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Holographic Entropy Bound in Two-Dimensional Gravity

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Bousso's entropy bound for two-dimensional gravity is investigated in the lightcone gauge. It is shown that due to the Weyl anomaly, the null component of the energy-momentum tensor takes a nonvanishing value, and thus, combined with the conditions that were recently proposed by Bousso, Flanagan and Marolf, a holographic entropy bound similar to Bousso's is expected to hold in two dimensions. A connection of our result to that of Strominger and Thompson is also discussed.

§1. Introduction

The ultimate goal of today's high energy physics is to unify all the interactions including gravity. Although string theory is the most promising candidate for such "theory of everything", a constructive definition of string theory is still under development. For this purpose, it is relevant to grasp the fundamental dynamical variables (even if they are strings) that describe gravity systems in a consistent manner and also in such a way that all observations which have been made on gravity are naturally accounted for.

The holographic principle[1),2)] appears in the developments of the study of thermodynamics of black holes and various entropy bounds, and states that a spacelike or lightlike region of a system including gravity can be equivalently described by the system on the boundary of the region (nice review articles are Refs. 3) and 4)). In order to check this principle quantum mechanically, we need to have a full quantum theory of gravity, but the current status of string theory is not at the level to check in a satisfactory manner whether the assertion holds or not. On the other hand, gravity can be treated fully quantum mechanically in two dimensions, so that it is worthwhile to study the holographic nature in two-dimensional gravity.

Here we first make a historical review of the holographic principle. In the early 70's, Bekenstein proposed that the entropy of a black hole, $S_{\rm BH}$, is proportional to the area of the horizon,[5),6)] and subsequently, Bardeen, Cartar and Hawking[7)] and Hawking[8)] determined the coefficient to be $1/4G_{\rm N}$, i.e., $S_{\rm BH} = A/4G_{\rm N}$, where A and $G_{\rm N}$ are the area of the horizon and the Newton constant, respectively.[†)] On the basis of their proposal and by requiring the generalized second law of thermodynamics[9)]

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†) We will set $G_{\rm N} = 1$ in the following discussions.

(matters' entropy plus black holes' entropy should increase in time), Bekenstein[10] showed that there must be a bound on the entropy of matters in the following form:

$$S \leq 2\pi E R, \tag{1·1}$$

where S, E and R are the entropy, the energy and the linear size of the system, respectively. This entropy bound is expected to hold for systems with weak gravity. On the other hand, 't Hooft[1] and Susskind[2] pointed out that, for strongly gravitating systems, we must use another entropy bound

$$S \leq \frac{1}{4} A, \tag{1·2}$$

where A is the area of the boundary of a given region. The bound saturates when the region is filled with a black hole whose horizon coincides with the boundary. This bound (1·2) is much more radical than the Bekenstein bound, because it asserts that the entropy of gravity system is not extensive and should be bounded by area. It thus suggests that the fundamental dynamical variables including gravity are not ordinary local variables but correspond to some quantities living on the boundary. This is called the holographic principle.

Originally, the bound (1·2) is assumed to hold for the entropy of a spacelike region, but there are many counterexamples to such spacelike entropy bound. One such counterexample is the flat Friedmann-Robertson-Walker (FRW) universe. Fischler and Susskind[11] pointed out that if the entropy is estimated for a lightlike region (null hypersurface) then the same form of entropy bound does hold even for the flat FRW universe. Subsequently Bousso made their proposal into a precise form,[12,13] introducing the idea of "lightsheet", which is a null hypersurface characterized by the condition that the expansion θ is nonpositive along the generator of the null hypersurface. His entropy bound requires the entropy S_L on the lightsheet L to be bounded by the difference of the areas $A(0)$ and $A(1)$ of the boundaries of the lightsheet (see Fig. 1):[*]

$$S_L \leq \frac{1}{4}\big(A(0) - A(1)\big). \tag{1·3}$$

This bound does not have a serious counterexample so far and is believed to hold for any classical gravity.

