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Weyl Groups in AdS_3/CFT_2

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A system of D1- and D5-branes with a Kaluza-Klein momentum is re-investigated using the five-dimensional U-duality group $E_{6(+6)}(\boldsymbol{Z})$. We show that the residual U-duality symmetry that keeps this D1-D5-KK system intact is generically given by a lift of the Weyl group of $F_{4(+4)}$, embedded as a finite subgroup in $E_{6(+6)}(\boldsymbol{Z})$. We also show that the residual U-duality group is enhanced to $F_{4(+4)}(\boldsymbol{Z})$ when all three charges coincide. We then apply the analysis to the AdS_3/CFT_2 correspondence and discuss that among 28 marginal operators of CFT_2 which couple to massless scalars of AdS_3 gravity at the boundary, 16 would behave as exactly marginal operators for generic D1-D5-KK systems. This is shown by analyzing possible three-point couplings among the 42 Kaluza-Klein scalars with the use of their transformation properties under the residual U-duality group.

§1. Introduction

Since the AdS/CFT conjecture was proposed,[1)] brane systems of various types have been investigated to examine the AdS_{p+1}/CFT_p correspondence.[2)] Among these, AdS_3/CFT_2 has been studied extensively, because CFT_2 is easy to investigate due to its infinite-dimensional symmetry, which in turn allows a detailed comparison between the CFT_2 and the AdS_3 supergravity in the near horizon. There, systems with D1- and D5-branes play important roles, since they have an AdS_3 geometry in the noncompact directions after the near-horizon limit is taken appropriately.

For example, D1-D5-brane systems without a Kaluza-Klein (KK) momentum wrapped on $S^1 \times \mathrm{K3}$ or $S^1 \times T^4$ describe $AdS_3 \times S^3 \times \mathcal{N}$ in the near horizon with $\mathcal{N}$ being K3 or T^4.[3)] Such D1-D5 systems have also drawn much attention because their dimensional reductions are related to five-dimensional black holes.[4)] In fact, if we compactify the type IIB string theory on a five-dimensional torus T^5 with internal coordinates $(y^1, \cdots, y^5)$, and wrap Q_5 D5-branes on T^5 and Q_1 D1-branes along the y^5-direction, then this D1-D5-brane system describes a five-dimensional black hole in the noncompact directions $(x^0, \cdots, x^4)$ with vanishing horizon area, i.e. with vanishing Bekenstein-Hawking entropy.

Black holes with nonvanishing horizon area can also be obtained by adding KK momenta in the D1-direction. The near-horizon limit of this D1-D5-KK system then

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becomes BTZ $\times S^3 \times T^4$. The BTZ black hole is known to be locally equivalent to AdS_3.[5)]

Systems with nonvanishing KK momentum charge $Q_K \neq 0$ have quite different near-horizon behavior from that with $Q_K = 0$. This can be seen for the 42 KK scalars that appear when type IIB strings are compactified on T^5. In fact, for $Q_K = 0$, these 42 scalars will be split into three parts: 20 minimal scalars with $m^2 = 0$, 16 intermediate scalars with $m^2 = 3$, and 6 fixed scalars with $m^2 = 8$.[3)] Here we have set the radius of AdS_3 to unity. The 16 intermediate scalars, G_{a5}, B_{a5}, C_{a5} and C_{abc5} $(a, b, c = 1, \cdots, 4)$, come as KK scalars out of the metric, NS-NS 2-form, R-R 2-form and R-R self-dual 4-form, respectively. The mass of these intermediate scalars near the horizon is expected to change discontinuously if the KK charge is turned on. Klebanov, Rajaraman and Tseytlin[6)] actually showed, setting $B_{a5} = C_{abc5} = 0$, that the fluctuations of G_{a5} and C_{a5} all have the same mass $(m^2 = 3)$ at the horizon if $Q_K = 0$, while for $Q_K \neq 0$, they split into two groups, $(G_{a5} + C_{a5})$ and $(G_{a5} - C_{a5})$, with the mass squared at the horizon given by $m^2 = 0$ and $m^2 = 8$, respectively. It thus seems natural that when taking into account the fluctuations of all the intermediate scalars with $Q_K \neq 0$, they will also be split into two parts, one half joining minimal scalars $(m^2 = 0)$ and another half joining fixed scalars $(m^2 = 8)$.

The moduli space of the corresponding supersymmetric system was analyzed in Ref. 7), showing that the D1-D5-KK systems actually have 28 massless scalars and 14 massive scalars $(m^2 = 8)$ at the horizon. From the AdS_3/CFT_2 correspondence, the 28 massless scalars may couple to marginal operators $(\Delta = 2)$. However, it is possible that these marginal operators behave differently under the renormalization group if higher-order corrections are taken into account. In particular, it is of great interest to know how many among the 28 marginal operators are exactly marginal.

The type II string theory compactified on T^5 is known to have the U-duality group $E_{6(+6)}(\boldsymbol{Z})$.[8)-10)] Thus, it may be useful if we can investigate the D1-D5-KK systems using the U-duality as a complementary tool. Although U-duality transformations generally transform the D1-D5-KK systems into some other brane systems, if the subgroup that keeps the systems intact is sufficiently rich, then it may constrain the system purely from the symmetry principle, giving us useful information on the system.

One of the main aims of the present article is to show that such "residual U-duality symmetry" for three-charge systems is given by a lift of the Weyl group of $F_{4(+4)}$, $\widetilde{W}(F_{4(+4)})$, in $E_{6(+6)}(\boldsymbol{Z})$. We also show that use of the lifted Weyl group enables us to obtain further information on the 42 scalars.

In general, the KK scalars live on a coset manifold and transform non-linearly under the U-duality if all auxiliary fields are removed. This sometimes makes it difficult to study the scalar sector group-theoretically. However, as far as the lifted Weyl group is concerned, there exists a parametrization of the coset manifold such that scalars transform linearly. Since the residual U-duality group is of this type, we can resort to representation theory to determine possible three-point couplings of these scalars, and to discuss which massless scalars couple to exactly marginal operators.

In §2, we discuss the general U-duality group for type II strings compactified on a d-dimensional torus T^d and show that it can be expressed as the semidirect product of the lifted Weyl group and the Borel group of $E_{d+1(d+1)}$. In §3, we treat the $d=5$ case in detail. We show that, when the three charges (Q_1, Q_5, Q_K) take positive integer values generically, the residual U-duality group G is given by the lifted Weyl group of $F_{4(+4)}$, $G = \widetilde{W}(F_{4(+4)})$, while G is enhanced to $F_{4(+4)}(\boldsymbol{Z})$ in the special case when $Q_1 = Q_5 = Q_K$. In §4, after introducing a convenient parametrization of the coset manifold, we determine possible three-point couplings of scalars by using a representation of G and give evidence that 16 operators would behave as exactly marginal operators. Section 5 is devoted to discussion.

§2. *U*-duality and the Weyl-Borel group

We start our discussion by recalling a general property of type II strings compactified on a d-dimensional torus T^d. We will decompose the 10-dimensional coordinates as $(x^{\hat{\mu}}) = (x^\mu, y^i) = (x^0, \cdots, x^{9-d}, y^1, \cdots, y^d)$ with y^i $(i = 1, \cdots, d)$ denoting the coordinates on T^d. This system is conjectured to have the U-duality group $E_{d+1\,(d+1)}(\boldsymbol{Z})$ as an exact symmetry.[9),10)] It is known that this group can be generated by a set of generators $\{\exp(E_{\pm\alpha})\}$.[11),12)] Here $E_{\pm\alpha}$ are step operators of $E_{d+1\,(d+1)}$ (the normal real form of E_{d+1}) and α are the positive roots. For our purposes, however, it is convenient to take another set of generators, which we will call Weyl and Borel generators, and express the U-duality group as the following semidirect product:

$$E_{d+1(d+1)}(\boldsymbol{Z}) = \widetilde{W}(E_{d+1(d+1)}) \ltimes B(E_{d+1(d+1)}). \tag{2·1}$$

