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Higher-Derivative Gravity and the AdS/CFT Correspondence

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We investigate the AdS/CFT correspondence for higher-derivative gravity systems and develop a formalism in which the generating functional of the boundary field theory is given as a functional that depends only on the boundary values of bulk fields. We also derive a Hamilton-Jacobi-like equation that uniquely determines the generating functional, and give an algorithm calculating the Weyl anomaly. Using the expected duality between a higher-derivative gravity system and $N = 2$ superconformal field theory in four dimensions, we demonstrate that the resulting Weyl anomaly is consistent with the field theoretic anomaly.

§1. Introduction

Over the past few years, many attempts have been made to check the AdS/CFT correspondence.[1)-3)] (For a review, see Ref. 4)). As an example, it is shown in Ref. 3) that the spectrum of chiral operators of $\mathcal{N} = 4$ super Yang-Mills in four dimensions coincides with that of the Kaluza-Klein modes of type IIB supergravity on $AdS_5 \times S^5$. Also, the computation of anomalies via bulk gravity has been shown to exactly reproduce the results of the super Yang-Mills theory.[3)-6)] However, this matching of the anomalies is valid only in the regime where $N \to \infty$, $\lambda = g_{\rm YM}^2 N \gg 1$, since the analysis is based on a classical supergravity computation. At present, it remains an important issue to test the duality beyond this regime.

There have been several attempts to confirm the validity of the duality beyond the classical gravity approximation.[7)-11)] Among these, Ref. 8) treats $\mathcal{N} = 2$ $G = USp(N)$ superconformal field theory (SCFT) in four dimensions. This SCFT can be realized on the world volume of D3-branes situated inside eight D7-branes coincident with an $O7^-$ brane, and is known[12)] to be dual to type IIB string on $AdS_5 \times S^5/\boldsymbol{Z}_2$. The authors of Ref. 8) showed that this duality reproduces the $1/N$ correction to the $U(1)_{\mathcal{R}}$ chiral anomaly correctly. In Refs. 9) and 10), the $1/N$ correction to the Weyl anomaly of the SCFT is computed using a higher-derivative gravity theory in which a curvature square term is added.

However, higher-derivative gravity theories[†] exhibit some features in the AdS/CFT correspondence that differ from those in Einstein gravity. To see this, we first recall that the equation of motion for Einstein gravity is a second-order differential equation in time r. Thus, a classical solution can be totally specified by prescribing the value at the boundary if we further impose the regular behavior of the

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solution inside the bulk,[3] and the boundary value can be identified with an external field coupled to an operator in the dual CFT.[2),3)] The situation changes drastically if we consider higher-derivative theories. In fact, a higher-derivative system with Lagrangian density $\mathcal{L}(g,\dot{g},\cdots,g^{(N+1)})$, where g_{ij} is the metric and $\cdot=\partial/\partial r$, generically gives an equation of motion that is a differential equation of $2(N+1)$-order in r. We then would need $(N+1)$ boundary conditions for each field to specify a classical solution, even if we require its regular behavior inside the bulk.

The main aim of the present paper is to formulate higher-derivative gravity systems in accordance with the holographic principle. In this paper, we say that the holographic principle holds when the following two conditions are satisfied: (1) the classical solution of a higher-derivative system is specified uniquely by the boundary value of each bulk field, and (2) the bulk geometry becomes AdS-like near the boundary. In order to satisfy the first condition, we first note that the system $\mathcal{L}(g,\dot{g},\cdots,g^{(N+1)})$ can be transformed into a Hamilton system with $(N+1)$ pairs of canonical variables (g,Q^a), (p,P_a) $(a=1,\cdots,N)$ by defining $Q^a_{ij}=\partial^a g_{ij}/\partial r^a$. (See the next section for details.) Thus, by setting boundary conditions that are of the Dirichlet type for g and the Neumann type for Q^a, the classical solution of this system can be specified only by the boundary value of g. Note also that the classical action of this system, which is obtained by plugging this solution into the action, becomes a functional of these boundary values of bulk fields. The second condition ensures the existence of a UV fixed point of the dual theory at the boundary, and such a fixed point enables us to take the continuum limit.[14] We see below that appropriate boundary terms need to be added to the bulk action in order for the bulk metric to exhibit such asymptotic behavior when higher-derivative terms exist.

For a systematic treatment of these issues, we employ the Hamilton-Jacobi formulation, as introduced by de Boer, Verlinde and Verlinde[15] to investigate the holographic RG structure of Einstein gravity. (See Refs. 16)–24) for more details of the holographic RG.) This formulation is further elaborated in Refs. 25)–31). In particular, a systematic prescription for calculating the Weyl anomaly in arbitrary dimensions is developed in Ref. 26). In this paper, we show that the Hamilton-Jacobi equation is quite a useful tool also to study the holographic RG structure in higher-derivative systems. Actually, we can derive a Hamilton-Jacobi-like equation that determines the classical action in accordance with the holographic principle. That is, the classical action can be solved as a functional of a boundary value for each bulk field. As a check of our formulation, we compute $1/N$ corrections to the Weyl anomaly of the $\mathcal{N}=2$ SCFT by solving the Hamilton-Jacobi-like equation. In the course of this analysis, we find that the prescription developed in Ref. 26) is again helpful. We show that our result can reproduce that of Refs. 9) and 10).

The organization of this paper is as follows. In §2, we formulate the Hamilton-Jacobi equation for a higher-derivative system with emphasis on applications to the AdS/CFT correspondence. In §3, we apply the formulation to higher-derivative gravity and derive an equation that determines the classical action. In §4, we solve the equation following the prescription given in Ref. 26), and demonstrate how to calculate the Weyl anomaly. We show that the resulting Weyl anomaly correctly reproduces that given in Refs. 9) and 10). Section 6 is devoted to a conclusion.

There, a comment is given on the holographic RG structure in higher-derivative gravity systems. Some useful results are summarized in the appendices.

§2. Hamilton-Jacobi equation for a higher-derivative Lagrangian

In this section, we give a prescription for determining the classical action when higher-derivative terms are added. We start our discussion for a system of point particles with the action

$$\boldsymbol{S}[q(r)] = \int_{t'}^{t} dr\, L\left(q, \dot{q}, \cdots, q^{(N+1)}\right) \quad \left(q^{(n)}(r) \equiv d^n q(r)/dr^n\right) . \tag{2·1}$$

The extension of our argument to gravitational systems is straightforward and will be carried out in the next section.*)

The action (2·1) can be rewritten into the first-order form in the following way. We first introduce the Lagrange multipliers $p, P_1, \cdots, P_{N-1}$, so that $q, Q^1 = \dot{q}, \cdots, Q^N = q^{(N)}$ can be regarded as independent canonical variables:

$$\begin{aligned} &L\left(q, Q^1, \cdots, Q^N, \dot{Q}^N; p, P_1, \cdots, P_{N-1}\right) \\ &= p(\dot{q} - Q^1) + P_1(\dot{Q}^1 - Q^2) + \cdots + P_{N-1}(\dot{Q}^{N-1} - Q^N) \\ &\quad + L(q, Q^1, \cdots, Q^N, \dot{Q}^N). \end{aligned} \tag{2·2}$$

We then carry out a Legendre transformation from $(Q^N, \dot{Q}^N)$ to (Q^N, P_N) through

$$P_N = \frac{\partial L}{\partial \dot{Q}^N}\left(q, Q^1, \cdots, Q^N, \dot{Q}^N\right) . \tag{2·3}$$

We here assume that this equation can be solved with respect to $\dot{Q}^N$ $\Big(\equiv f(q, Q^1, \cdots, Q^N; P_N)\Big)$, and thus obtain the following action that is equivalent to (2·1) classically:

$$\boldsymbol{S}[q, Q^1, \cdots, Q^N; p, P_1, \cdots, P_N] = \int_{t'}^{t} dr \left[p\,\dot{q} + \sum_{a=1}^{N} P_a \dot{Q}^a - H(q, Q^a; p, P_a)\right], \tag{2·4}$$

where $\dot{Q}^N$ is now the time-derivative of the independent variable Q^N, and the Hamiltonian is given by

$$\begin{aligned} H(q, Q^a; p, P_a) = p\,Q^1 + P_1 Q^2 + \cdots + P_{N-1} Q^N + P_N\, f(q, Q^a;\, P_N) \\ - L\left(q, Q^1, \cdots, Q^N, f(q, Q^a;\, P_N)\right). \end{aligned} \tag{2·5}$$

The variation of the action (2·4) is given by

*) See also Ref. 32), where higher-derivative systems are discussed from the viewpoint of string theories.