A derivation of the Bousso bound (1·3) was given by Flanagan, Marolf and Wald (FMW),[14] who assumed a condition that is essentially equivalent to the Bekenstein bound on a null hypersurface:

$$s \equiv -s^\mu\, k_\mu \leq 2\pi\,(1-\lambda)\, T_{\mu\nu}\, k^\mu k^\nu. \tag{1·4}$$

Here s^μ and s are the entropy current and the entropy density over a lightsheet L, respectively, and the null vector $k \equiv d/d\lambda$ is the generator of the lightsheet with affine parameter λ (for detailed definitions, see the next sections).

[*] Bousso originally proposed the bound as $S_L \leq (1/4)\,A(0)$. The entropy bound in the form of Eq. (1·3) was first presented by Flanagan, Marolf and Wald.[14]

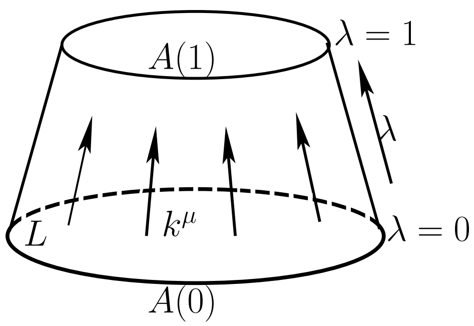

Fig. 1. A null hypersurface L generated by a null vector $k = d/d\lambda$ with affine parameter λ. L is parametrized from $\lambda = 0$ to $\lambda = 1$. Letting $A(\lambda)$ be the cross-sectional area at λ, the expansion θ is defined as $\theta(\lambda) \equiv A'(\lambda)/A(\lambda)$. If the expansion is always non-positive along λ, $\theta(\lambda) \le 0$, the null hypersurface L is called (part of) a lightsheet. The Bousso entropy bound states that the entropy S_L on the lightsheet is bounded by the difference of the areas $A(0)$ and $A(1)$ of the boundaries of the lightsheet, $S_L \le (1/4)\big(A(0) - A(1)\big)$.

The FMW condition (1·4) is not given in a completely local form, as can be seen, for example, from the presence of λ in the expression. Recently, Bousso, Flanagan and Marolf (BFM)[15)] gave another condition which is completely local:

$$s' \equiv \frac{ds}{d\lambda} \le 2\pi\, T_{\mu\nu}\, k^\mu k^\nu. \tag{1·5}$$

This is essentially a differential form of the FMW condition. As discussed in the next section, this condition leads to the Bousso bound in $(n+2)$ dimensions $(n \ge 1)$. The essential step in the derivation there is to use the identity

$$T_{\mu\nu}\, k^\mu k^\nu(\lambda) = -\frac{n}{8\pi}\,\frac{G''(\lambda)}{G(\lambda)} - \frac{1}{8\pi}\,\sigma_{\mu\nu}\,\sigma^{\mu\nu}, \tag{1·6}$$

which can be proven by the Raychaudhuri equation (see the next section). Here $\sigma_{\mu\nu}$ is the shear tensor, and

$$G(\lambda) \equiv \exp\Big[\frac{1}{n}\int_0^\lambda d\bar\lambda\, \theta(\bar\lambda)\Big]. \tag{1·7}$$

In the present article, we discuss two-dimensional gravity $(n = 0)$ which is coupled to conformal matters of central charge c, treating gravity as a background for quantum matters. If we set $n = 0$, the right-hand side of Eq. (1·6) vanishes since the shear tensor does not exist in two dimensions, so that Eqs. (1·5) and (1·6) do not seem to give a physically meaningful bound on the entropy. The main purpose of this article is to show that due to the Weyl anomaly, an equality similar to Eq. (1·6) holds for two dimensions:

$$T_{\mu\nu}\, k^\mu k^\nu(\lambda) = -4\beta\,\frac{\bar G''(\lambda)}{\bar G(\lambda)}, \tag{1·8}$$

where the coefficient is related to the central charge of matters as $\beta = (c-26)/48\pi$, and $\bar G(\lambda)$ is related to an effective "area" of point at λ. This equality allows us to prove a nontrivial bound on the entropy. In order to discuss fully quantum aspects of the Bousso bound, we further need to quantize gravity. This is now under investigation and will be reported elsewhere.

