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On the supersingular reduction of $K3$ surfaces with complex multiplication

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Abstract. We study the good reduction modulo $p$ of $K3$ surfaces with complex multiplication (CM). We determine when the good reduction is supersingular. Moreover, for almost all $p$, we calculate its Artin invariant. Our results generalize Shimada’s results on complex projective $K3$ surfaces with Picard number 20.

1. Introduction

Let $X_C$ be a projective $K3$ surface over $C$. Let $T(X_C) := \operatorname{Pic}(X_C)^{\perp} \subset H^2(X_C, \mathbb{Z}(1))$ be the transcendental lattice. Let $E$ be a CM field with maximal totally real subfield $F$. Assume that $X_C$ has complex multiplication (CM) by $E$, i.e.

$E \simeq \operatorname{End}_{\text{Hdg}}(T(X_C)) \otimes \mathbb{Q}$

and

$\dim E(T(X_C) \otimes \mathbb{Q}) = 1$.

Pjateckiĭ-Šapiro and Šafarevič showed that $X_C$ has a model $X_K$ over a number field $K \subset C$ which contains $E$.

Let $v$ be a finite place of $K$ whose residue characteristic is $p$. We assume that the $K3$ surface $X_K$ has good reduction at $v$.

- We say the geometric special fiber $X_v$ is (Shioda-)supersingular if $\rho(X_v) = \operatorname{rank}_\mathbb{Z} \operatorname{Pic}(X_v) = 22$.
- The Artin invariant $a$ of $X_v$ is defined by $\operatorname{disc} \operatorname{Pic}(X_v) = -p^{2a} \quad (1 \leq a \leq 10)$.

2. Main Theorem

Theorem 2.1 ([1]). Recall $F \subset E \subset K \subset C$. Let $q$ be the finite place of $F$ below $v$. For almost all finite places $v$, the following hold.

1. The $K3$ surface $X_v$ is supersingular if and only if $q$ does not split in $E$.
2. If $q$ does not split in $E$, the Artin invariant of $X_v$ is equal to $[k(q) : \mathbb{F}_p]$. Here $k(q)$ is the residue field of $q$.

Our results generalize Shimada’s results [5] for $K3$ surfaces with $\rho(X_C) = 20$. See below.

3. Some examples of $K3$ surfaces with CM

$K3$ surfaces with Picard number 20. Projective $K3$ surfaces $X_C$ over $C$ with $\rho(X_C) = 20$ are classified by Shioda-Inose. They have CM by imaginary quadratic fields.

Kummer surfaces. Let $A_C$ be a simple abelian surface over $C$ with CM, i.e. $\operatorname{End}(A_C) \otimes \mathbb{Q}$ is a CM field of degree 4. Then the Kummer surface $\text{Km}(A_C)$ has CM by a CM field of degree 4.

$K3$ surfaces with automorphisms. A projective $K3$ surface $X_C$ over $C$ has CM by the cyclotomic field $\mathbb{Q}(\zeta_N)$ if it has an automorphism $f \in \operatorname{Aut}(X_C)$ such that the order of $f^* \in \operatorname{Aut}(T(X_C))$ is $N$ with $\phi(N) = \operatorname{rank}_\mathbb{Z} T(X_C)$, where $\phi$ is the Euler’s totient function. For each $N \in \{3, 5, 7, 9, 11, 12, 13, 17, 19, 25, 27, 28, 36, 42, 44, 66\}$, Kondō proved that there exists a projective $K3$ surface $X_C$ over $C$ with CM by $\mathbb{Q}(\zeta_N)$ [3].

4. Sketch of the proof

- First, we use the main theorem of CM for $K3$ surfaces (Rizov [4]) to calculate the Frobenius action on the Galois module $H_2^c(X_{K\bar{v}}, \mathbb{Q}_\ell) \quad (\ell \neq p)$.
- We describe the Breuil-Kisin module $\Omega(H^2_\ell(X_{K\bar{v}}, \mathbb{Z}_\ell))$ by a description of the Breuil-Kisin modules of Lubin-Tate characters (Andreatta-Goren-Howard-Madapusi Pera).
- We use the integral $p$-adic Hodge theory (Bhatt-Morrow-Scholze [2]) $H^2_{\text{crys}}(X_v/W) \simeq \varphi^*(\Omega(H^2_\ell(X_{K\bar{v}}, \mathbb{Z}_\ell))/\omega \Omega(H^2_\ell(X_{K\bar{v}}, \mathbb{Z}_\ell)))$.
- Finally, we calculate the length of the cokernel of the crystalline Chern class map $\operatorname{Pic}(X_v) \otimes \mathbb{Z} W \to H^2_{\text{crys}}(f(X_v/W)$, which is equal to the Artin invariant $a$.

References