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<td>Ito, Kazuhiro</td>
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Kyoto University
On the supersingular reduction of $K3$ surfaces with complex multiplication

Kazuhiro Ito
(Department of Mathematics, Kyoto University)

ABSTRACT. We study the good reduction modulo $p$ of $K3$ surfaces with complex multiplication (CM). We determine when the good reduction is supersingular. Moreover, for almost all $p$, we calculate its Artin invariant. Our results generalize Shimada’s results on complex projective $K3$ surfaces with Picard number 20.

1. Introduction

Let $X_C$ be a projective $K3$ surface over $C$. Let $T(X_C) := \text{Pic}(X_C) \cap H^2(X_C, Z(1))$ be the transcendental lattice. Let $E$ be a CM field with maximal totally real subfield $F$. Assume that $X_C$ has complex multiplication (CM) by $E$, i.e.

$E \simeq \text{End}_{\text{Hdg}}(T(X_C)) \otimes Z \mathbb{Q}$

and

$\dim E(T(X_C) \otimes Z \mathbb{Q}) = 1$.

Pjateckiĭ-Šapiro and Šafarevič showed that $X_C$ has a model $X_K$ over a number field $K \subset C$ which contains $E$.

Let $v$ be a finite place of $K$ whose residue characteristic is $p$. We assume that the $K3$ surface $X_K$ has good reduction at $v$.

- We say the geometric special fiber $\mathcal{X}_v$ is (Shioda-)supersingular if $\rho(\mathcal{X}_v) = \text{rank}_Z \text{Pic}(\mathcal{X}_v) = 22$.
- The Artin invariant $a$ of $\mathcal{X}_v$ is defined by $\text{disc} \text{Pic}(\mathcal{X}_v) = -p^2a \quad (1 \leq a \leq 10)$.

2. Main Theorem

THEOREM 2.1 ([1]). Recall $F \subset E \subset K \subset C$. Let $q$ be the finite place of $F$ below $v$. For almost all finite places $v$, the following hold.

1. The $K3$ surface $\mathcal{X}_v$ is supersingular if and only if $q$ does not split in $E$.
2. If $q$ does not split in $E$, the Artin invariant of $\mathcal{X}_v$ is equal to $[k(q) : \mathbb{F}_p]$. Here $k(q)$ is the residue field of $q$.

Our results generalize Shimada’s results [5] for $K3$ surfaces with $\rho(X_C) = 20$. See below.

3. Some examples of $K3$ surfaces with CM

K3 surfaces with Picard number 20. Projective $K3$ surfaces $X_C$ over $C$ with $\rho(X_C) = 20$ are classified by Shioda-Inose. They have CM by imaginary quadratic fields.

Kummer surfaces. Let $A_C$ be a simple abelian surface over $C$ with CM, i.e. $\text{End}(A_C) \otimes Z \mathbb{Q}$ is a CM field of degree 4. Then the Kummer surface $\text{Km}(A_C)$ has CM by a CM field of degree 4.

K3 surfaces with automorphisms. A projective $K3$ surface $X_C$ over $C$ has CM by the cyclotomic field $\mathbb{Q}(\zeta_N)$ if it has an automorphism $f \in \text{Aut}(X_C)$ such that the order of $f^* \in \text{Aut}(T(X_C))$ is $N$ with $\phi(N) = \text{rank}_Z T(X_C)$, where $\phi$ is the Euler’s totient function. For each $N \in \{3, 5, 7, 9, 11, 12, 13, 17, 19, 25, 27, 28, 36, 42, 44, 66\}$, Kondō proved that there exists a projective $K3$ surface $X_C$ over $C$ with CM by $\mathbb{Q}(\zeta_N)$ [3].

4. Sketch of the proof

- First, we use the main theorem of CM for $K3$ surfaces (Rizov [4]) to calculate the Frobenius action on the Galois module $H^2_\text{cris}(X_{\overline{K}}, \mathbb{Q}_\ell) \quad (\ell \neq p)$.
- We describe the Breuil-Kisin module $\Omega(H^2_\text{cris}(X_{\overline{K}}, \mathbb{Z}_p))$ by a description of the Breuil-Kisin modules of Lubin-Tate characters (Andreatta-Goren-Howard-Madapusi Pera).
- We use the integral $p$-adic Hodge theory (Bhatt-Morrow-Scholze [2]) $H^2_\text{cris}(\mathcal{X}_v/W)$ $\simeq \varphi^*(\Omega(H^2_\text{cris}(X_{\overline{K}}, \mathbb{Z}_p))/u\Omega(H^2_\text{cris}(X_{\overline{K}}, \mathbb{Z}_p)))$.
- Finally, we calculate the length of the cokernel of the crystalline Chern class map $\text{Pic}(\mathcal{X}_v) \otimes Z W \rightarrow H^2_\text{cris}(\mathcal{X}_v/W)$, which is equal to the Artin invariant $a$.

References