This paper is organized as follows. In §2, we first review the BFM conditions in $(n+2)$-dimensional space-time, which leads to a classical bound on the entropy over a lightsheet. The derivation does not work in two dimensions when gravity systems are treated classically. However, in §3, we show that the same manner of derivation is also possible in two dimensions when the Weyl anomaly is taken into account correctly. In §4, we compare our result with the energy-momentum tensor of two-dimensional dilatonic gravity, which can be obtained by compactifying an $(n+2)$-dimensional space-time on n-dimensional sphere, S^n. This section was inspired by work of Strominger and Thompson.[16] Section 5 is devoted to conclusion and outlook.

§2. BFM conditions

Let (M, g) be an $(n+2)$-dimensional space-time with metric $g = (g_{\mu\nu})$. Let L be a null hypersurface which is generated by a null vector k with affine parameter λ,[*]

$$k = \frac{d}{d\lambda}. \tag{2·1}$$

We assume that L is parametrized from $\lambda = 0$ to $\lambda = 1$ with two boundaries. Denoting the cross-sectional area at λ by $A(\lambda)$, and its ratio to $A(0)$ by $a(\lambda)$,

$$a(\lambda) \equiv \frac{A(\lambda)}{A(0)}, \tag{2·2}$$

we introduce the expansion $\theta(\lambda)$ as

$$\theta(\lambda) \equiv \frac{1}{A(\lambda)} \frac{dA(\lambda)}{d\lambda} = \frac{1}{a(\lambda)} \frac{da(\lambda)}{d\lambda}. \tag{2·3}$$

Then L is called (part of) a lightsheet if the expansion θ is always non-positive along λ, $\theta(\lambda) \leq 0$.

By denoting the entropy current by s^μ and the entropy density by $s \equiv -s_\mu k^\mu$ on the lightsheet L, Bousso, Flanagan and Marolf[15] showed that the Bousso bound (1·3) can be derived if the following two conditions are satisfied:

$$\text{(i)} \quad s'(\lambda) \equiv \frac{ds(\lambda)}{d\lambda} \leq 2\pi\, T_{\mu\nu}\, k^\mu\, k^\nu, \tag{2·4}$$

$$\text{(ii)} \quad s(0) \leq -\frac{1}{4}\, a'(0) = -\frac{1}{4} \frac{A'(0)}{A(0)}. \tag{2·5}$$

The first condition is a differential form of the FMW condition, which is an analogue of the Bekenstein bound and thus is supposed to hold for any normal matters. The second one is an initial condition, whose physical meaning is described in Ref. 16).

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[*] λ is called an affine parameter if $k = d/d\lambda = k^\mu \partial_\mu$ satisfies the geodesic equation of the form $k^\nu \nabla_\nu k^\mu = 0$.

We here demonstrate that the above two conditions actually lead to the Bousso bound (1·3) in arbitrary $(n+2)$ dimensions $(n \geq 1)$. We first introduce the function $G(\lambda)$ as

$$G(\lambda) \equiv \big[a(\lambda)\big]^{1/n}. \tag{2·6}$$

Then the Raychaudhuri equation*)

$$\frac{d\theta}{d\lambda} = -\frac{1}{n}\,\theta^2 - \sigma_{\mu\nu}\,\sigma^{\mu\nu} + \omega_{\mu\nu}\,\omega^{\mu\nu} - 8\pi\,T_{\mu\nu}\,k^\mu k^\nu \tag{2·7}$$

gives an inequality on $T_{\mu\nu}\,k^\mu k^\nu$ as

$$T_{\mu\nu}\,k^\mu k^\nu(\lambda) = -\frac{n}{8\pi}\,\frac{G''(\lambda)}{G(\lambda)} - \frac{1}{8\pi}\,\sigma_{\mu\nu}\,\sigma^{\mu\nu} \leq -\frac{n}{8\pi}\,\frac{G''(\lambda)}{G(\lambda)} \leq -\frac{n}{8\pi}\left(\frac{G'(\lambda)}{G(\lambda)}\right)'. \tag{2·8}$$