$$\delta \boldsymbol{S} = \int_{t'}^{t} dr \left[\delta p \left(\dot{q} - \frac{\partial H}{\partial p} \right) + \sum_a \delta P_a \left(\dot{Q}^a - \frac{\partial H}{\partial P_a} \right) \right. \\ \left. - \delta q \left(\dot{p} + \frac{\partial H}{\partial q} \right) - \sum_a \delta Q^a \left(\dot{P}_a + \frac{\partial H}{\partial Q^a} \right) \right] + \left(p\, \delta q + \sum P_a\, \delta Q^a \right) \Big|_{t'}^{t} , \tag{2·6}$$

and thus the equation of motion consists of the usual Hamilton equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{Q}^a = \frac{\partial H}{\partial P_a}, \qquad \dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{P}_a = -\frac{\partial H}{\partial Q^a} , \tag{2·7}$$

and the following constraint, which must hold at the boundary, $r=t$ and $r=t'$:

$$p\, \delta q + \sum_a P_a\, \delta Q^a = 0 \quad (r=t, t') . \tag{2·8}$$

The latter requirement, (2·8), can be satisfied when we use either Dirichlet boundary conditions,

$$\underline{\text{Dirichlet}} : \qquad \delta q = 0 , \quad \delta Q^a = 0 \quad (r=t, t') , \tag{2·9}$$

or Neumann boundary conditions,

$$\underline{\text{Neumann}} : \qquad p = 0 , \quad P_a = 0 \quad (r=t, t') , \tag{2·10}$$

for each variable q and Q^a ($a = 1, \cdots, N$). If, for example, we take the classical solution $(\bar{q}, \bar{Q}^a, \bar{p}, \bar{P}_a)$ that satisfies the Dirichlet boundary conditions for all (q, Q^a) with the specified boundary values as

$$\bar{q}(r=t) = q, \quad \bar{Q}^a(r=t) = Q^a, \quad \text{and} \quad \bar{q}(r=t') = q', \quad \bar{Q}^a(r=t') = Q'^a , \tag{2·11}$$

then after plugging the solution into the action, we obtain the classical action that is a function of these boundary values,

$$S(t, q, Q^a;\, t', q', Q'^a) = \boldsymbol{S}\left[\bar{q}(r), \bar{Q}^a(r);\, \bar{p}(r), \bar{P}_a(r) \right] . \tag{2·12}$$

However, as we discussed in the Introduction, this classical action is not of great interest to us in the context of the AdS/CFT correspondence, since the holographic principle requires that the bulk be specified by only the values q and q' at the boundary. This leads us to use mixed boundary conditions:

$$\delta q = P_a = 0 \quad (r=t,\, t') . \tag{2·13}$$

That is, we impose Dirichlet boundary conditions for q and Neumann boundary conditions for Q^a. In this case, the classical action (to be called the *reduced classical action*) becomes a function only of the boundary values q and q':

$$S = S(t, q;\, t', q') . \tag{2·14}$$

A renormalization group interpretation of this condition is discussed briefly in the concluding section, and will be discussed in detail in a forthcoming paper.[34]

Now we derive a Hamilton-Jacobi-like equation that determines the reduced classical action (2·14). This can be derived in two ways, and we start with the more complicated way, since this gives us a deeper understanding of the mathematical structure. To this end, we first change the polarization of the system by performing the canonical transformation[*]

$$\widehat{\boldsymbol{S}} \equiv \boldsymbol{S} - \int_{t'}^{t} dF\,, \tag{2·15}$$

with the generating function

$$F = \sum_a P_a Q^a\,. \tag{2·16}$$

Although the Hamilton equation does not change under this transformation, the boundary conditions at $r=t$ and $r=t'$ become

$$p\,\delta q - \sum_a Q^a \delta P_a = 0 \quad (r=t,t')\,. \tag{2·17}$$

These boundary conditions can be satisfied by imposing the Dirichlet boundary conditions for both $\bar{q}$ and $\bar{P}_a$:

$$\bar{q}(r=t) = q,\quad \bar{P}_a(r=t) = P_a\,,\quad \text{and}\quad \bar{q}(r=t') = q',\quad \bar{P}_a(r=t') = P'_a\,. \tag{2·18}$$

Substituting this solution into $\widehat{\boldsymbol{S}}$, we obtain a new classical action that is a function of these boundary values,

$$\widehat{S}\left(t,q,P_a;\,t',q',P'_a\right) = \widehat{\boldsymbol{S}}\left[\bar{q}(r),\bar{Q}^a(r);\,\bar{p}(r),\bar{P}_a(r)\right]. \tag{2·19}$$

By taking the variation of $\widehat{\boldsymbol{S}}$ and using the equation of motion, we can easily show that the new classical action $\widehat{S}$ obeys the Hamilton-Jacobi equation:

$$\frac{\partial \widehat{S}}{\partial t} = -H\left(q, -\frac{\partial \widehat{S}}{\partial P_a};\, +\frac{\partial \widehat{S}}{\partial q}, P_a\right),$$
$$\frac{\partial \widehat{S}}{\partial t'} = +H\left(q', +\frac{\partial \widehat{S}}{\partial P'_a};\, -\frac{\partial \widehat{S}}{\partial q'}, P'_a\right). \tag{2·20}$$

The reduced classical action $S(t,q;t',q')$ is then obtained by setting $P_a = 0$ in $\widehat{S}$:

$$S\left(t,q;t',q'\right) = \widehat{S}\left(t,q,P_a=0;\,t',q',P'_a=0\right). \tag{2·21}$$

[*] The following procedure corresponds to a change of representation from the Q-basis to the P-basis in the WKB approximation:

$$\Psi(t,q,Q) = e^{iS(t,q,Q)/\hbar} \to \widehat{\Psi}(t,q,P) = e^{i\widehat{S}(t,q,P)/\hbar} \equiv \int dQ\, e^{-iP_aQ^a/\hbar}\, \Psi(t,q,Q)\,.$$

Note that the generating function F vanishes at the boundary when we set $P_a = 0$.

Here we briefly describe how the Hamilton-Jacobi equation (2·20) is solved. For simplicity, we consider the case $N = 1$ and focus only on the upper boundary at $r = t$. Motivated by the gravitational system considered in the next section, we assume that the Lagrangian takes the form

$$L(q, \dot{q}, \ddot{q}) = L_0(q, \dot{q}) + cL_1(q, \dot{q}, \ddot{q}), \tag{2·22}$$

where

$$\begin{aligned} L_0(q, \dot{q}) &= \frac{1}{2} m_{ij}(q)\dot{q}^i\dot{q}^j - V(q), \\ L_1(q, \dot{q}, \ddot{q}) &= \frac{1}{2} n_{ij}(q)\ddot{q}^i\ddot{q}^j - A_i(q, \dot{q})\ddot{q}^i - \phi(q, \dot{q}), \end{aligned} \tag{2·23}$$

with

$$\begin{aligned} A_i(q, \dot{q}) &= a^{(2)}_{ijk}(q)\dot{q}^j\dot{q}^k + a^{(0)}_i(q), \\ \phi(q, \dot{q}) &= \phi^{(4)}_{ijkl}(q)\dot{q}^i\dot{q}^j\dot{q}^k\dot{q}^l + \phi^{(2)}_{ij}(q)\dot{q}^i\dot{q}^j + \phi^{(0)}(q). \end{aligned} \tag{2·24}$$

We further assume that the determinants of the matrices $m_{ij}(q)$ and $n_{ij}(q)$ have the same signature.*) Following the procedure discussed above, this Lagrangian can be rewritten into the first-order form

$$L = p\,\dot{q} + P\dot{Q} - H(q, Q; p, P)\,, \tag{2·25}$$

with the Hamiltonian

$$\begin{aligned} H(q, Q; p, P) = {} & p_iQ^i - \frac{1}{2} m_{ij}(q)Q^iQ^j + V(q) \\ & + \frac{1}{2c} n^{ij}(q)\Big(P_i + cA_i(q, Q)\Big)\Big(P_j + cA_j(q, Q)\Big) + c\,\phi(q, Q), \end{aligned} \tag{2·26}$$

where $n^{ij} = (n_{ij})^{-1}$. The Hamilton-Jacobi equation (2·20) is solved as a double expansion with respect to c and P by assuming that the classical action takes the form

$$\begin{aligned} \widehat{S}(t, q, P) = {} & \frac{1}{\sqrt{c}}\,\widehat{S}_{-1/2}(t, q, P) + \widehat{S}_0(t, q, P) + \sqrt{c}\,\widehat{S}_{1/2}(t, q, P) + c\,\widehat{S}_1(t, q, P) \\ & + \mathcal{O}(c^{3/2}). \end{aligned} \tag{2·27}$$