Here the Borel subgroup $B(E_{d+1(d+1)})$ consists of the Borel generators of the form $\{\exp(E_\alpha)\}$ with positive roots $\alpha > 0$, while the lifted Weyl group $\widetilde{W}(E_{d+1(d+1)})$ is obtained by lifting the elements of the Weyl group $W(E_{d+1})$ into $E_{d+1\,(d+1)}(\boldsymbol{Z})$.[13),14)] Here the Weyl group $W(E_{d+1})$ is generated by Weyl reflections w_α that act on weights λ as $w_\alpha(\lambda) = \lambda - (2\lambda\cdot\alpha/\alpha\cdot\alpha)\,\alpha$, and $\widetilde{W}(E_{d+1(d+1)})$ is generated by the lift of w_α defined by

$$\widetilde{w}_\alpha \equiv \exp\left(\frac{\pi}{2}(E_\alpha - E_{-\alpha})\right). \tag{2·2}$$

Note that such lifts belong to the maximal compact subgroup K of $E_{d+1\,(d+1)}$,

$$\widetilde{w}_\alpha \in K \equiv \{\exp(\theta^\alpha(E_\alpha - E_{-\alpha}))\}, \tag{2·3}$$

and have an adjoint action on generators of the Lie algebra: $X \to \widetilde{w}_\alpha X \widetilde{w}_\alpha^{-1}$. Although a Weyl reflection w_α satisfies $(w_\alpha)^2 = 1$, the corresponding lifted Weyl generator $\widetilde{w}_\alpha$ generally does not, and only satisfies $(\widetilde{w}_\alpha)^4 = 1$. Note that $\widetilde{w}_{-\alpha} = \widetilde{w}_\alpha^{-1}$. Furthermore, while the Cartan generators transform canonically under the action of the lifted Weyl transformation,

$$\widetilde{w}_\alpha\ (\lambda\cdot H)\ \widetilde{w}_\alpha^{-1} = w_\alpha(\lambda)\cdot H, \tag{2·4}$$

this is not always the case for the step operators:

$$\widetilde{w}_\alpha \; E_\beta \; \widetilde{w}_\alpha^{-1} = C_{\alpha,\beta} \, E_{w_\alpha(\beta)}. \tag{2·5}$$

The constants $C_{\alpha,\beta}$ can be taken to be real if the structure constants $N_{\alpha,\beta}$ of $[E_\alpha, E_\beta] = N_{\alpha,\beta} \, E_{\alpha+\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,\,-\beta} \in \boldsymbol{R}$. Furthermore, for simply-laced Lie algebras with the normalization $\alpha^2 = 2$ for the roots, one can easily see that $N_{\alpha,\beta}$ only takes the value 0 or ± 1, and

$$\widetilde{w}_\alpha \; E_\beta \; \widetilde{w}_\alpha^{-1} = \begin{cases} N_{\alpha,\beta} \, E_{\alpha+\beta}, & (\alpha \cdot \beta = -1) \\ -N_{\alpha,-\beta} \, E_{-\alpha+\beta}, & (\alpha \cdot \beta = +1) \\ E_\beta, & (\alpha \cdot \beta = 0, \, \pm 2) \end{cases} \tag{2·6}$$

and also

$$\widetilde{w}_\alpha \; \widetilde{w}_\beta \; \widetilde{w}_\alpha^{-1} = \begin{cases} (\widetilde{w}_{\alpha+\beta})^{N_{\alpha,\beta}}, & (\alpha \cdot \beta = -1) \\ (\widetilde{w}_{-\alpha+\beta})^{N_{\alpha,-\beta}}, & (\alpha \cdot \beta = +1) \\ \widetilde{w}_\beta. & (\alpha \cdot \beta = 0, \, \pm 2) \end{cases} \tag{2·7}$$

The simplest example is the U-duality (or S-duality itself) of 10-dimensional type IIB strings, $SL(2; \boldsymbol{Z})$, which is generated by the Weyl and the Borel generators S and T:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp\left(\frac{\pi}{2}(E_+ - E_-)\right), \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp\left(E_+\right), \tag{2·8}$$

with

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2·9}$$

Note that $S^2 = -1$ and $S^4 = 1$.

The Weyl-Borel-group structure of the general $E_{d+1(d+1)}(\boldsymbol{Z})$ can be understood easily by decomposing it with respect to the T-duality subgroup $O(d, d; \boldsymbol{Z})$. Introducing an orthonormal basis e_i $(i = 1, \cdots, d)$ in the weight space, $e_i \cdot e_j = \delta_{ij}$, we choose the positive roots of $O(d,d)$ as $\{\alpha_{ij}^{(1)} \equiv -e_i + e_j, \;\; \alpha_{ij}^{(2)} \equiv e_i + e_j\}_{1 \le i < j \le d}$, with the simple roots $\{\alpha_1, \cdots, \alpha_d\}$ as $\alpha_1 \equiv \alpha_{12}^{(2)}$ and $\alpha_i \equiv \alpha_{(i-1)\,i}^{(1)}$ $(i = 2, \cdots, d)$. The lifted Weyl generators then correspond to the following T-duality transformations:

$$\begin{aligned} \widetilde{w}_{\alpha_{ij}^{(1)}} &= R_{ij}, \\ \widetilde{w}_{\alpha_{ij}^{(2)}} &= T'_{ij} \, \equiv \, T_{ij} \, R_{ij}, \end{aligned} \tag{2·10}$$

where R_{ij} is a $\pi/2$–rotation in the (y^i, y^j)-plane, and T_{ij} is the T-duality transformation for y^i- and y^j-directions. Note that T_{ij} and R_{ij} commute because $\alpha_{ij}^{(1)} \cdot \alpha_{ij}^{(2)} = 0$. On the other hand, the Borel generator $\exp\left(E_{\alpha_{ij}^{(1)}}\right)$ yields a linear transformation among the KK scalars from the metric and NS-NS 2-form:

$$\begin{aligned} G_{kl} &\to G_{kl} + \delta_{kj} G_{il} + \delta_{lj} G_{ki} + \delta_{kj} \delta_{lj} G_{ii}, \\ B_{kl} &\to B_{kl} + \delta_{kj} B_{il} + \delta_{lj} B_{ki}. \end{aligned} \tag{2·11}$$

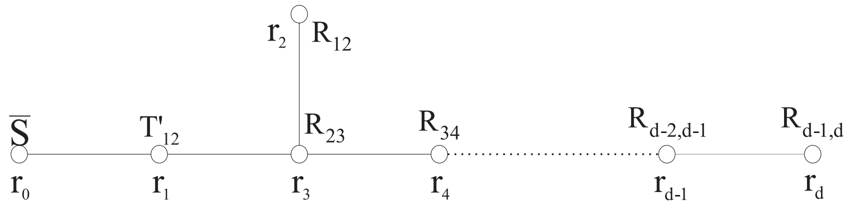

Fig. 1. Dynkin diagram of the U-duality group $E_{d+1(d+1)}$. The lifts of simple reflections, $r_s \equiv w_{\alpha_s}$, are also depicted with $T'_{12} = T_{12}\,R_{12}$. The subsystem $\{\alpha_1, \cdots, \alpha_d\}$ gives the simple roots of $D_{d(+d)} = SO(d,d)$.

