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On the supersingular reduction of $K3$ surfaces with complex multiplication

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Abstract. We study the good reduction modulo $p$ of $K3$ surfaces with complex multiplication (CM). We determine when the good reduction is supersingular. Moreover, for almost all $p$, we calculate its Artin invariant. Our results generalize Shimada’s results on complex projective $K3$ surfaces with Picard number 20.

1. Introduction

Let $X_\mathbb{C}$ be a projective $K3$ surface over $\mathbb{C}$. Let

$$T(X_\mathbb{C}) := \text{Pic}(X_\mathbb{C})^\perp \subset H^2(X_\mathbb{C}, \mathbb{Z}(1))$$

be the transcendental lattice. Let $E$ be a CM field with maximal totally real subfield $F$. Assume that $X_\mathbb{C}$ has complex multiplication (CM) by $E$, i.e.

$$E \cong \text{End}_{\text{Hdg}}(T(X_\mathbb{C})) \otimes \mathbb{Q}$$

and

$$\dim(E(T(X_\mathbb{C}) \otimes \mathbb{Q})) = 1.$$

Pjateckiĭ-Šapiro and Šafarevič showed that $X_\mathbb{C}$ has a model $X_K$ over a number field $K \subset \mathbb{C}$ which contains $E$.

Let $v$ be a finite place of $K$ whose residue characteristic is $p$. We assume that the $K3$ surface $X_K$ has good reduction at $v$.

- We say the geometric special fiber $X_\mathbb{F}_p$ is (Shioda-)supersingular if $\rho(X_\mathbb{F}_p) = \text{rank}_2 \text{Pic}(X_\mathbb{F}_p) = 22$.
- The Artin invariant $a$ of $X_\mathbb{F}_p$ is defined by $\text{disc}(\text{Pic}(X_\mathbb{F}_p)) = -p^{2a} \quad (1 \leq a \leq 10)$.

2. Main Theorem

Theorem 2.1 ([1]). Recall $F \subset E \subset K \subset \mathbb{C}$. Let $q$ be the finite place of $F$ below $v$. For almost all finite places $v$, the following hold.

1. The $K3$ surface $X_\mathbb{F}_q$ is supersingular if and only if $q$ does not split in $E$.
2. If $q$ does not split in $E$, the Artin invariant of $X_\mathbb{F}_q$ is equal to $[k(q) : \mathbb{F}_p]$. Here $k(q)$ is the residue field of $q$.

Our results generalize Shimada’s results [5] for $K3$ surfaces with $\rho(X_\mathbb{C}) = 20$. See below.

3. Some examples of $K3$ surfaces with CM

$K3$ surfaces with Picard number 20. Projective $K3$ surfaces $X_\mathbb{C}$ over $\mathbb{C}$ with $\rho(X_\mathbb{C}) = 20$ are classified by Shioda-Inose. They have CM by imaginary quadratic fields.

Kummer surfaces. Let $A_\mathbb{C}$ be a simple abelian surface over $\mathbb{C}$ with CM, i.e. $\text{End}(A_\mathbb{C}) \otimes \mathbb{Q}$ is a CM field of degree 4. Then the Kummer surface $\text{Km}(A_\mathbb{C})$ has CM by a CM field of degree 4.

$K3$ surfaces with automorphisms. A projective $K3$ surface $X_\mathbb{C}$ over $\mathbb{C}$ has CM by the cyclotomic field $\mathbb{Q}((\zeta_N))$ if it has an automorphism $f \in \text{Aut}(X_\mathbb{C})$ such that the order of $f^* \in \text{Aut}(T(X_\mathbb{C}))$ is $N$ with

$$\phi(N) = \text{rank}_2 T(X_\mathbb{C}),$$

where $\phi$ is the Euler’s totient function. For each

$$N \in \{3, 5, 7, 9, 11, 12, 13, 17, 19, 25, 27, 28, 36, 42, 44, 66\},$$

Kondō proved that there exists a projective $K3$ surface $X_\mathbb{C}$ over $\mathbb{C}$ with CM by $\mathbb{Q}((\zeta_N))$ [3].

4. Sketch of the proof

- First, we use the main theorem of CM for $K3$ surfaces (Rizov [4]) to calculate the Frobenius action on the Galois module $H^2_{\text{cris}}(X_{\mathbb{F}_p}, \mathbb{Q}_p)$ ($\ell \neq p$).
- We describe the Breuil-Kisin module $\mathfrak{N}(H^2_{\text{cris}}(X_{\mathbb{F}_p}, \mathbb{Z}_p))$ by a description of the Breuil-Kisin modules of Lubin-Tate characters (Andreatta-Goren-Howard-Madapusi Pera).
- We use the integral $p$-adic Hodge theory (Bhatt-Morrow-Scholze [2]) $H^2_{\text{cris}}(X_{\mathbb{F}_p}/W) \cong \varphi^*(\mathfrak{N}(H^2_{\text{cris}}(X_{\mathbb{F}_p}, \mathbb{Z}_p)))/\mathfrak{N}(H^2_{\text{cris}}(X_{\mathbb{F}_p}, \mathbb{Z}_p))).$.
- Finally, we calculate the length of the cokernel of the crystalline Chern class map $\text{Pic}(X_{\mathbb{F}_p}) \otimes \mathbb{Z}W \to H^2_{\text{cris}}(X_{\mathbb{F}_p}/W)$, which is equal to the Artin invariant $a$.

References