After some simple algebra, the coefficients are found to be

$$\begin{aligned} \widehat{S}_{-1/2} &= \frac{1}{2} u^{ij}(q)P_iP_j + \mathcal{O}(P^3), \\ \widehat{S}_0 &= S_0(t, q) - P_i\,\partial^i S_0 + \mathcal{O}(P^2), \end{aligned}$$

*) In fact, it is easy to see that this is the case in the higher-derivative gravity system considered below.

$$\widehat{S}_{1/2} = P_i\, u^{ij}(q) n_{jk}(q) \left[\boldsymbol{\Gamma}^k_{lm}\, \partial^l S_0\, \partial^m S_0 + \partial^k V(q) + n^{kl}(q) A_l\Big(q, \frac{\partial S_0}{\partial q}\Big)\right] + \mathcal{O}(P^2). \tag{2·28}$$

Here,

$$\partial_i \equiv \frac{\partial}{\partial q^i}, \qquad \partial^i \equiv m^{ij}\partial_i, \tag{2·29}$$

and $\boldsymbol{\Gamma}^i_{jk}$ is the affine connection defined by m_{ij}. Also u^{ij} is defined by the relation

$$u^{ik}(q) u^{jl}(q) m_{kl}(q) = n^{ij}(q). \tag{2·30}$$

Furthermore, $S_0(t,q) = \widehat{S}_0(t,q,P=0)$ and $S_1(t,q) = \widehat{S}_1(t,q,P=0)$ satisfy the equations

$$\begin{aligned}
-\frac{\partial S_0}{\partial t} &= \frac{1}{2} m_{ij}(q) \frac{\partial S_0}{\partial q^i}\frac{\partial S_0}{\partial q^j} + V(q),\\
-\frac{\partial S_1}{\partial t} &= m_{ij}(q)\frac{\partial S_1}{\partial q^i}\frac{\partial S_0}{\partial q^j}\\
&\quad -\frac{1}{2} n_{ij}(q) \Big(\boldsymbol{\Gamma}^i_{kl}\, \partial^k S_0\, \partial^l S_0 + \partial^i V(q)\Big)\Big(\boldsymbol{\Gamma}^j_{mn}\, \partial^m S_0\, \partial^n S_0 + \partial^j V(q)\Big)\\
&\quad - A_i\Big(q, \frac{\partial S_0}{\partial q}\Big)\Big(\boldsymbol{\Gamma}^i_{kl}\, \partial^k S_0\, \partial^l S_0 + \partial^i V(q)\Big) + \phi\Big(q, \frac{\partial S_0}{\partial q}\Big)\,,
\end{aligned} \tag{2·31}$$

which can be expressed as a Hamilton-Jacobi-like equation for the reduced classical action $S(t,q) = S_0(t,q) + c\, S_1(t,q) + \mathcal{O}(c^2)$:

$$-\frac{\partial S}{\partial t} = \widetilde{H}(q,p), \qquad p_i = \frac{\partial S}{\partial q^i}\,, \tag{2·32}$$

where

$$\begin{aligned}
\widetilde{H}(q,p) &= \frac{1}{2} m^{ij}(q) p_i p_j + V(q)\\
&\quad + c\left[-\frac{1}{2} n_{ij}(q)\Big(\boldsymbol{\Gamma}^i_{kl}\, p^k p^l + \partial^i V(q)\Big)\Big(\boldsymbol{\Gamma}^j_{mn}\, p^m p^n + \partial^j V(q)\Big)\right.\\
&\qquad \left. - A_i(q,p)\Big(\boldsymbol{\Gamma}^i_{kl}\, p^k p^l + \partial^i V(q)\Big) + \phi(q,p)\right].
\end{aligned} \tag{2·33}$$

It is important to note that $\widetilde{H}$ is not the Hamiltonian. In fact, the Hamilton equation for $\widetilde{H}$ does not coincide with that obtained from (2·26).

In solving the full Hamilton-Jacobi equation (2·20) for $\widehat{S}(t,q,P)$, we imposed the condition that everything becomes regular around $c=0$ when we set $P=0$. This is because in most interesting cases (like those of the gravity systems we discuss in the following sections) the higher-derivative term is regarded as a perturbation, so that the reduced classical action must have a finite limit for $c \to 0$. Once such a regularity condition is imposed, we have an alternative way to derive this *pseudo-Hamiltonian* $\widetilde{H}$ with greater ease. In fact, for any Lagrangian of the form

$$L(q^i, \dot{q}^i, \ddot{q}^i) = L_0(q^i, \dot{q}^i) + c\, L_1(q^i, \dot{q}^i, \ddot{q}^i)\,, \tag{2·34}$$

one can prove the following theorem, assuming that the classical solution can be expanded around $c=0$:*)

Theorem

Let $H_0(q,p)$ be the Hamiltonian corresponding to $L_0(q,\dot{q})$. Then the reduced classical action $S(t,q;\,t',q') = S_0(t,q;\,t',q') + c\,S_1(t,q;\,t',q') + \mathcal{O}(c^2)$ satisfies the following equation up to $\mathcal{O}(c^2)$:

$$-\frac{\partial S}{\partial t} = \widetilde{H}(q,p), \quad p_i = \frac{\partial S}{\partial q^i}, \quad \text{and} \quad +\frac{\partial S}{\partial t'} = \widetilde{H}(q',p'), \quad p'_i = -\frac{\partial S}{\partial q'^i}, \tag{2·35}$$

where

$$\begin{aligned}
\widetilde{H}(q,p) &\equiv H_0(q,p) - c\,L_1(q, f_1(q,p), f_2(q,p)),\\
f_1^i(q,p) &\equiv \Big\{H_0, q^i\Big\} = \frac{\partial H_0}{\partial p_i},\\
f_2^i(q,p) &\equiv \Big\{H_0, \Big\{H_0, q^i\Big\}\Big\} = \frac{\partial^2 H_0}{\partial p_i \partial q^j}\frac{\partial H_0}{\partial p_j} - \frac{\partial^2 H_0}{\partial p_i \partial p_j}\frac{\partial H_0}{\partial q^j}.\\
&\left(\{F(q,p), G(q,p)\} \equiv \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q^i}\right)
\end{aligned} \tag{2·36}$$

A proof of this theorem is given in Appendix A. It can easily be confirmed that this correctly reproduces (2·32) and (2·33) for the Lagrangian given in (2·22)–(2·24).

§3. Application to higher-derivative gravity

In this section, following the prescription developed in the previous section, we derive an equation that determines the reduced classical action for a higher-derivative gravity system.

We first recall the holographic description of RG flows in the dual boundary field theory. We parametrize the bulk metric with the Euclidean ADM decomposition. (For more details of the ADM decomposition, see Appendix B.) We then have

$$\begin{aligned}
ds^2 &= \widehat{g}_{\mu\nu}\, dX^\mu dX^\nu\\
&= N(x,r)^2 dr^2 + g_{ij}(x,r)\Big(dx^i + \lambda^i(x,r)dr\Big)\Big(dx^j + \lambda^j(x,r)dr\Big).
\end{aligned} \tag{3·1}$$

*) As long as we think of L_1 as a perturbation, any classical solution can be expanded as

$$\bar{q}(r) = \bar{q}_0(r) + c\,\bar{q}_1(r) + \mathcal{O}(c^2)\,.$$

Here $\bar{q}_0$ is the classical solution for L_0, and $\bar{q}_1$ is obtained by solving a second-order differential equation. Note that we can, in particular, enforce the boundary conditions

$$\bar{q}_0(r\!=\!t) = q, \quad \bar{q}_1(r\!=\!t) = 0 \quad \text{and} \quad \bar{q}_0(r\!=\!t') = q', \quad \bar{q}_1(r\!=\!t') = 0\,.$$

In this case, due to the equation of motion for $\bar{q}_0(r)$, the classical action is simply given by

$$S(q,t;q',t') = \int_{t'}^{t} dr\Big[L_0(\bar{q}_0,\dot{\bar{q}}_0) + c\,L_1(\bar{q}_0,\dot{\bar{q}}_0,\ddot{\bar{q}}_0)\Big] + \mathcal{O}(c^2)\,.$$

This corresponds to the classical action considered in Ref. 10).