By contrast, $\exp\left(E_{\alpha^{(2)}_{ij}}\right)$ generates a constant shift of the KK scalar of the NS-NS 2-form:

$$\begin{aligned} G_{kl} &\to G_{kl}, \\ B_{kl} &\to B_{kl} + \delta_{kj}\delta_{li} - \delta_{ki}\delta_{lj}\,. \end{aligned} \tag{2·12}$$

It is known that $SO(d,d;\boldsymbol{Z})$ can be generated by these generators. [15]

$E_{d+1\,(d+1)}$ is obtained by extending $O(d,d)$ with the addition of one more simple root*) α_0 such that $\alpha_0 \cdot \alpha_1 = -1$ and $\alpha_0 \cdot \alpha_i = 0$ for $i = 2, \cdots, d$ (see Fig. 1). For $d \le 5$, $E_{d+1(d+1)}$ then has three types of positive roots:**)

$$\begin{aligned} &\text{(i)} && \alpha^{(1)}_{ij} = -\,e_i + e_j\,, && (1 \le i < j \le d) \\ &\text{(ii)} && \alpha^{(2)}_{ij} = +\,e_i + e_j\,, && (1 \le i < j \le d) \\ &\text{(iii)} && \alpha(\{n_i\}) = \sum_{i=1}^{d} \left(n_i - \frac{1}{2}\right) e_i + \sqrt{2 - \frac{d}{4}}\, e_0\,. \end{aligned} \tag{2·13}$$

Here, $n_i = 0, 1$ with $\sum n_i = $ even, and e_s $(s = 0, 1, \cdots, d)$ is now an orthonormal basis of the $(d+1)$-dimensional weight space. The simple roots are given by $\alpha_0 = \alpha(\{0\})$, $\alpha_1 = \alpha^{(2)}_{12}$ and $\alpha_i = \alpha^{(1)}_{(i-1)\,i}$ $(i = 2, \cdots, d)$. We define the $(10-d)$-dimensional S-duality transformation $\bar{S}$ as the lift of the Weyl reflection with respect to this α_0: $\bar{S} = \widetilde{w}_{\alpha_0}$. While the usual S-duality exchanges the NS-NS 2-form B_2 with the R-R 2-form C_2 in type IIB strings, this $\bar{S}$ exchanges B_2 with $D_2 \equiv C_2 + B_2\,C_0$, which was introduced in Ref. 16).

§3. $d = 5$ and the residual U-duality group G

Now we discuss the five-dimensional U-duality group $E_{6(+6)}(\boldsymbol{Z})$. There, the 27 vector fields $\boldsymbol{A}^I_\mu$ and the corresponding charges z^I $(I = 1, \cdots, 27)$ transform as **27** of $E_{6(+6)}$ (see Fig. 2). This representation has the cubic invariant $I_3(\mathbf{27}) = c_{IJK} z^I z^J z^K$. It is known that for $I_3(\mathbf{27}) \ne 0$, the 27 charges can be rotated by

*) This root is *not* the lowest root of $SO(d,d)$ that is used in extending D_d to $D^{(1)}_d$.

**) For E_7 we have an extra positive root, $\alpha' = \sqrt{2}\, e_0$.

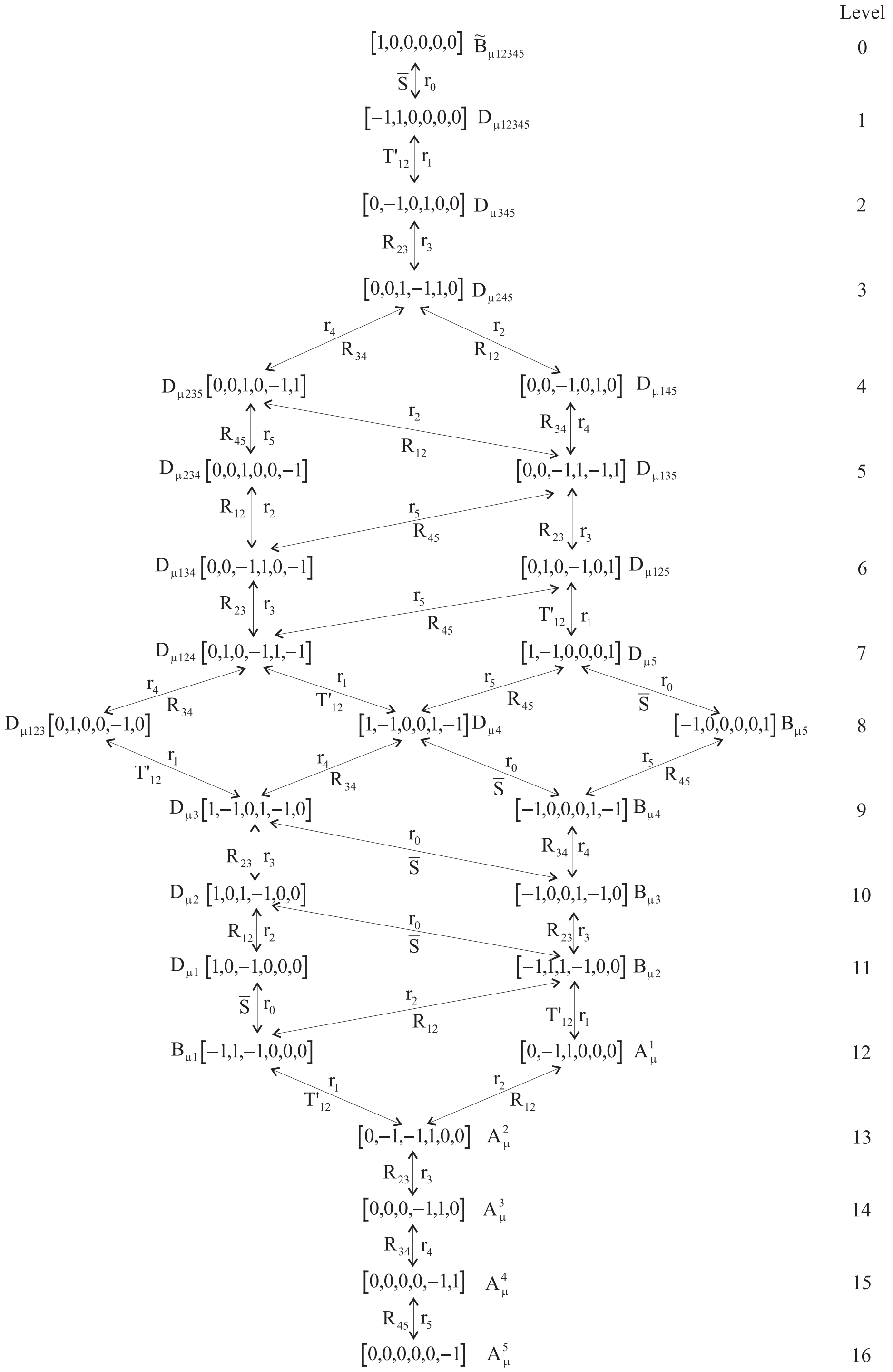

Fig. 2. Weight diagram of **27** for $E_{6(6)}$. The weight λ is indicated by the Dynkin indices $[q_s] = [q_0, q_1, \cdots, q_5]$ with $\lambda = \sum_{s=0}^{5} q_s\, \mu^s$ (μ^s: the fundamental weights). The corresponding gauge fields $\boldsymbol{A}_\mu^I$ are also depicted. r_s again denotes the Weyl reflection with respect to the simple root α_s: $r_s \equiv w_{\alpha_s}$.

$F_{4(+4)}$ into systems where only three of them (Q_1, Q_2, Q_3) do not vanish:[17)]

$$z^I = (0, \cdots, 0, Q_1, 0, \cdots, 0, Q_2, 0, \cdots, 0, Q_3, 0, \cdots, 0), \qquad (3\cdot1)$$

with $I_3(\mathbf{27}) \propto Q_1 Q_2 Q_3$. The corresponding weights λ_1, λ_2 and λ_3 must satisfy the condition $\lambda_1 + \lambda_2 + \lambda_3 = 0$. In the following, we exclusively consider these three-charge systems.

The 42 KK scalars, on the other hand, live on the coset space $E_{6(+6)}(\boldsymbol{R})/USp(8)$, where $USp(8)$ is the maximal compact subgroup of $E_{6(+6)}(\boldsymbol{R})$, and the equivalence relation for $L \in E_{6(+6)}(\boldsymbol{R})$ is introduced as $L \sim L'$ for $L' = g \cdot L$ with an element $g \in USp(8)$. One parametrization of L is given by the Iwasawa decomposition:[*)]

$$L(\varphi^i, \varphi^\alpha) = \exp\left(\sum H_i \varphi^i\right) \exp\left(\sum_{\alpha>0} E_\alpha \varphi^\alpha\right). \qquad (3\cdot2)$$

Another parametrization, which respects the structure of the Weyl group, is given by

$$L(\phi^i, \phi^\alpha) = \exp\left(\sum H_i \phi^i\right) \exp\left(\sum_{\alpha>0} (E_\alpha + E_{-\alpha}) \phi^\alpha\right). \qquad (3\cdot3)$$

We assume that the scalars have their classical values only at the Cartan part.