We thus have the following inequality on the entropy density:

$$\begin{aligned} s(\lambda) &= \int_0^\lambda d\bar{\lambda}\,s'(\bar{\lambda}) + s(0) \\ &\leq 2\pi \int_0^\lambda d\bar{\lambda}\,T_{\mu\nu}\,k^\mu k^\nu(\bar{\lambda}) + s(0) \\ &\leq -\frac{n}{4}\,\frac{G'(\lambda)}{G(\lambda)} + \frac{n}{4}\,\frac{G'(0)}{G(0)} + s(0) \\ &\leq -\frac{n}{4}\,\frac{G'(\lambda)}{G(\lambda)}, \qquad \left(\frac{n}{4}\,\frac{G'(0)}{G(0)} = \frac{1}{4}\,a'(0) \leq -s(0)\right) \end{aligned} \tag{2·9}$$

which gives a bound on the entropy over the lightsheet L $(0 \leq \lambda \leq 1)$ as

$$\begin{aligned} S_L &= A(0) \int_0^1 d\lambda\,s(\lambda)\,\big[G(\lambda)\big]^n \\ &\leq -\frac{n}{4}\,A(0) \int_0^1 d\lambda\,G'(\lambda)\,\big[G(\lambda)\big]^{n-1} \\ &= -\frac{1}{4}\,A(0)\,\big[a(\lambda)\big]_0^1 \\ &= \frac{1}{4}\big(A(0) - A(1)\big). \end{aligned} \tag{2·10}$$

§3. 2D gravity in the lightcone gauge

In the preceding section, we have shown that the classical identity

$$T_{\mu\nu}\,k^\mu k^\nu(\lambda) = -\frac{n}{8\pi}\,\frac{G''(\lambda)}{G(\lambda)} - \frac{1}{8\pi}\,\sigma_{\mu\nu}\,\sigma^{\mu\nu}, \tag{3·1}$$

*) Here, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are the shear and the twist tensors, respectively.[17] The latter vanishes when the vector k^μ generates a family of null hypersurfaces, to which k^μ is normal (and tangent).

plays an essential role in deriving the Bousso bound in $(n+2)$-dimensional spacetime. One might think that the equality makes no sense in two dimensions, since the shear tensor does not exist in two dimensions and thus the right-hand side vanishes when we set $n=0$. The main purpose of this section is to show that if we take into account the Weyl anomaly correctly, the right-hand side is rewritten into the desired form with a nonvanishing coefficient, and that the coefficient is essentially the central charge of the conformal matter to which gravity is coupled.

We consider the effective action $\Gamma_{\rm eff}[g(x)]$ defined as

$$e^{i\Gamma_{\rm eff}[g(x)]}=\int[d\phi(x)]\,e^{iS[\phi(x),g(x)]}, \qquad (3\cdot2)$$

where $\phi(x)$ stands for a set of conformal matters. Assuming that the path integral is regularized in such a way that two-dimensional diffeomorphism (Diff_2) is respected, the effective action can be calculated by integrating the Weyl anomaly equation:

$$\langle T(x)\rangle_g\equiv\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta\Gamma_{\rm eff}}{\delta g_{\mu\nu}(x)}=\frac{c-26}{24\pi}\,R. \qquad (3\cdot3)$$

The result is[18)]

$$\Gamma_{\rm eff}[g]=\frac{\beta}{2}\int d^2x\,\sqrt{-g}\,R\,\frac{1}{\nabla^2}\,R \qquad (3\cdot4)$$

with[*)]

$$\beta\equiv\frac{c-26}{48\pi}. \qquad (3\cdot5)$$

There are two popular parametrizations (or gauges) of metric; one is the conformal gauge and the other is the lightcone gauge. Although the former has an advantage in its manifest covariance, the lightcone gauge would be more convenient in order to analyze holographic behavior over lightsheets. We thus write the metric as

$$ds^2=g_{\mu\nu}(x)\,dx^\mu dx^\nu=-dx^+\big(dx^-+h(x^+,x^-)\,dx^+\big). \qquad (3\cdot6)$$