Here $X^\mu = (x^i, r)$, with $i, j = 1, 2, \cdots, d$, and N and λ^i are the lapse and the shift function, respectively. The signature of the metric g_{ij} is taken to be $(+\cdots+)$. By assuming that the geometry becomes AdS-like in the limit $r \to -\infty$, the Euclidean time r is identified with the RG parameter of the d-dimensional boundary theory, and the time evolution of other bulk fields (such as scalars) is interpreted as an RG flow of the coupling constants with a UV fixed point at the boundary. To avoid a singularity of the metric g_{ij} at $r = -\infty$, we restrict the region of r such that $r_0 \le r < \infty$.[2),3),33)] This corresponds to the introduction of a UV cutoff to the boundary field theory. In the following, we consider a $(d+1)$-dimensional manifold $M_{d+1} = \{(x^i, r)\}$ that has a topology given by $M_{d+1} \sim (\boldsymbol{R}^d \cup \infty) \times \boldsymbol{R}_+$, with $r_0 \le r < \infty$.

We consider classical gravity on M_{d+1} with the action

$$\boldsymbol{S} = \boldsymbol{S}_B + \boldsymbol{S}_b \,. \tag{3·2}$$

Here $\boldsymbol{S}_B$ is the bulk action given by

$$\boldsymbol{S}_B = \int_{M_{d+1}} d^{d+1}X \sqrt{\hat{g}}\, \mathcal{L}_B \,, \tag{3·3}$$

$$\mathcal{L}_B = 2\Lambda - \widehat{R} - a\widehat{R}^2 - b\widehat{R}^2_{\mu\nu} - c\widehat{R}^2_{\mu\nu\rho\sigma} \,, \tag{3·4}$$

where a, b and c are some given constants. $\boldsymbol{S}_b$ contains boundary terms defined on the boundary $\Sigma_d = \partial M_{d+1}$ at $r = r_0$. The form of $\boldsymbol{S}_b$ can be determined by requiring that it is invariant under the diffeomorphism

$$X^\mu \to X'^\mu = f^\mu(X), \tag{3·5}$$

with the condition

$$f^r(r = r_0, x) = r_0. \tag{3·6}$$

Equation (3·6) implies that the diffeomorphism does not change the location of the boundary. It is then easy to verify that $\boldsymbol{S}_b$ takes the form (for details see Appendix C)

$$\boldsymbol{S}_b = \int_{\Sigma_d} d^d x \sqrt{g}\, \mathcal{B}, \tag{3·7}$$

with

$$\mathcal{B} = 2K + x_1\, RK + x_2\, R_{ij}K^{ij} + x_3\, K^3 + x_4\, KK^2_{ij} + x_5\, K^3_{ij}, \tag{3·8}$$

where K_{ij} is the extrinsic curvature of Σ_d given by

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i \lambda_j - \nabla_j \lambda_i\right), \tag{3·9}$$

and $K = g^{ij}K_{ij}$. ∇_i and R_{ijkl} are, respectively, the covariant derivative and the Riemann tensor defined by g_{ij}. The first term in $\mathcal{B}$ ensures that the Dirichlet boundary conditions can be imposed in the Einstein theory[35)] and also plays an important role in the context of the AdS/CFT correspondence.[36)] We argue below that the coefficients $x_1, \cdots, x_5$ must obey some relations so that the holography holds even for higher-derivative gravity.*)

The action (3·2) is expressed in terms of the ADM parametrization as

$$\begin{aligned} \boldsymbol{S} &= \int_{M_{d+1}} d^{d+1}X \left[\sqrt{g}\,\mathcal{L}_B - \frac{\partial}{\partial r}\left(\sqrt{g}\,\mathcal{B}\right) \right] \\ &= \int_{r_0}^{\infty} dr \int d^dx \sqrt{g}\, \left[\boldsymbol{\mathcal{L}}_0(g, K;\, N, \lambda) + \boldsymbol{\mathcal{L}}_1(g, K, \dot{K};\, N, \lambda) \right], \end{aligned} \tag{3·10}$$

where **)

$$\frac{1}{N}\,\boldsymbol{\mathcal{L}}_0 = 2\Lambda - R + K_{ij}^2 - K^2, \tag{3·11}$$

$$\begin{aligned} \frac{1}{N}\,\boldsymbol{\mathcal{L}}_1 = &-aR^2 - bR_{ij}^2 - cR_{ijkl}^2 + \Big[(-6a + 2x_1)K_{ij}^2 + (2a - x_1)K^2\Big]R \\ &+\Big[-2(2b + 4c - x_2)(K^2)_{ij} + (2b + 2x_1 - x_2)KK_{ij}\Big]R^{ij} \\ &+ 2(6c + x_2)K_{ik}K_{jl}R^{ijkl} \\ &- 2(2b + c - 3x_5)K_{ij}^4 + (4b + 4x_4 - x_5)KK_{ij}^3 \\ &- (9a + b + 2c - 2x_4)\left(K_{ij}^2\right)^2 + (6a - b + 6x_3 - x_4)K^2K_{ij}^2 \\ &- (a + x_3)K^4 \\ &- (4b + 2x_1 - x_2)K_{ij}\nabla^i\nabla^jK + 2(b - 4c + x_2)K_{ij}\nabla^j\nabla_kK^{ki} \\ &+ (8c + x_2)K_{ij}\nabla^2K^{ij} + 2(b + x_1)K\nabla^2K \\ &-\Big[(4a + b)g^{ij}g^{kl} + (b + 4c)g^{ik}g^{jl}\Big]L_{ij}L_{kl} \\ &+\Big[\Big\{(4a - x_1)R + (12a + 2b - x_4)K_{kl}^2 - (4a + 3x_3)K^2\Big\}g^{ij} \\ &\quad + (2b - x_2)R^{ij} + (4b + 8c - 3x_5)(K^2)^{ij} - 2(b + x_4)KK^{ij}\Big]L_{ij}, \end{aligned} \tag{3·12}$$

with

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i\lambda_j - \nabla_j\lambda_i\right), \tag{3·13}$$

$$L_{ij} = \frac{1}{N}\left(\dot{K}_{ij} - \lambda^k\,\nabla_kK_{ij} - \nabla_i\lambda^k\,K_{kj} - \nabla_j\lambda^k\,K_{ik} + \nabla_i\nabla_jN\right). \tag{3·14}$$

By regarding g_{ij} and K_{ij} as independent canonical variables,***) the action (3·10)

*) See, e.g., Refs. 37) and 38) for another discussion of boundary terms in higher-derivative gravity.

**) We here use the following abbreviated notation: $K_{ij}^n \equiv K_{i_1}^{i_2}K_{i_2}^{i_3}\cdots K_{i_n}^{i_1}$, $(K^2)_{ij} \equiv K_{ik}K_j^k$.

***) The correspondences between the variables in §2 are as follows: $q \leftrightarrow g_{ij}$, $p \leftrightarrow \sqrt{g}\,\pi^{ij}$, $Q \leftrightarrow K_{ij}$, $P \leftrightarrow \sqrt{g}\,P^{ij}$.

can be further rewritten into the first-order form

$$\boldsymbol{S} = \int_{r_0}^{\infty} dr \int d^d x \sqrt{g} \left[\pi^{ij} \left(\dot{g}_{ij} - 2NK_{ij} - \nabla_i \lambda_j - \nabla_j \lambda_i \right) + \boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{L}}_1 \right]$$
$$= \int_{r_0}^{\infty} dr \int d^d x \sqrt{g} \left[\pi^{ij} \dot{g}_{ij} + P^{ij} \dot{K}_{ij} - \boldsymbol{\mathcal{H}}(g, K; \pi, P; N, \lambda) \right] . \qquad (3\cdot15)$$