Under the $SO(5,5)$ subgroup, $\mathbf{27}$ of $E_{6(+6)}$ is decomposed as $\mathbf{1} + \mathbf{10}_{\boldsymbol{v}} + \mathbf{16}_{\boldsymbol{c}}$. We denote these singlet, vector and Majorana-Weyl (conjugate-) spinor charges of $SO(5,5)$ by u, v^A $(A = 1, \cdots, 10)$ and s^α $(\alpha = 1, \cdots, 16)$, respectively. We also denote the fundamental weights of $E_{6\,(+6)}$ by μ^r $(r = 0, \cdots, 5)$, with $2\,(\mu^r \cdot \alpha_s)/(\alpha_s \cdot \alpha_s) = \mu^r \cdot \alpha_s = \delta^r_s$. Throughout the paper, we let the long roots have length squared equal to two. Then, under the convention of Fig. 2, the singlet charge u corresponds to the highest weight μ^0 of $\mathbf{27}$ and comes from $\widetilde{B}_{\mu 12345}$, the electromagnetic dual of the singlet NS-NS 2-form $B_{\mu\nu}$ in the 5 noncompact dimensions. The vector charges all come from the NS-NS gauge fields $(B_{\mu\, i},\, A^i_\mu)$ $(i = 1, \cdots, 5)$, where A^i_μ is the KK gauge field. The 16 spinor charges come from the R-R gauge fields $D_{\mu\,\alpha} = (D_{\mu\,1}, \cdots, D_{\mu\,5}, D_{\mu\,123}, \cdots, D_{\mu\,345}, D_{\mu\,12345})$, where D is the modified R-R potential that is obtained by suitably combining the original R-R potential with the NS-NS 2-form.[16)] The cubic invariant of $\mathbf{27}$ is then decomposed with respect to $SO(5,5)$ as

$$I_3(\mathbf{27}) = c_{IJK} z^I z^J z^K = u J_{AB} v^A v^B + \frac{1}{2\sqrt{2}} v^A (C\varGamma_A)_{\alpha\beta} s^\alpha s^\beta. \qquad (3\cdot4)$$

Here J_{AB} is the $SO(5,5)$ invariant tensor, $\varGamma_A$ represents the gamma matrices of $SO(5,5)$ satisfying $\{\varGamma_A, \varGamma_B\} = 2J_{AB}$, and C is the charge conjugation matrix.

Since Borel generators move the classical values for scalars out of the Cartan, we only need to consider the lifted Weyl group $\widetilde{W}(E_{6(+6)})$, looking for the subgroup that keeps the generic three-charge system intact. Since an element of $\widetilde{W}(E_{6(+6)})$ maps one three-charge system to another, we can set the three charges with any

[*)] See Refs. 18) and 19) for recent discussion on the parametrization respecting the solvability.

weights $(\lambda_1, \lambda_2, \lambda_3)$ we like, as long as they satisfy the condition $\lambda_1 + \lambda_2 + \lambda_3 = 0$. There are two types of choices in view of $SO(5,5)$.[20] The first one is to set the charges only in the first term of (3·4). As is clear from Fig. 2, all of the nonvanishing charges are related to NS-NS fields. We thus call this choice the gauge of NS type. Another choice is to set the charges in the second term of (3·4), and this will be called the gauge of R type. In particular, if we give charges to the gauge fields $(D_{\mu 12345}, D_{\mu 5}, A^5_\mu)$ of the D1-D5-KK system, then this is given by an R-type gauge. We will especially call this choice the R(amond) gauge. On the other hand, its $\bar{S}$-dual with charges for $(\widetilde{B}_{\mu 12345}, B_{\mu 5}, A^5_\mu)$ is given by an NS-type gauge, to be called especially the NS gauge. Their weights can be easily seen from Fig. 2:

gauge	gauge fields	weights
R gauge	$(D_{\mu 12345}, D_{\mu 5}, A^5_\mu)$	$(\lambda_1,\, \lambda_2,\, \lambda_3) = (-\mu^0 + \mu^1,\, \mu^0 - \mu^1 + \mu^5,\, -\mu^5)$
NS gauge	$(\widetilde{B}_{\mu 12345}, B_{\mu 5}, A^5_\mu)$	$(\lambda'_1,\, \lambda'_2,\, \lambda'_3) = (\mu^0,\, -\mu^0 + \mu^5,\, -\mu^5)$

(3·5)

The residual U-duality group G that does not change the charge vector up to permutations of the three charges (Q_1, Q_2, Q_3) can be easily determined by the following consideration. First, we take the NS gauge above, and note that the three weights λ'_1, λ'_2 and λ'_3 span a two-dimensional subspace $\boldsymbol{R}^2$ in the six-dimensional weight space of E_6. Since Weyl reflections induce orthogonal transformations in the weight space, the elements of the residual-symmetry group should be the lifts of those elements of $W(E_6)$ that transform the subspace $\boldsymbol{R}^2$ and its orthogonal complement $\boldsymbol{R}^4$ into themselves, respectively. On the other hand, it is easy to see that this $\boldsymbol{R}^4$ is spanned by $\alpha_1, \cdots, \alpha_4$, since μ^r is the dual basis of α_r. Furthermore, these simple roots $\alpha_1, \cdots, \alpha_4$ constitute a simple-root system of the $D_{4(+4)}$ subalgebra of $E_{6(+6)}$. Thus, these transformations will induce automorphisms of the root lattice of $D_{4(+4)}$, which is a sublattice of the weight lattice of $E_{6(+6)}$.*) The group consisting of such automorphisms is given by the semidirect product of the outer automorphism group S_3 and the Weyl group $W(D_4)$, and it is known to be isomorphic to the Weyl group of F_4, $W(F_4) = S_3 \bowtie W(D_4)$. This F_4 is embedded in $E_{6(+6)}$ as a subalgebra $F_{4(+4)}$, as is shown in the Appendix. Interestingly, elements of this outer automorphism have a one-to-one correspondence with the permutations among the three weights λ'_1, λ'_2 and λ'_3. Explicit calculation using (2·4)–(2·7) shows that permutations on the triplet $(\alpha_1, \alpha_2, \alpha_4)$ correspond to those on $(\lambda'_3, \lambda'_2, \lambda'_1)$. We thus conclude that the residual U-duality group G of the three-charge system is generically given by the lifted Weyl group of the subalgebra $F_{4(+4)}$, embedded in the lifted Weyl group of $E_{6(+6)}$:

$$G = \widetilde{W}(F_{4(+4)}) \left(\subset \widetilde{W}(E_{6(+6)}) \right). \tag{3·6}$$

We comment that this conclusion needs to be modified in the case $Q_1 = Q_2 = Q_3$. In fact, as can be easily seen from the folding procedure in the Appendix, the little

*) For its $\bar{S}$-dual D1-D5-KK system, the corresponding simple-root system of the isotropy $D_{4(+4)}$ is given by $\{\alpha_0 + \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

group of the charge vector z^I is enlarged for this case so as to include some Borel generators, which constitute the Borel subgroup of $F_{4(+4)}(\boldsymbol{Z})$. Thus, for this special case, the residual U-duality group can be thought to be enhanced to $G = F_{4(+4)}(\boldsymbol{Z})$, as far as the charge vectors are concerned.[21)]

Our group $G = \widetilde{W}(F_{4(+4)})$ is generated by the lifts of the Weyl reflections with respect to the simple roots β'_a $(a = 1, \cdots, 4)$ of the subalgebra $F_{4(+4)}$. The step operators associated with the simple roots are determined in the Appendix, and are given in terms of $E_{6(+6)}$ generators by

$$\begin{aligned}
e'_{\beta'_1} &= E_{\alpha_3}, \\
e'_{\beta'_2} &= E_{\alpha_4}, \\
e'_{\beta'_3} &= E_{\alpha_0+\alpha_1+\alpha_2+\alpha_3} + E_{-(\alpha_0+\alpha_1+\alpha_3+\alpha_4)}, \\
e'_{\beta'_4} &= E_{\alpha_1+\alpha_3+\alpha_4+\alpha_5} + E_{-(\alpha_2+\alpha_3+\alpha_4+\alpha_5)}.
\end{aligned} \tag{3·7}$$