For this, $x^+=\mathrm{const}$ gives a null hyper "surface", which is generated by $k\equiv\partial_-=\partial/\partial x^-$ with affine parameter x^-. Then $\Gamma_{\rm eff}[g]=\Gamma_{\rm eff}[h]$ is expressed as[18)]

$$\Gamma_{\rm eff}[h]=\beta\int d^2x\left[\frac{\partial_-^2 f\,\partial_-\partial_+ f}{\left(\partial_- f\right)^2}-\frac{\left(\partial_-^2 f\right)^2\partial_+ f}{\left(\partial_- f\right)^3}\right]. \qquad (3\cdot7)$$

*) Here -26 comes from the Jacobian when we reduce the measure over $g_{\mu\nu}$ to the measure over h:[19)]

$$\frac{[dg_{\mu\nu}]}{\mathrm{Vol}(\mathrm{Diff}_2)}=[dh]\times\big(\mathrm{Jacobian}\big),$$

although we do not make an integration over h in this article.

Here we have introduced the function $f(x^+, x^-)$ through the relation

$$h(x) = \frac{\partial_+ f(x)}{\partial_- f(x)}. \tag{3·8}$$

From Eq. (3·7) the energy-momentum tensor is calculated as

$$\begin{aligned} T_{--} &\equiv T_{\mu\nu}\, k^\mu k^\nu \\ &= \frac{2}{\sqrt{-g}}\, \frac{\delta \Gamma_{\text{eff}}}{\delta g_{\mu\nu}}\, k_\mu k_\nu \\ &= -4\beta \sqrt{\partial_- f}\, \partial_-^2 \frac{1}{\sqrt{\partial_- f}}. \end{aligned} \tag{3·9}$$

This has the same form with Eq. (3·1) under the identification

$$\begin{array}{ccc} \lambda & \leftrightarrow & x^-, \\ G & \leftrightarrow & 1/\sqrt{\partial_- f}, \\ n & \leftrightarrow & (2/3)\,(c-26). \end{array} \tag{3·10}$$

Note that for arbitrary function f of x, the expression $\sqrt{f'(x)}\left(1/\sqrt{f'(x)}\right)''$ is essentially the Schwarzian differential:

$$\{f,\, x\} \equiv \frac{f'(x)\, f'''(x) - (3/2)\left(f''(x)\right)^2}{\left(f'(x)\right)^2} = -2\sqrt{f'(x)}\left(\frac{1}{\sqrt{f'(x)}}\right)''. \tag{3·11}$$

From this, we can easily understand why T_{--} has the form (3·9). In fact, consider the diffeomorphism F defined by

$$F:\, \left(x^+,\, x^-\right) \to \left(\tilde{x}^+,\, \tilde{x}^-\right) = \left(x^+,\, f(x^+, x^-)\right). \tag{3·12}$$

This is actually a conformal isometry from $ds^2 = -dx^+\left(dx^- + h\, dx^+\right)$ to $d\tilde{s}^2 \equiv -d\tilde{x}^+ \, d\tilde{x}^-$ since

$$\begin{aligned} F^*\left(d\tilde{s}^2\right) &= -\, d\tilde{x}^+(x)\, d\tilde{x}^-(x) = -\, dx^+ \left(\left(\partial_+ f\right) dx^+ + \left(\partial_- f\right) dx^-\right) \\ &= -\left(\partial_- f\right) dx^+ \left(dx^- + h\, dx^+\right) = \left(\partial_- f\right) ds^2. \end{aligned} \tag{3·13}$$

