p).$
- We describe the Breuil-Kisin module
  $\mathfrak{M}(H^2_{\text{ét}}(X_{\overline{K_v}}, \mathbb{Z}_p))$
  by a description of the Breuil-Kisin modules of Lubin-Tate characters (Andreatta-Goren-Howard-Madapusi Pera).
- We use the integral p-adic Hodge theory (Bhatt-Morrow-Scholze [**2**])
  $$H^2_{\mathrm{cris}}(\mathscr{X}_{\overline{v}}/W) \simeq \varphi^*(\mathfrak{M}(H^2_{\text{ét}}(X_{\overline{K_v}}, \mathbb{Z}_p))/u\mathfrak{M}(H^2_{\text{ét}}(X_{\overline{K_v}}, \mathbb{Z}_p))).$$
- Finally, we calculate the length of the cokernel of the crystalline Chern class map
  $\mathrm{Pic}(\mathscr{X}_{\overline{v}}) \otimes_{\mathbb{Z}} W \to H^2_{\mathrm{cris}}(\mathscr{X}_{\overline{v}}/W),$
  which is equal to the Artin invariant a.

References

[1] Ito, K., *On the supersingular reduction of $K3$ surfaces with complex multiplication*, preprint, 2017. arXiv 1710.00225.
[2] Bhatt, B., Morrow, M., Scholze, P., *Integral p-adic Hodge theory*, preprint, 2016, arXiv: 1602.03148.
[3] Kondō, S., *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75-98.
[4] Rizov, J., *Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic*, J. Reine Angew. Math. **648** (2010), 13-67.
[5] Shimada, I., *Transcendental lattices and supersingular reduction lattice of a singular K3 surface*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 909-949.