The lifts of the Weyl reflections with respect to them, $\widetilde{w}'_{\beta'_a} = \exp\left(\frac{\pi}{2}\left(e'_{\beta'_a} - e'_{-\beta'_a}\right)\right)$, are thus

$$\begin{aligned}
\widetilde{w}'_{\beta'_1} &= \widetilde{w}_{\alpha_3}, \\
\widetilde{w}'_{\beta'_2} &= \widetilde{w}_{\alpha_4}, \\
\widetilde{w}'_{\beta'_3} &= \widetilde{w}_{\alpha_0+\alpha_1+\alpha_2+\alpha_3} \cdot \left(\widetilde{w}_{\alpha_0+\alpha_1+\alpha_3+\alpha_4}\right)^{-1}, \\
\widetilde{w}'_{\beta'_4} &= \widetilde{w}_{\alpha_1+\alpha_3+\alpha_4+\alpha_5} \cdot \left(\widetilde{w}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}\right)^{-1}.
\end{aligned} \tag{3·8}$$

All the above can be translated into our original D1-D5-KK system in the R gauge by further taking the $\bar{S}$-dual of the system in the NS gauge. The step operators associated with the simple roots β_a of $F_{4(+4)}$ can be calculated by $e_{\beta_a} = \widetilde{w}_{\alpha_0} \cdot e'_{\beta'_a} \cdot \widetilde{w}_{\alpha_0}^{-1}$, and are written in terms of $E_{6(+6)}$ generators as

$$\begin{aligned}
e_{\beta_1} &= E_{\alpha_3}, \\
e_{\beta_2} &= E_{\alpha_4}, \\
e_{\beta_3} &= E_{\alpha_1+\alpha_2+\alpha_3} + E_{-(\alpha_1+\alpha_3+\alpha_4)}, \\
e_{\beta_4} &= E_{\alpha_0+\alpha_1+\alpha_3+\alpha_4+\alpha_5} + E_{-(\alpha_2+\alpha_3+\alpha_4+\alpha_5)}.
\end{aligned} \tag{3·9}$$

The generators of the residual U-duality symmetry in the R gauge are thus

$$\begin{aligned}
\widetilde{w}_{\beta_1} &= \widetilde{w}_{\alpha_3}, \\
\widetilde{w}_{\beta_2} &= \widetilde{w}_{\alpha_4}, \\
\widetilde{w}_{\beta_3} &= \widetilde{w}_{\alpha_1+\alpha_2+\alpha_3} \cdot \left(\widetilde{w}_{\alpha_1+\alpha_3+\alpha_4}\right)^{-1}, \\
\widetilde{w}_{\beta_4} &= \widetilde{w}_{\alpha_0+\alpha_1+\alpha_3+\alpha_4+\alpha_5} \cdot \left(\widetilde{w}_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}\right)^{-1}.
\end{aligned} \tag{3·10}$$

The remaining generators of the subalgebra $F_{4(+4)}$ in the R gauge are obtained by taking the commutators of (3·9), which yield 12 long and 12 short, positive roots. The step operators e_{β_L} for long roots are generally given by $\pm E_\alpha$ with a root α

of $E_{6(+6)}$ (not necessarily positive), while the step operators e_{β_S} for short roots generally have the form $E_\alpha \pm E_{\alpha'}$ with some roots α and α' of $E_{6(+6)}$ (again not necessarily positive). The Cartan generators $h_a = [e_{\beta_a}, e_{-\beta_a}]$ are given by

$$h_1 = H_3, \qquad h_2 = H_4, \qquad h_3 = H_2 - H_4, \qquad h_4 = H_0 + H_1 - H_2. \tag{3·11}$$

Their orthogonal complement in the Cartan subalgebra of $E_{6(+6)}$ with respect to the Killing metric are spanned by

$$\begin{aligned} j_1 &= H_0 + H_1 + H_2 + 2H_3 + 2H_4 + 2H_5, \\ j_2 &= 2H_1 + H_2 + 2H_3 + H_4. \end{aligned} \tag{3·12}$$

In general, if the step operators associated with the simple roots β_3 and β_4 are written as

$$e_{\beta_3} = E_{\alpha_{(3)}} \pm E_{\alpha'_{(3)}}\,, \qquad e_{\beta_4} = E_{\alpha_{(4)}} \pm E_{\alpha'_{(4)}}\,, \tag{3·13}$$

then the linear basis of the orthogonal complement, j_r, can be given by

$$\begin{aligned} j_1 &= \left[E_{\alpha_{(3)}},\, E_{-\alpha_{(3)}}\right] - \left[E_{\alpha'_{(3)}},\, E_{-\alpha'_{(3)}}\right]\,, \\ j_2 &= \left[E_{\alpha_{(4)}},\, E_{-\alpha_{(4)}}\right] - \left[E_{\alpha'_{(4)}},\, E_{-\alpha'_{(4)}}\right]\,, \end{aligned} \tag{3·14}$$

as can be easily checked.

§4. Classification of scalar multiplets

We now classify the 42 scalars with respect to the residual U-duality group $G = \widetilde{W}(F_{4(+4)})$. We take the parametrization (3·3) for the scalar manifold $E_{6(+6)}(\boldsymbol{R})$ $/USp(8)$. In general, the scalar fields (ϕ^i, ϕ^α) transform non-linearly under the U-duality transformations. However, if we restrict the U-duality to the lifted Weyl group $\widetilde{W}(E_{6(+6)})$, these scalars transform linearly. In fact for $\widetilde{w} \in \widetilde{W}(E_{6(+6)})$, we have

$$\begin{aligned} L(\phi^i, \phi^\alpha) \cdot \widetilde{w}^{-1} &= \widetilde{w}^{-1} \cdot \widetilde{w} \cdot L(\phi^i, \phi^\alpha) \cdot \widetilde{w}^{-1} \\ &\equiv \widetilde{w}^{-1} \cdot L\left(\phi'^i, \phi'^\alpha\right) \\ &\sim L\left(\phi'^i, \phi'^\alpha\right), \end{aligned} \tag{4·1}$$

since the lifted Weyl group of $E_{6(+6)}$ is a subgroup of the maximal compact subgroup $USp(8)$: $\widetilde{w} \in \widetilde{W}(E_{6(+6)}) \subset USp(8)$. We comment that the transformation of the fields ϕ^i under the lifted Weyl group always reduces to a (linear) representation of the Weyl group $W(E_6)$, since the Cartan generators $H_i = [E_{\alpha_i}, E_{-\alpha_i}]$ transform as $\tilde{w}\, H_i\, \tilde{w}^{-1} = \sum_j H_j\, w_{ji}$ $\left(w(\alpha_i) = \sum_j \alpha_j\, w_{ji}\right)$ without any extra factors [see (2·4)–(2·7)]. These scalars ϕ^i correspond to the "dilatonic scalars" considered in Ref. 22).