The energy-momentum tensor vanishes in the $\tilde{x}$ coordinates ($\widetilde{T}_{--}(\tilde{x}) = 0$), and thus, by using the transformation law of the energy-momentum tensor, $T_{--}(x)$ can be calculated as follows:

$$\begin{aligned} T_{--}(x) &= \left(\partial_- f\right)^2 \widetilde{T}_{--}(\tilde{x}) + \frac{c-26}{24\pi}\left\{f,\, x^-\right\} \\ &= -\frac{c-26}{12\pi} \sqrt{\partial_- f}\, \partial_-^2 \frac{1}{\sqrt{\partial_- f}} \\ &= -\, 4\beta \sqrt{\partial_- f}\, \partial_-^2 \frac{1}{\sqrt{\partial_- f}}. \end{aligned} \tag{3·14}$$

Equation (3·10) gives us an interpretation that in two dimensions, the coefficient of $G^{-1}\,\partial_-^2\,G$ is shifted from the classical value $n=0$ to $n_{\rm eff}\equiv(2/3)(c-26)$. This in turn implies that the radiative corrections from matter fields keep the form of the conditions.

In the lightcone quantization, the coordinate x^+ is regarded as time. Thus, defining the effective area of a point $(x^+{=}0,x^-)$ on the time-slice $x^+=0$ as

$$A_{\rm eff}(x^-)\equiv A_{\rm eff}(0)\,\bigl[G(x^+{=}0,x^-)\bigr]^{n_{\rm eff}}=A_{\rm eff}(0)\left(\frac{\partial_-f(x^+{=}0,0)}{\partial_-f(x^+{=}0,x^-)}\right)^{(c-26)/3} \tag{3·15}$$

with unknown constant $A_{\rm eff}(0)$, and following the derivation of the Bousso bound given in the preceding section with an appropriate initial condition, we expect that the entropy on the lightsheet L $(0\le x^-\le 1)$ at time $x^+=0$ would be bounded as

$$\begin{aligned}S_L&\equiv A_{\rm eff}(0)\int_0^1 dx^-\,s(x^+{=}0,x^-)\,\bigl[G(x^+{=}0,x^-)\bigr]^{n_{\rm eff}}\\&\le-\frac{1}{4}\,A_{\rm eff}(0)\,\Bigl[\bigl[G(x^+{=}0,x^-)\bigr]^{n_{\rm eff}}\Bigr]_0^1\\&=\frac{1}{4}\,\bigl(A_{\rm eff}(0)-A_{\rm eff}(1)\bigr).\end{aligned} \tag{3·16}$$

§4. 2D dilatonic gravity

In this section, we discuss two-dimensional dilatonic gravity, which can be obtained by compactifying an $(n+2)$-dimensional space-time on n-dimensional sphere, S^n. We demonstrate that $n_{\rm eff}=(2/3)(c-26)$ can be naturally identified with the dimensionality of the compactified space, $n_{\rm eff}\sim n$. Our discussion was inspired by work of Strominger and Thompson.[16)]

We consider an $(n+2)$-dimensional space-time M_{n+2} with topology $M_{n+2}=M_2\times S^n$ and with coordinates $X^M=(x^\mu,\,y^i)$ ($\mu=0,1$ (or $+,-$) and $i=1,\cdots,n$). The metric is then written as

$$\begin{aligned}ds^2_{n+2}&=G_{MN}\,dX^MdX^N\\&=g_{\mu\nu}(x)\,dx^\mu dx^\nu+e^{-2\phi(x)}\,\tilde g_{ij}(y)dy^i\,dy^j,\end{aligned} \tag{4·1}$$

where $d\tilde s_n^2\equiv\tilde g_{ij}(y)\,dy^idy^j$ is a metric of unit sphere, which can be taken, for example, to be $\tilde g_{ij}(y)=\delta_{ij}+y_iy_j/(1-y^2)$ with $|y|^2\le 1$. If we take only the zero mode of harmonic functions on S^n, the Einstein-Hilbert action reduces to the action of dilatonic gravity:

$$\begin{aligned}S_{n+2}\bigl[G_{MN}(x,y)\bigr]&=\frac{1}{16\pi G_{n+2}}\int d^2x\,d^ny\,\sqrt{-G}\,R_G\\&\to\frac{1}{16\pi G_2}\int d^2x\,\sqrt{-g}\,e^{-n\phi}\,\Bigl[R+n(n-1)\Bigl(\bigl(\nabla\phi\bigr)^2+e^{2\phi}\Bigr)\Bigr]\\&\equiv S^{\rm DG}\bigl[g_{\mu\nu}(x),\phi(x)\bigr].\end{aligned} \tag{4·2}$$