Here the Hamiltonian density $\boldsymbol{\mathcal{H}}$ can be evaluated as

$$\boldsymbol{\mathcal{H}} = \pi^{ij} \left(2NK_{ij} + \nabla_i \lambda_j + \nabla_j \lambda_i \right) + P^{ij} \dot{K}_{ij} - \boldsymbol{\mathcal{L}}_0 - \boldsymbol{\mathcal{L}}_1$$
$$= N\mathcal{H}(g, K; \pi, P) + \lambda_i \mathcal{P}^i(g, K; \pi, P), \qquad (3\cdot16)$$

with

$$\begin{aligned}
\mathcal{H}(g, K; \pi, P) = {} & 2\pi^{ij} K_{ij} - \frac{1}{4(b+4c)} P_{ij}^2 + \frac{4a+b}{4(b+4c)\left(4da+(d+1)b+4c\right)} P^2 \\
& - \nabla_i \nabla_j P^{ij} + \left[A_1 \, R^{ij} + A_2 \, (K^2)^{ij} + A_3 \, K K^{ij} \right] P_{ij} \\
& + \left[A_4 \, R + A_5 \, K_{ij}^2 + A_6 \, K^2 \right] P \\
& - 2\Lambda + R - K_{ij}^2 + K^2 \\
& + B_1 \, R^2 + B_2 \, R_{ij}^2 + B_3 \, R_{ijkl}^2 \\
& + \left(C_1 \, K_{ij}^2 + C_2 \, K^2 \right) R + \left[C_3 \, (K^2)_{ij} + C_4 K K_{ij} \right] R^{ij} \\
& + C_5 \, K_{ik} K_{jl} R^{ijkl} \\
& + D_1 \, K_{ij}^4 + D_2 \, K K_{ij}^3 + D_3 \, (K_{ij}^2)^2 + D_4 \, K^2 K_{ij}^2 + D_5 \, K^4 \\
& + E_1 \, K_{ij} \nabla^i \nabla^j K + E_2 \, K_{ij} \nabla^j \nabla_k K^{kj} \\
& + E_3 \, K_{ij} \nabla^2 K^{ij} + E_4 \, K \nabla^2 K, \qquad (3\cdot17) \\
\mathcal{P}^i(g, K; \pi, P) = {} & - 2\nabla_j \pi^{ij} + P_{kl} \nabla^i K^{kl} - 2\nabla^k (K^{ij} P_{jk}). \qquad (3\cdot18)
\end{aligned}$$

The coefficients $A_1, \cdots, E_4$ are not important in the following discussion, and are listed in Appendix D. The classical equivalence between the two actions (3·10) and (3·15) can be easily established by noting that the latter gives the following equation of motion for π^{ij}:

$$\begin{aligned}
P^{ij} = {} & - 2 \left((4a+b) \, g^{ij} g^{kl} + (b+4c) \, g^{ik} g^{jl} \right) L_{kl} \\
& + \left[(4a - x_1) R + (12a + 2b - x_4) K_{kl}^2 - (4a + 3x_3) K^2 \right] g^{ij} \\
& + (2b - x_2) R^{ij} + (4b + 8c - 3x_5) (K^2)^{ij} - 2(b + x_4) \, K K^{ij} \, . \qquad (3\cdot19)
\end{aligned}$$

This correctly reproduces the original action (3·10) when substituted into (3·15).

Following the prescription given in §2, we now make a canonical transformation that changes the polarization of $\boldsymbol{S}$ from (g_{ij}, K_{ij}) to (g_{ij}, P^{ij}):

$$\widehat{\boldsymbol{S}} \equiv \boldsymbol{S} - \int_{M_{d+1}} d^{d+1} X \, \frac{\partial}{\partial r} \left(\sqrt{g} \, K_{ij} P^{ij} \right)$$
$$= \int_{r_0}^{\infty} \int d^d x \, \sqrt{g} \left(\pi^{ij} \dot{g}_{ij} - K_{ij} \dot{P}^{ij} - N \widehat{\mathcal{H}} - \lambda_i \widehat{\mathcal{P}}^i \right), \qquad (3\cdot20)$$

with

$$\begin{aligned}\widehat{\mathcal{H}}(g,K;\pi,P) &\equiv \mathcal{H}(g,K;\pi,P) + KK_{ij}P^{ij},\\ \widehat{\mathcal{P}}^i(g,K;\pi,P) &\equiv \mathcal{P}^i(g,K;\pi,P) - \nabla^i(K_{jk}P^{jk})\\ &= -2\nabla_j\pi^{ij} - \nabla^i P^{jk}\,K_{jk} - 2\nabla^k(K^{ij}P_{jk}),\end{aligned} \tag{3·21}$$

where we have used the relation

$$\partial_r\sqrt{g} = \sqrt{g}\left(NK + \nabla^i\lambda_i\right). \tag{3·22}$$

Since N and λ_i are the Lagrange multipliers, we obtain the Hamiltonian and momentum constraints

$$\frac{1}{\sqrt{g}}\frac{\delta\widehat{\boldsymbol{S}}}{\delta N} = \widehat{\mathcal{H}}(g,K;\,\pi,P) \ = \ 0, \tag{3·23}$$

$$\frac{1}{\sqrt{g}}\frac{\delta\widehat{\boldsymbol{S}}}{\delta\lambda_i} = \widehat{\mathcal{P}}^i(g,K;\,\pi,P) \ = \ 0. \tag{3·24}$$

We now let $\bar{g}_{ij}$ and $\bar{P}^{ij}$ represent the solution to the equation of motion for $\widehat{\boldsymbol{S}}$ that obeys the boundary conditions

$$\bar{g}_{ij}(x,r=r_0) = g_{ij}(x), \qquad \bar{P}^{ij}(x,r=r_0) = P^{ij}(x). \tag{3·25}$$

We also require that the solution be regular or be set to some specific value inside the bulk ($r\to\infty$), and assume that the above boundary condition is sufficient to specify the classical solution completely.[3] Plugging the solution into $\widehat{\boldsymbol{S}}$, we obtain the classical action $\widehat{S}[g(x),P(x)]$, which satisfies the following Hamilton-Jacobi equation:*)

$$\frac{1}{\sqrt{g}}\frac{\delta\widehat{S}}{\delta g_{ij}} = -\pi^{ij}\,, \qquad \frac{1}{\sqrt{g}}\frac{\delta\widehat{S}}{\delta P^{ij}} = +K_{ij}\,, \tag{3·26}$$

$$\widehat{\mathcal{H}}(g,K;\pi,P) = 0\,, \tag{3·27}$$

$$\widehat{\mathcal{P}}^i(g,K;\pi,P) = 0\,. \tag{3·28}$$

*) The last equation demonstrates the invariance of $\widehat{S}$ under a d-dimensional diffeomorphism,

$$\begin{aligned}0 &= -\int_{\Sigma_d} d^dx\sqrt{g}\,\epsilon_i\widehat{\mathcal{P}}^i\\ &= \int_{\Sigma_d} d^dx\left[(\nabla_i\epsilon_j+\nabla_j\epsilon_i)\frac{\delta\widehat{S}}{\delta g_{ij}} + (-\partial_k\epsilon^i\,P^{kj} - \partial_k\epsilon^j\,P^{ik} + \epsilon^k\,\partial_k P^{ij})\frac{\delta\widehat{S}}{\delta P^{ij}}\right],\end{aligned}$$

with $\epsilon^i(x)$ an arbitrary function. This also demonstrates the invariance of the reduced classical action,

$$0 = \int_{\Sigma_d} d^dx\,(\nabla_i\epsilon_j+\nabla_j\epsilon_i)\,\frac{\delta S}{\delta g_{ij}}\,,$$

for arbitrary $\epsilon^i(x)$.

Since the Hamiltonian density is a linear combination of the constraints, the classical action $\widehat{S}$ does not depend on the coordinate of the lower boundary:

$$\frac{\partial}{\partial r_0}\widehat{S} = \int d^dx\sqrt{g}\left(N\widehat{\mathcal{H}} + \lambda_i\widehat{\mathcal{P}}^i\right) = 0\,. \tag{3·29}$$

This implies that the reduced classical action

$$S[g(x)] \equiv \widehat{S}[g(x), P(x) = 0] \tag{3·30}$$

is also independent of r_0:

$$\frac{\partial}{\partial r_0}S = 0. \tag{3·31}$$

The Hamiltonian and the momentum constraints (3·27) and (3·28) can be translated into equations for the reduced classical action, as we sketched for point-particle systems in Eqs. (2·27)–(2·33). However, the resulting equation can be derived most easily by using the Theorem, (2·35) and (2·36), as follows: We first rewrite the Lagrangian density of zero-th order, $\boldsymbol{\mathcal{L}}_0$, into the first-order form

$$\boldsymbol{\mathcal{L}}_0 \to \pi^{ij}\dot{g}_{ij} - \boldsymbol{\mathcal{H}}_0\,, \tag{3·32}$$

where the zero-th order Hamiltonian density $\boldsymbol{\mathcal{H}}_0$ is given by

$$\boldsymbol{\mathcal{H}}_0(g,\pi;\,N,\lambda) = N\left(\pi_{ij}^2 - \frac{1}{d-1}\pi^2 - 2\Lambda + R\right) - 2\lambda_i\,\nabla_j\pi^{ij}\,. \tag{3·33}$$

Then by using the Theorem, the pseudo-Hamiltonian density is given by

$$\widetilde{\boldsymbol{\mathcal{H}}}(g,\pi;\,N,\lambda) = \boldsymbol{\mathcal{H}}_0(g,\pi;\,N,\lambda) - \boldsymbol{\mathcal{L}}_1(g, K^0(g,\pi), K^1(g,\pi);\,N,\lambda)\,. \tag{3·34}$$