It is known that the scalar manifold of $d = 5$ maximal supergravity has the following $N = 2$ decomposition:[23)]

$$E_{6(+6)}/USp(8) = F_{4(+4)}/USp(6) \times USp(2) + SU^*(6)/USp(6). \tag{4·2}$$

The first term is expected to correspond to 28 massless scalars and the second to 14 massive scalars. With our parametrization, this decomposition is expressed as

$$L = \exp\left(\sum_{a=1}^{4} h_a \phi^a + \sum_{r=1}^{2} j_r \psi^r\right)$$
$$\times \exp\left(\sum_{\beta\in\Delta_L^+}(e_\beta + e_{-\beta})\,\phi_L^\beta + \sum_{\beta\in\Delta_S^+}(e_\beta + e_{-\beta})\,\phi_S^\beta + \sum_{\beta\in\Delta_S^+}(x_\beta + x_{-\beta})\psi_S^\beta\right). \qquad (4\cdot3)$$

Here Δ_L^+ and Δ_S^+ are the set of long and short, positive roots of $F_{4(+4)}$, respectively. When e_β with $\beta \in \Delta_S^+$ is written as $e_\beta = E_\alpha \pm E_{\alpha'}$ in terms of $E_{6(+6)}$ generators (α and α' need not be positive), we define its complement x_β by $x_\beta \equiv E_\alpha \mp E_{\alpha'}$. For them, the fields ϕ^α and $\phi^{\alpha'}$ in (3·3) are mapped into

$$\phi_S^\beta = \frac{1}{2}\left(\phi^\alpha \pm \phi^{\alpha'}\right), \qquad \psi_S^\beta = \frac{1}{2}\left(\phi^\alpha \mp \phi^{\alpha'}\right), \qquad (4\cdot4)$$

as can be checked easily by equating $(E_\alpha + E_{-\alpha})\,\phi^\alpha + (E_{\alpha'} + E_{-\alpha'})\,\phi^{\alpha'}$ with $(e_\beta + e_{-\beta})\,\phi_S^\beta + (x_\beta + x_{-\beta})\,\psi_S^\beta$. The Cartans h_a $(a = 1, \cdots, 4)$ are again defined by $h_a \equiv [e_{\beta_a}, e_{-\beta_a}]$ for the simple roots β_a of $F_{4(+4)}$, and j_r are defined as their orthogonal complements with respect to the Killing metric [see (3·12)]. We denote the 28 scalars $(\phi^a, \phi_L^\beta, \phi_S^\beta)$ which parametrize $F_{4(+4)}/USp(6)\times USp(2)$ by $(\mathbf{4}^-, \mathbf{12}_{\boldsymbol{L}}^-, \mathbf{12}_{\boldsymbol{S}}^-)$, and the 14 scalars (ψ^r, ψ_S^β) for $SU^*(6)/USp(6)$ by $(\mathbf{2}^+, \mathbf{12}_{\boldsymbol{S}}^+)$. Here we assign "parity" to the fields. This will be useful in considering the three-point functions.

These scalars transform linearly under the lifted Weyl group $\widetilde{W}(F_{4(+4)})$. Let ϕ and G_{ij} $(1 \le i, j \le 5)$ be the ten-dimensional dilaton and the metric of T^5 in the string frame, respectively. We have made the following identification for the dilatonic scalars:

$$\begin{aligned}
\phi^a\ (\mathbf{4}^-):\quad & e^{-2\phi^1} = e^{-2\phi}\left(G_{33}G_{44}\right),\\
& e^{-2\phi^2} = e^{-4\phi}\left(G_{22}G_{33}G_{44}^2\right),\\
& e^{-2\phi^3} = e^{-3\phi}\left(G_{22}G_{33}G_{44}\right),\\
& e^{-2\phi^4} = e^{-2\phi}\left(G_{11}G_{22}G_{33}G_{44}\right)^{1/2},\\[1ex]
\psi^r\ (\mathbf{2}^+):\quad & e^{-6\psi^1} = e^{-2\phi}\left(G_{11}G_{22}G_{33}G_{44}G_{55}^4\right)^{1/2},\\
& e^{-6\psi^2} = e^{-\phi}\left(G_{11}G_{22}G_{33}G_{44}G_{55}\right).
\end{aligned} \qquad (4\cdot5)$$

It is easy to check that this identification is actually consistent with various transformations of the U-duality. For the D1-D5-KK system in the R gauge, we regard these dilatonic scalar fields as the sum of nonvanishing classical backgrounds and fluctuations, while we regard other scalar fields as purely fluctuations. Using a linear approximation for the fluctuations, the scalar fields ϕ^α [see (3·3)] associated with the roots (2·13) can then be identified as follows:

$$\phi^{\alpha_{ij}^{(1)}} = G_{ij} + \cdots,$$

$$\phi^{\alpha_{ij}^{(2)}} = B_{ij} + \cdots ,$$
$$\phi^{\alpha(\{n_i\})} = D_{\alpha(\{n_i\})} + \cdots . \qquad (4\cdot6)$$

Here $\alpha(\{n_i\})$ is a multi-index determined by the set $\{n_i\} : \alpha(\{n_i\}) = 1^{n_1}2^{n_2}3^{n_3}4^{n_4}5^{n_5}$. For example, $\alpha(\{1,0,1,0,0\})$ corresponds to the KK scalar D_{13}. Thus, under the linear approximation, the scalars $\mathbf{12}_{\boldsymbol{L}}^-$, $\mathbf{12}_{\boldsymbol{S}}^-$ and $\mathbf{12}_{\boldsymbol{S}}^+$ can be identified as

$$\begin{array}{lllll}
\phi_L^\beta\ (\mathbf{12}_{\boldsymbol{L}}^-): & G_{ab}\,, & D_{ab}\,, & & \\
\phi_S^\beta\ (\mathbf{12}_{\boldsymbol{S}}^-): & G_{a5}+D_{a5}\,, & B_{a5}+\frac{1}{3!}\epsilon_{abcd}D_{bcd5}\,, & B_{ab}+\frac{1}{2}\epsilon_{abcd}B_{cd}\,, & D+D_{1234}\,, \\
\psi_S^\beta\ (\mathbf{12}_{\boldsymbol{S}}^+): & G_{a5}-D_{a5}\,, & B_{a5}-\frac{1}{3!}\epsilon_{abcd}D_{bcd5}\,, & B_{ab}-\frac{1}{2}\epsilon_{abcd}B_{cd}\,, & D-D_{1234}\,.
\end{array} \qquad (4\cdot7)$$

The range of indices a and ab should be understood as $a = 1, \cdots, 4$ and $1 \leq a < b \leq 4$. Note that this D_{a5} equals C_{a5} under this approximation. This identification is thus consistent with the splitting of G_{a5} and C_{a5} observed in Ref. 6).

If we schematically write the exponent of (4·3) as

$$\sum_{\boldsymbol{r}} \sum_{m=1}^{\dim \boldsymbol{r}} |\, \boldsymbol{r}, m\,\rangle\, \phi_m^{\boldsymbol{r}}\,, \qquad (4\cdot8)$$

with $\boldsymbol{r} = \mathbf{4}^-$, $\mathbf{12}_{\boldsymbol{L}}^-$, $\mathbf{12}_{\boldsymbol{S}}^-$, $\mathbf{2}^+$ and $\mathbf{12}_{\boldsymbol{S}}^+$, then one can introduce the analogue of $3j$-symbols that are defined as the coefficients in the following expansion when a singlet $\mathbf{1}$ exists in the tensor product $\boldsymbol{r_1} \otimes \boldsymbol{r_2} \otimes \boldsymbol{r_3}$:

$$|\,\mathbf{1}\,\rangle\!\rangle\!\rangle = \sum_{m_1,m_2,m_3} |\,\boldsymbol{r_1}, m_1\,\rangle \otimes |\,\boldsymbol{r_2}, m_2\,\rangle \otimes |\,\boldsymbol{r_3}, m_3\,\rangle \cdot \begin{pmatrix} \boldsymbol{r_1} & \boldsymbol{r_2} & \boldsymbol{r_3} \\ m_1 & m_2 & m_3 \end{pmatrix}. \qquad (4\cdot9)$$

These coefficients are set to zero when no singlets appear in the tensor product.

The $3j$-symbols can be explicitly calculated as follows. First, we represent the step operators of $E_{6(+6)}$ on $\mathbf{27}$. We then calculate the matrix representation of $\widetilde{w} \in \widetilde{W}(F_{4(+4)})$ and determine the $3j$-symbols by requiring that the expression on the right-hand side of (4·9) be invariant under the action of all $\widetilde{w}$. The result is that the $3j$-symbols can have nonvanishing values only for the following five cases:

$$(\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}) = (\mathbf{12}_{\boldsymbol{S}}^-,\, \mathbf{12}_{\boldsymbol{S}}^-,\, \mathbf{2}^+),\, (\mathbf{12}_{\boldsymbol{S}}^+,\, \mathbf{12}_{\boldsymbol{S}}^+,\, \mathbf{2}^+),\, (\mathbf{2}^+, \mathbf{2}^+,\, \mathbf{2}^+),$$
$$(\mathbf{12}_{\boldsymbol{S}}^-,\, \mathbf{12}_{\boldsymbol{S}}^-,\, \mathbf{12}_{\boldsymbol{S}}^+),\, (\mathbf{12}_{\boldsymbol{S}}^+,\, \mathbf{12}_{\boldsymbol{S}}^+,\, \mathbf{12}_{\boldsymbol{S}}^+). \qquad (4\cdot10)$$

This is actually consistent with the assignment of parity. Note that the $3j$-symbols vanish whenever $\mathbf{4}^-$ or $\mathbf{12}_{\boldsymbol{L}}^-$ enters the expression.