Here $G_2 \equiv G_{n+2}/\omega_n$ is the two-dimensional Newton constant ($\omega_n = \int d^n y \sqrt{\tilde{g}}$ is the volume of unit sphere) and will be set to unity in the following discussion.

We choose the metric $g_{\mu\nu}(x)$ as in the preceding section,

$$ds_2^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = -dx^+ \big(dx^- + h(x^+, x^-)\, dx^+\big). \tag{4·3}$$

Then the vectors, $k \equiv \partial_-$, $l \equiv \partial_+ - h\,\partial_-$ and $\eta_{(i)} \equiv \partial_{y^i}$, satisfy the following equations:

$$\begin{aligned} &k^2 = l^2 = 0, \quad k \cdot l = -1/2, \quad k \cdot \eta_{(i)} = l \cdot \eta_{(i)} = 0, \\ &k \cdot \nabla\, k^M = 0, \quad k \cdot \nabla\, l^M = 0, \quad k \cdot \nabla\, \eta^M_{(i)} = \hat{B}^M_{\ N}\, \eta^N_{(i)} \end{aligned} \tag{4·4}$$

with

$$\hat{B}^M_{\ N} = -\big(\partial_- \phi\big) \Big(\delta^M_N + 2\, k^M l_N + 2\, l^M k_N\Big). \tag{4·5}$$

From this, we find that x^- is an affine parameter of a null hypersurface $x^+ = \text{const}$, and that the expansion $\theta \equiv \hat{B}^M_{\ M}$ is given by

$$\theta = -n\, \partial_- \phi. \tag{4·6}$$

This can also be concluded by noting that the cross-sectional area at x^- is given by $A = \omega_n\, e^{-n\phi}$ so that the expansion is given as $\theta = \partial_- A/A = -n\,\partial_-\phi$.

The null component of the energy-momentum tensor is calculated as

$$\begin{aligned} T^{\rm DG}_{--} &\equiv T^{\rm DG}_{\mu\nu}\, k^\mu k^\nu \\ &= \frac{2}{\sqrt{-g}}\, \frac{\delta S^{\rm DG}}{\delta g_{\mu\nu}}\, k_\mu k_\nu \\ &= -\frac{n}{8\pi}\, e^{-n\phi}\, e^{\phi}\, \partial_-^2\, e^{-\phi}. \end{aligned} \tag{4·7}$$

This expression suggests that the null component of the energy-momentum tensor of dilatonic gravity is related to that of the preceding section, T_{--}, as

$$T^{\rm DG}_{--} = e^{-n\phi}\, T_{--}, \tag{4·8}$$

if we identify the quantities there as

$$e^{-\phi} \sim G = \frac{1}{\sqrt{\partial_- f}} \tag{4·9}$$

and

$$n_{\rm eff} \sim n. \tag{4·10}$$

This identification can be inferred from the observation made in Ref. 16), that quantities which are scalar on an n-dimensional submanifold of the lightsheet with fixed affine parameter x^- (like entropy density, $s^{(n+2)}$, and the null component of