Here $K^0_{ij}(g,\pi)$ is obtained by replacing $\dot{g}_{ij}(x)$ in (3·13) with $\left\{\int d^dy\sqrt{g}\,\boldsymbol{\mathcal{H}}_0(y),\,g_{ij}(x)\right\}$, and it is calculated to be

$$K^0_{ij} = \pi_{ij} - \frac{1}{d-1}\pi\,g_{ij}\,. \tag{3·35}$$

On the other hand, $K^1_{ij} \equiv \left\{\int d^dy\sqrt{g}\,\boldsymbol{\mathcal{H}}_0(y),\,K^0_{ij}\right\}$ is found to be equivalent to replacing L_{ij} in $\boldsymbol{\mathcal{L}}_1$ by

$$\begin{aligned} L^0_{ij} = &-\frac{1}{2(d-1)^2}\Big[2(d-1)\Lambda + (d-1)R + (d-1)\pi_{kl}^2 - 3\pi^2\Big]g_{ij} \\ &+ R_{ij} + 2(\pi^2)_{ij} - \frac{3}{d-1}\pi\pi_{ij}\,. \end{aligned} \tag{3·36}$$

Using Eqs. (3·31)–(3·36), we obtain the following Hamilton-Jacobi-like equation for the reduced classical action:

$$\begin{aligned} 0 &= \int d^dx\sqrt{g}\,\widetilde{\boldsymbol{\mathcal{H}}}\left(g(x),\pi(x);N,\lambda^i\right) \\ &= \int d^dx\sqrt{g}\,\left[N\,\widetilde{\mathcal{H}}(g(x),\pi(x)) + \lambda^i\,\widetilde{\mathcal{P}}_i(g(x),\pi(x))\right]\,, \end{aligned} \tag{3·37}$$

$$\pi^{ij}(x) = \frac{-1}{\sqrt{g}} \frac{\delta S}{\delta g_{ij}(x)}\,, \tag{3·38}$$

where *)

$$\begin{aligned}
\widetilde{\mathcal{H}}(g,\pi) \equiv{}& \pi_{ij}^2 - \frac{1}{d-1}\pi^2 - 2\Lambda + R \\
&+ \alpha_1\,\pi_{ij}^4 + \alpha_2\,\pi\pi_{ij}^3 + \alpha_3\left(\pi_{ij}^2\right)^2 + \alpha_4\,\pi^2\pi_{ij}^2 + \alpha_5\,\pi^4 \\
&+ \beta_1\,\Lambda\pi_{ij}^2 + \beta_2\,\Lambda\pi^2 + \beta_3\,R\,\pi_{ij}^2 + \beta_4\,R\,\pi^2 \\
&+ \beta_5\,R_{ij}(\pi^2)^{ij} + \beta_6\,R_{ij}\,\pi\pi^{ij} + \beta_7\,R_{ijkl}\,\pi^{ik}\pi^{jl} \\
&+ \gamma_1\,\Lambda^2 + \gamma_2\,\Lambda R + \gamma_3\,R^2 + \gamma_4\,R_{ij}^2 + \gamma_5\,R_{ijkl}^2\,, \qquad (3·39) \\
\widetilde{\mathcal{P}}_i(g,\pi) \equiv{}& -2\nabla^j\pi_{ij}\,, \qquad (3·40)
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1 ={}& 2c, \quad \alpha_2 = \frac{2x_5}{(d-1)}, \\
\alpha_3 ={}& \frac{1}{4(d-1)^2}\Big[4a + (d^2-3d+4)b + 4(d-2)(2d-3)c \\
&\qquad - 2(d-1)(dx_4+3x_5)\Big], \\
\alpha_4 ={}& \frac{1}{2(d-1)^3}\Big[-4a - (d^2-3d+4)b - 4(2d^2-5d+4)c \\
&\qquad - 3dx_3 + (2d^2-7d+2)x_4 - 3(2d-1)x_5\Big], \\
\alpha_5 ={}& \frac{1}{4(d-1)^4}\Big[4a + (d^2-3d+4)b + 4(2d^2-5d+4)c \\
&\qquad + 2(3d-4)x_3 - 2(d^2-6d+6)x_4 + 2(5d-6)x_5\Big], \qquad (3·41) \\
\beta_1 ={}& \frac{1}{(d-1)^2}\Big[4da - d(d-3)b - 4(d-2)c - (d-1)(dx_4+3x_5)\Big], \\
\beta_2 ={}& \frac{1}{(d-1)^3}\Big[-4da + d(d-3)b + 4(d-2)c \\
&\qquad - 3dx_3 + (d^2-2d-2)x_4 + 3(d-2)x_5\Big], \\
\beta_3 ={}& \frac{1}{2(d-1)^2}\Big[4a + (d^2-3d+4)b - 4(3d-4)c \\
&\qquad - (d-1)(dx_1 + x_2 - (d-2)x_4 + 3x_5)\Big], \\
\beta_4 ={}& \frac{1}{2(d-1)^3}\Big[-4a - (d^2-3d+4)b + 4(d-2)c
\end{aligned}$$

*) We have ignored those terms in $\widetilde{\mathcal{H}}$ that contain the covariant derivative ∇. This is justified when we consider the holographic Weyl anomaly in four dimensions. Actually, it turns out that they give only total derivative terms in the Weyl anomaly.

$$
\begin{aligned}
&- (d-1)(d-4)x_1 - 3(d-1)x_2 + 3(d-2)x_3 \\
&- (d^2-8d+10)x_4 + 3(3d-4)x_5\Big],
\end{aligned}
$$

$$
\beta_5 = 16c + 3x_5, \qquad \beta_6 = \frac{2(x_1+2x_2-x_4-3x_5)}{d-1}, \qquad \beta_7 = -12c - 2x_2, \tag{3·42}
$$

$$
\begin{aligned}
\gamma_1 &= \frac{d}{(d-1)^2}\Big[4da + (d+1)b + 4c\Big], \\
\gamma_2 &= \frac{1}{(d-1)^2}\Big[4da - d(d-3)b - 4(d-2)c - (d-1)(dx_1+x_2)\Big], \\
\gamma_3 &= \frac{1}{4(d-1)^2}\Big[4a + (d^2-3d+4)b - 4(3d-4)c + 2(d-1)((d-2)x_1 - x_2)\Big], \\
\gamma_4 &= 4c + x_2, \qquad \gamma_5 = c.
\end{aligned} \tag{3·43}
$$

Since the classical action $\widehat{S}[g(x), P(x)]$ is independent of the choice of N and λ^i (and, thus, so is $S[g(x)]$), from Eqs. (3·37)–(3·40) we finally obtain the following equation that determines the reduced classical action:

$$
\widetilde{\mathcal{H}}(g_{ij}(x), \pi^{ij}(x)) = 0\,, \quad \widetilde{\mathcal{P}}_i(g_{ij}(x), \pi^{ij}(x)) = 0\,, \quad \pi^{ij}(x) = \frac{-1}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}(x)}\,. \tag{3·44}
$$

We conclude this section by making a few comments on the possible form of the boundary action $\boldsymbol{S}_b$ and the cosmological constant Λ. As discussed above, in order that the boundary field theory has a continuum limit, the geometry must be asymptotically AdS:

$$
ds^2 \to dr^2 + e^{-2r/l}\gamma_{ij}(x)dx^i dx^j \quad \text{for} \quad r \to -\infty. \tag{3·45}
$$

This should be consistent with our boundary condition $P^{ij} = 0$. By investigating the equation of motion derived from the action (3·15) explicitly, it can easily be shown that this compatibility gives rise to the relation

$$
\begin{aligned}
x_1 &= 4a, \\
x_2 &= 2b, \\
d^2\, x_3 + d\, x_4 + x_5 &= -\frac{4}{3}\Big(d(d+1)a + d\,b + 2c\Big).
\end{aligned} \tag{3·46}
$$

It can also be shown that the asymptotic behavior (3·45) determines the cosmological constant Λ as

$$
\Lambda = -\frac{d(d-1)}{2l^2} + \frac{d(d-3)}{2l^4}\Big[d(d+1)a + db + 2c\Big]. \tag{3·47}
$$

§4. Solution to the flow equation and the Weyl anomaly

In this section, we solve the equation (3·44), using the derivative expansion that was developed in Ref. 26). We then apply the result to computing the holographic Weyl anomaly of $\mathcal{N} = 2$ superconformal field theory in four dimensions, which is dual to IIB supergravity on $AdS_5 \times S^5/\boldsymbol{Z}_2$.