§5. Discussion

The result obtained in the previous section leads to an interesting interpretation in the AdS_3/CFT_2 correspondence. On the AdS_3 supergravity side, suppose that we expand the action around the classical D1-D5-KK background and integrate over all the fields other than scalars. The resulting interaction terms for the 42 scalars should be a singlet for our residual U-duality group, and thus they can be expanded in the above scalar multiplets with the $3j$-symbols as coefficients. The vanishing of the $3j$-symbols including $\mathbf{4}^-$ or $\mathbf{12}_{\boldsymbol{L}}^-$ thus implies that there are no interaction terms for these 16 scalars. On the other hand, from the CFT$_2$ point of view, by regarding the scalar fields $\phi_m^{\boldsymbol{r}}$ as the sources of the scaling operators $\mathcal{O}_m^{\boldsymbol{r}}$ of the boundary CFT, the coefficients of the interaction term correspond to corrections to the scaling relation in the renormalization group equation,

$$\begin{aligned}\beta_m^{\boldsymbol{r}} &\equiv \Lambda \frac{d}{d\Lambda}\,\phi_m^{\boldsymbol{r}} \\ &= (\Delta^{\boldsymbol{r}} - 2)\,\phi_m^{\boldsymbol{r}} + \sum_{\boldsymbol{r_1},\boldsymbol{r_2}} C^{\boldsymbol{r}\,\boldsymbol{r_1}\boldsymbol{r_2}} \sum_{m_1,m_2} \begin{pmatrix} \boldsymbol{r} & \boldsymbol{r_1} & \boldsymbol{r_2} \\ m & m_1 & m_2 \end{pmatrix} \phi_{m_1}^{\boldsymbol{r_1}}\,\phi_{m_2}^{\boldsymbol{r_2}} + O(\phi^3)\,,\end{aligned} \tag{5·1}$$

with some constants $C^{\boldsymbol{r}\,\boldsymbol{r_1}\boldsymbol{r_2}}$. Thus, from the vanishing of the $3j$-symbols for $\boldsymbol{r} = \mathbf{4}^-$ or $\mathbf{12}_{\boldsymbol{L}}^-$ with $\Delta^{\boldsymbol{r}} = 2$, we may conclude that up to this order, these $\mathbf{4}^-$ and $\mathbf{12}_{\boldsymbol{L}}^-$ couple to exactly marginal operators and express the real moduli at the horizon.

We should make a comment here. When multiple couplings enter the renormalization group equation, the second coefficients can be highly nonuniversal, even for marginal operators with $\Delta^{\boldsymbol{r}} = 2$. However, one can easily show that vanishing coefficients still remain zero universally if the renormalization group respects the residual U-duality symmetry and we only allow field redefinitions which respect the symmetry. Nonvanishing coefficients, on the other hand, can change rather arbitrarily. This would ensure that our conclusion is universal.

In this article, we considered the residual U-duality group G for three-charge systems. When (Q_1, Q_2, Q_3) take generic positive integer values, G is found to be the lift of the Weyl group of $F_{4(+4)}$, $G = \widetilde{W}(F_{4(+4)})$, being a subgroup of $E_{6(+6)}(\boldsymbol{Z})$. We then classified the 42 scalars into $\mathbf{4}^-$, $\mathbf{12}_{\boldsymbol{L}}^-$, $\mathbf{12}_{\boldsymbol{S}}^-$, $\mathbf{2}^+$ and $\mathbf{12}_{\boldsymbol{S}}^+$. The splitting of G_{a5} and D_{a5} into $G_{a5} \pm D_{a5}$ in Ref. 6) can thus be naturally explained in the light of the lifted Weyl group $\widetilde{W}(F_{4(+4)})$ [see (4·4)]. We further considered the possible three-point couplings and showed that they always vanish when $\mathbf{4}^-$ or $\mathbf{12}_{\boldsymbol{L}}^-$ enters the expression. This implies that conformal invariance is preserved at least to this order under the perturbation with respect to the operators coupling to these fields at the boundary. This should be regarded as evidence that these 16 operators are exactly marginal. We are not able to determine whether the remaining 12 marginal operators (corresponding to $\mathbf{12}_{\boldsymbol{S}}^-$) are also exactly marginal from only an argument based on the Weyl group.

On the other hand, if all three charges coincide ($Q_1 = Q_2 = Q_3$), then the residual U-duality group is enhanced to $G = F_{4(+4)}(\boldsymbol{Z})$, as far as the charge vectors are concerned. Under the action of the full $F_{4(+4)}(\boldsymbol{Z})$, the three multiplets $\mathbf{4}^-$, $\mathbf{12}_{\boldsymbol{L}}^-$ and $\mathbf{12}_{\boldsymbol{S}}^-$ are combined into a single multiplet of 28 dimensions, and thus all the three-point couplings including $\mathbf{12}_{\boldsymbol{S}}^-$ also vanish. This may in turn imply that the operators corresponding to $\mathbf{12}_{\boldsymbol{S}}^-$ break the symmetry $F_{4(+4)}(\boldsymbol{Z})$ down to its subgroup $\widetilde{W}(F_{4(+4)})$. If this is the case, it would be interesting to investigate if this property of

$\mathbf{12}_L^-$ can be interpreted through the renormalization group or in terms of attractor.[24)]

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Appendix A
—— $F_{4(+4)}$ *in* $E_{6(+6)}$ ——

The embedding of $F_{4(+4)}$ in $E_{6(+6)}$ can be understood most easily if we use another R-type gauge (to be called the canonical gauge) with D3-D1-F1 charges at the weights all being at level 8 (see Fig. 2):

gauge	gauge fields	weights
canonical gauge	$(D_{\mu123}, D_{\mu4}, B_{\mu5})$	$(\bar\lambda_1, \bar\lambda_2, \bar\lambda_3) = (\mu^1-\mu^4, \mu^0-\mu^1+\mu^4-\mu^5, -\mu^0+\mu^5)$

In fact, we first note that $w_{\alpha_0}w_{\alpha_5}$ exchanges $\bar\lambda_2$ and $\bar\lambda_3$, and $w_{\alpha_1}w_{\alpha_4}$ exchanges $\bar\lambda_1$ and $\bar\lambda_2$. Thus, $w_{\alpha_0}w_{\alpha_5}$ and $w_{\alpha_1}w_{\alpha_4}$ generate permutations of the three charges. We then lift them together with w_{α_2} and w_{α_3}:

$$\widetilde{w}_1 \equiv \widetilde{w}_{\alpha_2}, \qquad \widetilde{w}_2 \equiv \widetilde{w}_{\alpha_3}, \qquad \widetilde{w}_3 \equiv \widetilde{w}_{\alpha_1}\widetilde{w}_{\alpha_4}, \qquad \widetilde{w}_4 \equiv \widetilde{w}_{\alpha_0}\widetilde{w}_{\alpha_5}. \tag{A·1}$$

Here $\widetilde{w}_3$ can be rewritten as

$$\begin{aligned}\widetilde{w}_3 &= \exp\left(\frac{\pi}{2}\,(E_{\alpha_1}-E_{-\alpha_1})\right)\cdot\exp\left(\frac{\pi}{2}\,(E_{\alpha_4}-E_{-\alpha_4})\right)\\ &= \exp\left(\frac{\pi}{2}\,\{(E_{\alpha_1}+E_{\alpha_4})-(E_{-\alpha_1}+E_{-\alpha_4})\}\right).\end{aligned} \tag{A·2}$$