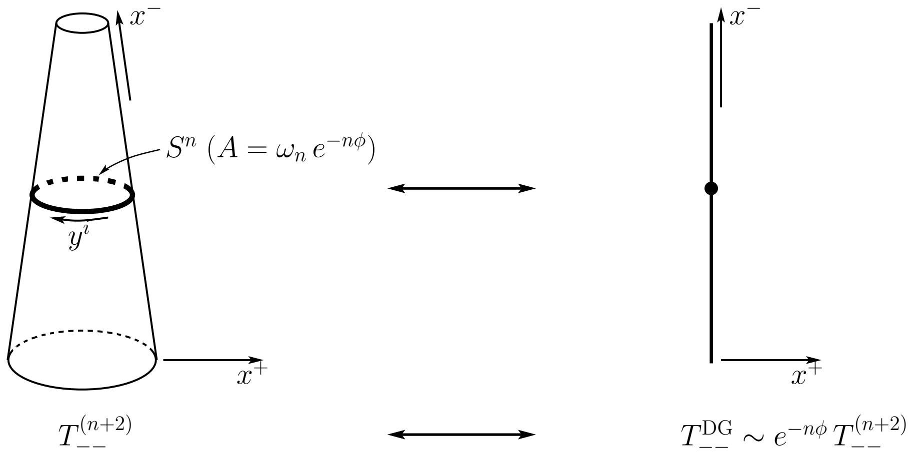

Fig. 2. Each sphere S^n on the lightsheet of the left figure corresponds to a point on the lightsheet of the right figure after the compactification is made from $(n+2)$-dimensional space-time to two-dimensional one. 2D energy momentum tensor $T^{\rm DG}_{--}$ thus corresponds to the $(n+2)$D energy momentum tensor $T^{(n+2)}_{--}$ if we multiply the latter by the cross-sectional area at x^-, $T^{\rm DG}_{--} = \left(\omega_n\, e^{-n\phi}\right) \times T^{(n+2)}_{--}$.

the energy-momentum tensor, $T^{(n+2)}_{--}$) should be multiplied by the area of the submanifold at x^- in order to interpret them as quantities in two-dimensional dilatonic gravity (see Fig. 2):

$$s^{\rm DG} \sim e^{-n\phi}\, s^{(n+2)}, \quad T^{\rm DG}_{--} \sim e^{-n\phi}\, T^{(n+2)}_{--}. \tag{4·11}$$

This implies that the two-dimensional objects like the entropy current and the energy-momentum tensor in the preceding section are directly related to $(n+2)$-dimensional objects through the relation $n_{\rm eff} \sim n$. The relation $n_{\rm eff} = (2/3)(c-26) \sim n$ is actually plausible, since, if we start with larger n, then two-dimensional gravity should feel conformal matters of larger central charge.

We can give another reasoning to the relation (4·8) that is purely based on two-dimensional consideration. We first note that in dilatonic gravity, the propagator of the field h is given by $\big\langle h(p)\, h(-p)\big\rangle \sim e^{n\phi}$. We further note that $S^{\rm DG}[h,\phi]$ can be interpreted as the classical part of the generating functional of amputated 1PI diagrams with vacuum expectation values $h(x)$ and $\phi(x)$, while $\Gamma_{\rm eff}[h]$ is interpreted as the generating functional with $h(x)$ as the source of the energy-momentum tensor. This consideration leads to the desired relation

$$T_{--} \sim \frac{\delta \Gamma_{\rm eff}}{\delta h} \sim \big\langle h\, h\big\rangle\, \frac{\delta S^{\rm DG}}{\delta h} \sim e^{n\phi}\, T^{\rm DG}_{--}. \tag{4·12}$$

§5. Conclusion and outlook

In this paper, we considered two-dimensional gravity, treating metric as a background for quantum conformal matters, and argued that a nontrivial holographic bound on entropy holds even for two-dimensional case due to the Weyl anomaly.

We also discussed dilatonic gravity to demonstrate the naturalness of our relation $n_{\rm eff} \sim (2/3)(c-26)$.

We further need to quantize the metric h in order to see how Bousso's entropy bound is realized in full quantum gravity in two dimensions. The quantization would be carried out with the use of representation theory of $SL(2,\boldsymbol{R})$ Kac-Moody algebra, which is the residual gauge symmetry after the lightcone gauge is taken.[18),20)] Work towards this direction is now in progress.

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