We first note that the basic equation, (3·44), can be rewritten as a flow equation of the form

$$\{S,S\} + \{S,S,S,S\} = \mathcal{L}_d, \tag{4·1}$$

with

$$\begin{aligned}(\sqrt{g})^2\ \{S,S\} \equiv\ & \Bigg[\left(\frac{\delta S}{\delta g_{ij}}\right)^2 - \frac{1}{d-1}\left(g_{ij}\frac{\delta S}{\delta g_{ij}}\right)^2 \\ & + \beta_1\, \mathit{\Lambda}\left(\frac{\delta S}{\delta g_{ij}}\right)^2 + \beta_2\, \mathit{\Lambda}\left(g_{ij}\frac{\delta S}{\delta g_{ij}}\right)^2 + \beta_3\, R\left(\frac{\delta S}{\delta g_{ij}}\right)^2 \\ & + \beta_4\, R\left(g_{ij}\frac{\delta S}{\delta g_{ij}}\right)^2 + \beta_5\, R_{ij} g_{kl}\frac{\delta S}{\delta g_{ik}}\frac{\delta S}{\delta g_{jl}} \\ & + \beta_6\, R_{ij}\frac{\delta S}{\delta g_{ij}}\, g_{kl}\frac{\delta S}{\delta g_{kl}} + \beta_7\, R_{ijkl}\frac{\delta S}{\delta g_{ik}}\frac{\delta S}{\delta g_{jl}}\Bigg],\end{aligned} \tag{4·2}$$

$$\begin{aligned}(\sqrt{g})^4\,\{S,S,S,S\} \equiv & \Bigg[\alpha_1\left(\frac{\delta S}{\delta g_{ij}}\right)^4 + \alpha_2\left(g_{kl}\frac{\delta S}{\delta g_{kl}}\right)\left(\frac{\delta S}{\delta g_{ij}}\right)^3 + \alpha_3\left(\left(\frac{\delta S}{\delta g_{ij}}\right)^2\right)^2 \\ & + \alpha_4\left(g_{kl}\frac{\delta S}{\delta g_{kl}}\right)^2\left(\frac{\delta S}{\delta g_{ij}}\right)^2 + \alpha_5\left(g_{ij}\frac{\delta S}{\delta g_{ij}}\right)^4\Bigg],\end{aligned} \tag{4·3}$$

$$\mathcal{L}_d \equiv 2\mathit{\Lambda} - R - \gamma_1\mathit{\Lambda}^2 - \gamma_2\mathit{\Lambda}R - \gamma_3 R^2 - \gamma_4 R_{ij}^2 - \gamma_5 R_{ijkl}^2. \tag{4·4}$$

Following Refs. 15) and 26), we then assume that the reduced classical action $S[g(x)]$ takes the form

$$\frac{1}{2\kappa_{d+1}^2}S[g(x)] = \frac{1}{2\kappa_{d+1}^2}S_{\rm loc}[g(x)] + \mathit{\Gamma}[g(x)]\,, \tag{4·5}$$

where $2\kappa_{d+1}^2$ is the $(d+1)$-dimensional Newton constant. The functional $\mathit{\Gamma}[g]$ is identified with the generating functional of the boundary field theory in the background metric $g_{ij}(x)$, with any local sources set to zero, and $S_{\rm loc}[g]$ is the local counterterm in $S[g]$:

$$\begin{aligned}S_{\rm loc}[g(x)] &= \int d^dx\,\sqrt{g(x)}\,\mathcal{L}_{\rm loc}(x) \\ &= \int d^dx\,\sqrt{g(x)}\sum_{w=0,2,4,\cdots}\Big[\mathcal{L}_{\rm loc}(x)\Big]_w.\end{aligned} \tag{4·6}$$

Here we have arranged the sum over local terms according to the weight w,[26] which is defined additively from the following rule:

	weight
$g_{ij}(x)$, $\Gamma[g]$	0
∂_i	1
R, $R_{ij}, \cdots$	2
$\delta\Gamma/\delta g_{ij}(x)$	d

We then substitute (4·5) into the flow equation (4·1) and rearrange the resulting equation with respect to the weight. The parts of weight 0 and 2 give

$$2\Lambda - \gamma_1\Lambda^2 = \Big[\{S_{\rm loc},\, S_{\rm loc}\}\Big]_0 + \Big[\{S_{\rm loc},\, S_{\rm loc}, S_{\rm loc}, S_{\rm loc}\}\Big]_0, \tag{4·7}$$

$$-R - \gamma_2\Lambda R = \Big[\{S_{\rm loc},\, S_{\rm loc}\}\Big]_2 + \Big[\{S_{\rm loc},\, S_{\rm loc}, S_{\rm loc}, S_{\rm loc}\}\Big]_2. \tag{4·8}$$

These two equations determine $[\mathcal{L}_{\rm loc}]_0$ and $[\mathcal{L}_{\rm loc}]_2$ as

$$[\mathcal{L}_{\rm loc}]_0 = W\,, \qquad [\mathcal{L}_{\rm loc}]_2 = -\Phi\, R, \tag{4·9}$$

$$\begin{aligned} W =& -\frac{2(d-1)}{l} + \frac{1}{l^3}\Big[-4d(d+1)a - 4db - 8c + d(d^2x_3 + dx_4 + x_5)\Big], \\ \Phi =& \frac{l}{d-2} - \frac{2}{(d-1)(d-2)\,l}\Big[d(d+1)a + d\,b + 2c\Big] \\ &+\frac{1}{l}\left[d\,x_1 + x_2 + \frac{3(d^2x_3 + d\,x_4 + x_5)}{2(d-1)}\right], \end{aligned} \tag{4·10}$$

where (3·47) has been used. It is worthwhile to note that W and Φ can be written in terms of only a, b and c upon substituting into (3·46):

$$\begin{aligned} W =& -\frac{2(d-1)}{l} - \frac{4(d+3)}{3l^3}\Big[d(d+1)a + db + 2c\Big], \\ \Phi =& \frac{l}{d-2} + \frac{2}{(d-2)\,l}\Big[d(d-5)a - 2b - 2c\Big]. \end{aligned} \tag{4·11}$$

For $d > 4$, the flow equation of weight 4 simply determines $\Big[\mathcal{L}_{\rm loc}\Big]_4$ in the local counterterm, as in the case of Einstein gravity (cf. Ref. 26)), while for $d = 4$ this gives an equation that characterizes the generating functional $\Gamma[g(x)]$:

$$\begin{aligned} &2\Big[\{S_{\rm loc},\, \Gamma\}\Big]_4 + 4\Big[\{S_{\rm loc},\, S_{\rm loc},\, S_{\rm loc},\, \Gamma\}\Big]_4 \\ &= -\frac{1}{2\kappa_5^2}\bigg(\Big[\{S_{\rm loc},\, S_{\rm loc}\}\Big]_4 + \Big[\{S_{\rm loc},\, S_{\rm loc},\, S_{\rm loc},\, S_{\rm loc}\}\Big]_4 \\ &\qquad + \gamma_3 R^2 + \gamma_4 R_{ij}^2 + \gamma_5 R_{ijkl}^2\bigg). \end{aligned} \tag{4·12}$$

From this, we can evaluate the trace of the stress tensor for the boundary field theory:

$$\langle T^i_i \rangle_g \equiv \frac{-2}{\sqrt{g}} g_{ij} \frac{\delta \Gamma}{\delta g_{ij}}. \tag{4·13}$$

In fact, using the values in (4·10), we can show that the trace is given by

$$\langle T^i_i \rangle_g = \frac{2l^3}{2\kappa_5^2} \left[\left(\frac{-1}{24} + \frac{5a}{3l^2} + \frac{b}{3l^2} + \frac{c}{3l^2} \right) R^2 + \left(\frac{1}{8} - \frac{5a}{l^2} - \frac{b}{l^2} - \frac{3c}{2l^2} \right) R_{ij}^2 + \frac{c}{2l^2} R_{ijkl}^2 \right]. \tag{4·14}$$

This correctly reproduces the result*) obtained in Refs. 9) and 10), where the Weyl anomaly was calculated by perturbatively solving the equation of motion near the boundary and by looking at the logarithmically divergent term, as in Ref. 6).