$\widetilde{w}_4$ can also be rewritten similarly. We thus can express $\widetilde{w}_a$ $(a=1,\cdots,4)$ as

$$\widetilde{w}_a = \exp\left(\frac{\pi}{2}\left(e_{\bar\beta_a}-e_{-\bar\beta_a}\right)\right), \tag{A·3}$$

with

$$e_{\pm\bar\beta_1} \equiv E_{\pm\alpha_2}\,, \quad e_{\pm\bar\beta_2} \equiv E_{\pm\alpha_3}\,, \quad e_{\pm\bar\beta_3} \equiv E_{\pm\alpha_1}+E_{\pm\alpha_4}, \quad e_{\pm\bar\beta_4} \equiv E_{\pm\alpha_0}+E_{\pm\alpha_5}. \tag{A·4}$$

These $e_{\pm\bar\beta_a}$ generate $F_{4(+4)}$ as the invariant Lie subalgebra of $E_{6(+6)}$ under the $\boldsymbol{Z}_2$ automorphism of the E_6 Dynkin diagram: $\alpha_0 \leftrightarrow \alpha_5$, $\alpha_1 \leftrightarrow \alpha_4$, $\alpha_2 \leftrightarrow \alpha_2$, $\alpha_3 \leftrightarrow \alpha_3$. What we did here can thus be understood as the folding procedure to obtain $F_{4(+4)}$ from $E_{6(+6)}$ with the $\boldsymbol{Z}_2$ automorphism (see Fig. 3). The operators $e_{\pm\bar\beta_a}$ then

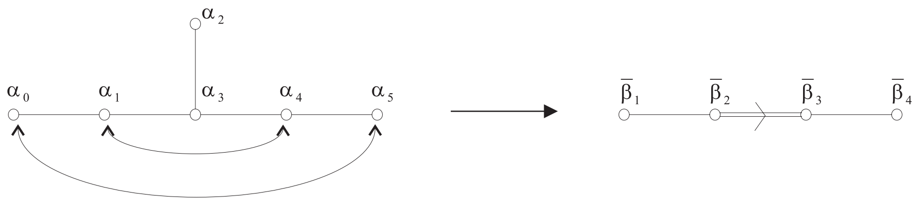

Fig. 3. Folding procedure to obtain F_4 from E_6 with the $\boldsymbol{Z}_2$ outer automorphism.

correspond to the step operators associated with the $F_{4(+4)}$ simple roots $\bar{\beta}_a$ with lengths $\bar{\beta}_1^2 = \bar{\beta}_2^2 = 2$ and $\bar{\beta}_3^2 = \bar{\beta}_4^2 = 1$.

One can easily show that the NS gauge with $(\lambda_1', \lambda_2', \lambda_3')$ is related to this canonical gauge as $\lambda_i' = w'(\bar{\lambda}_i)$ with the Weyl group element

$$w' \equiv w_{\alpha_0} w_{\alpha_1} w_{\alpha_3} w_{\alpha_4} w_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} w_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}\,, \tag{A·5}$$

while the R gauge with $(\lambda_1, \lambda_2, \lambda_3)$ is related to the canonical gauge with

$$w = w_{\alpha_0} \cdot w' = w_{\alpha_1} w_{\alpha_3} w_{\alpha_4} w_{\alpha_2+\alpha_3+\alpha_4+\alpha_5} w_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}. \tag{A·6}$$

The step operators associated with the simple roots $\beta_a' \equiv w'(\bar{\beta}_a)$ (NS gauge) or $\beta_a \equiv w(\bar{\beta}_a)$ (R gauge) can then be calculated by

$$\begin{aligned} e'_{\beta_a'} &= \widetilde{w}'\ e_{\bar{\beta}_a}\ \widetilde{w}'^{-1}, \qquad &\text{(NS gauge)} \\ e_{\beta_a} &= \widetilde{w}\ e_{\bar{\beta}_a}\ \widetilde{w}^{-1}. \qquad &\text{(R gauge)} \end{aligned} \tag{A·7}$$

By representing all the step operators of $E_{6(+6)}$ as 27×27 matrices in the representation **27**, one can carry out the above calculation explicitly to obtain (3·7) and (3·9). Ambiguities may arise when lifting the elements of the Weyl group, but they can be essentially fixed by requiring that the lifted elements rotate positive charges to positive charges.

References

1) J. Maldacena, Adv. Theor. Math. Phys. **2** (1998), 231, hep-th/9711200.
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **B428** (1998), 105, hep-th/9802109.
E. Witten, Adv. Theor. Math. Phys. **2** (1998), 253, hep-th/9802150.
2) O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, hep-th/9905111, and references therein.
3) J. de Boer, Nucl. Phys. **B548** (1999), 139, hep-th/9806104.
J. David, G. Mandal and S. Wadia, Nucl. Phys. **B544** (1999), 590, hep-th/9808168; hep-th/9906112; hep-th/9907075.
4) A. Strominger and C. Vafa, Phys. Lett. **B379** (1996), 99, hep-th/9601029.
5) M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69** (1992), 1849, hep-th/9204099.
M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. **D48** (1993), 1506, gr-qc/9302012.
6) I. R. Klebanov, A. Rajaraman and A. Tseytlin, Nucl. Phys. **B503** (1997), 157, hep-th/9704112.
7) F. Larsen and E. Martinec, J. High Energy Phys. **06** (1999), 19, hep-th/9905064; ibid. **11** (1999), 2, hep-th/9909088.

8) E. Cremmer, in *Superspace and supergravity*, ed. S. W. Hawking and M. Roček (Cambridge University Press, 1981), p. 267.
B. Julia, ibid., p. 331.
9) C. M. Hull and P. K. Townsend, Nucl. Phys. **B438** (1995), 109, hep-th/9410167.
10) N. A. Obers and B. Pioline, Phys. Rep. **318** (1999), 113, hep-th/9809039, and references therein.
11) H. Matsumoto, *Proceedings of Symposia in Pure Mathematics, American Mathematical Society*, Vol. 9 (1966), p. 99; Ann. scient. Éc. Norm. Sup., 4^e série **2** (1969), 1.
12) S. Mizoguchi and G. Schröder, hep-th/9909150.
13) V. G. Kac, *Infinite-dimensional Lie Algebras,* 3rd ed. (Cambridge University Press, 1990).
14) A. Schellekens and N. Warner, Nucl. Phys. **B308** (1988), 397; **B313** (1989), 41.
W. Lerche, A. Schellekens and N. Warner, Phys. Rep. **177** (1989), 1.
15) A. Giveon, M. Porrati and E. Ravinovich, Phys. Rep. **244** (1994), 77, hep-th/9401139.
16) M. Fukuma, T. Oota and H. Tanaka, Prog. Theor. Phys. **103** (2000), 425, hep-th/9907132.
17) S. Ferrara and M. Günaydin, Int. J. Mod. Phys. **A13** (1998), 2075, hep-th/9708025.
18) L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, Nucl. Phys. **B496** (1997), 617, hep-th/9611014.
19) M. Trigiante, PhD thesis, hep-th/9707087, and references therein.
20) S. Ferrara and J. Maldacena, Class. Quant. Grav. **15** (1998), 749, hep-th/9706097.
21) D. Kutasov, F. Larsen and R. Leigh, Nucl. Phys. **B550** (1999), 183, hep-th/9812027.
22) H. Lü, C. N. Pope and K. S. Stelle, Nucl. Phys. **B476** (1996), 89, hep-th/9602140.
23) L. Andrianopoli, R. D'Auria and S. Ferarra, Phys. Lett. **B411** (1997), 39, hep-th/9705024.
24) S. Ferrara and R. Kallosh, Phys. Rev. **D54** (1996), 1514, hep-th/9602136; ibid., 1525, hep-th/9603090.
G. Moore, hep-th/9807056; hep-th/9807087.
R. Dijkgraaf, Nucl. Phys. **B543** (1999), 545, hep-th/9810210.
R. Kallosh, A. Linde and M. Shmakova, J. High Energy Phys. **11** (1999), 010, hep-th/9910021.