For the case of $\mathcal{N} = 2$ superconformal $USp(N)$ gauge theory in four dimensions, we choose $2\kappa_5^2$ such that

$$\frac{1}{2\kappa_5^2} = \frac{\mathrm{Vol}(S^5/\boldsymbol{Z}_2)\,(\text{radius of } S^5/\boldsymbol{Z}_2)^5}{2\kappa^2}, \tag{4·15}$$

where $2\kappa^2 = (2\pi)^7 g_s^2$ is the ten-dimensional Newton constant, [39] and the radius of $S^5/\boldsymbol{Z}_2$ could be set to $(8\pi g_s N)^{1/4}$. [8] In this relation, we note the replacement $N \to 2N$ as compared to the $AdS_5 \times S^5$ case. This is because here we must quantize the RR 5-form flux over $S_5/\boldsymbol{Z}_2$ instead of over S^5. [12] For the AdS_5 radius l, we may also set $l = (8\pi g_s N)^{1/4}$. Setting the values $a = b = 0$ and $c/2l^2 = 1/32N + \mathcal{O}(1/N^2)$, as determined in Ref. 10), we find that the Weyl anomaly (4·14) takes the form

$$\langle T^i_i \rangle_g = \frac{N^2}{2\pi^2} \left[\left(\frac{-1}{24} + \frac{1}{48N} \right) R^2 + \left(\frac{1}{8} - \frac{3}{32N} \right) R_{ij}^2 + \frac{1}{32N} R_{ijkl}^2 \right] + \mathcal{O}(N^0). \tag{4·16}$$

This is different from the field theoretical result, [40]

$$\langle T^i_i \rangle_g = \frac{N^2}{2\pi^2} \left[\left(\frac{-1}{24} - \frac{1}{32N} \right) R^2 + \left(\frac{1}{8} + \frac{1}{16N} \right) R_{ij}^2 + \frac{1}{32N} R_{ijkl}^2 \right] + \mathcal{O}(N^0). \tag{4·17}$$

As was pointed out in Ref. 10), the discrepancy could be accounted for by possible corrections to the radius l as well as to the five-dimensional Newton constant. In

*) The authors of Refs. 9) and 10) parametrized the cosmological constant Λ as

$$\Lambda = -\frac{d(d-1)}{2L^2},$$

so that their L is related to our l, the radius of asymptotic AdS, as

$$l^2 = L^2 \left[1 - \frac{(d-3)}{(d-1)L^2} \big(d(d+1)a + db + 2c \big) \right].$$

fact, if these corrections are

$$l = (8\pi g_s N)^{1/4}\left(1+\frac{\xi}{N}\right), \quad \frac{1}{2\kappa_5^2} = \frac{\mathrm{Vol}(S^5/\boldsymbol{Z}_2)\,(8\pi g_s N)^{5/4}}{2\kappa^2}\left(1+\frac{\eta}{N}\right), \tag{4·18}$$

then the field theoretical result is correctly reproduced for $3\xi+\eta = 5/4$.

§5. Conclusion

In this paper, we investigated higher-derivative gravity systems in the context of the AdS/CFT correspondence. Although higher-derivative gravity requires more boundary conditions than Einstein gravity, we pointed out that by choosing the Neumann boundary conditions for higher-derivative modes, the classical action can be made such that it depends only on the boundary values of bulk fields. We further derived a Hamilton-Jacobi-like equation that determines such a classical action. Using this equation, we computed the $1/N$ correction to the Weyl anomaly of $\mathcal{N}=2$ $G = USp(N)$ superconformal field theory in four dimensions on the basis of the holographic description in terms of type IIB string theory on $AdS_5\times S^5/\boldsymbol{Z}_2$.[12] We found that the resulting Weyl anomaly correctly reproduces the holographic Weyl anomaly given in Refs. 9) and 10), and is consistent with the field theoretical result if we take into account the possible corrections discussed in Ref. 10).

Finally, we comment on how our Neumann boundary condition $P=0$ can be interpreted in the context of the holographic RG. To this end, we consider a toy model with the Lagrangian

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\mu^2 q^2 + \frac{c}{2}\ddot{q}^2, \tag{5·1}$$

whose first-order form reads

$$L = p\dot{q} + P\dot{Q} - H(q,Q;\,p,P), \tag{5·2}$$

with

$$H(q,Q;\,p,P) = -\frac{1}{2}\mu^2 q^2 - \frac{1}{2}Q^2 + Qp + \frac{1}{2c}P^2. \tag{5·3}$$

By performing an almost diagonal canonical transformation,

$$\begin{pmatrix} q\\ Q\\ p\\ P\end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \frac{1}{m^2}a_3 & \frac{1}{M^2}a_4\\ a_3 & a_4 & a_1 & a_2\\ c\,M^2 a_3 & c\,m^2 a_4 & c\,M^2 a_1 & c\,m^2 a_2\\ c\,m^2 a_1 & c\,M^2 a_2 & c\,a_3 & c\,a_4 \end{pmatrix}\begin{pmatrix} q'\\ Q'\\ p'\\ P'\end{pmatrix}, \tag{5·4}$$

with

$$\begin{aligned} m^2 &= \frac{1-\sqrt{1-4c\,\mu^2}}{2c} = \mu^2(1+\mathcal{O}(c))\,,\\ M^2 &= \frac{1+\sqrt{1-4c\,\mu^2}}{2c} = \frac{1}{c}(1+\mathcal{O}(c)), \end{aligned} \tag{5·5}$$

$$a_1^2 = \frac{1}{m^2}a_3^2 + \frac{1}{1-2c\mu^2}\,, \qquad a_2^2 = \frac{1}{M^2}a_4^2 - \frac{1}{1-2c\mu^2}\,, \tag{5·6}$$

the Lagrangian can be rewritten into the following form with normalized kinetic term:

$$L = p'\dot{q}' + P'\dot{Q}' - H'(q', p';\, Q', P'), \tag{5·7}$$

where

$$H'(q', Q';\, p', P') = \frac{1}{2}p'^2 + \frac{1}{2}P'^2 - \frac{1}{2}m^2 q'^2 - \frac{1}{2}M^2 Q'^2. \tag{5·8}$$

Since a bulk mode with mass M is coupled to a scaling operator with scaling dimension $\Delta = \frac{1}{2}\left(d + \sqrt{d + 4M^2}\right)$,[2),3)] the relation (5·5) shows that the mode Q' is coupled to a highly irrelevant operator with large scaling dimension when $c \ll 1$. The essential point of this conclusion does not change even if the variable q corresponds to a bulk field with spin.

Turning to higher-derivative gravity systems, the above example shows that K_{ij} ($\sim Q \sim Q'$) is highly irrelevant in the dual CFT and is approximated well by assuming that it takes a constant value along the renormalized trajectory, as long as we consider the vicinity of the conformal fixed point. This is equivalent to demanding that the corresponding beta function vanishes along the renormalized trajectory. Since P^{ij}, the conjugate momentum of K_{ij}, can be regarded as the RG beta function of K_{ij}, this leads to our requirement, $P^{ij} = 0$. The holographic RG structure in higher-derivative systems will be explored in more detail in a subsequent paper.[34)]

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Appendix A
—— *Proof of Theorem* ——

In this appendix, we give a detailed proof of Theorem, (2·35) and (2·36), for the action

$$\boldsymbol{S} = \int_{t'}^{t} dr \left[L_0(q^i, \dot{q}^i) + c\, L_1(q^i, \dot{q}^i, \ddot{q}^i) \right], \tag{A·1}$$

where i runs over some values. In the following discussion, we focus only on the upper boundary, for simplicity.

We first rewrite the zero-th order Lagrangian L_0 into the first-order form by introducing the conjugate momentum p_{0i} of q^i as

$$\boldsymbol{S}[q(r), p_0(r)] = \int^{t} dr \left[p_{0i}\dot{q}^i - H_0(q, p_0) + c\, L_1(q, \dot{q}, \ddot{q}) \right], \tag{A·2}$$

through the Legendre transformation from $(q,\dot{q})$ to (q,p_0) defined by

$$p_{0i}=\frac{\partial L_0}{\partial \dot{q}^i}(q,\dot{q})\,. \tag{A·3}$$

From this, the equation of motion for p_{0i} and q^i is given by

$$\dot{q}^i=\frac{\partial H_0}{\partial p_{0i}}, \tag{A·4}$$

$$\dot{p_{0i}}=-\frac{\partial H_0}{\partial q^i}+c\left[\frac{\partial L_1}{\partial q^i}-\frac{d}{dr}\left(\frac{\partial L_1}{\partial \dot{q}^i}\right)+\frac{d^2}{dr^2}\left(\frac{\partial L_1}{\partial \ddot{q}^i}\right)\right]. \tag{A·5}$$

Let $\bar{q}(r)$, $\bar{p}_0(r)$ be the solution to this equation of motion that satisfies the boundary condition

$$\bar{q}^i(r\!=\!t)=q^i\,. \tag{A·6}$$

Since this condition determines the classical trajectory uniquely [together with the lower boundary values $\bar{q}^i(r\!=\!t')=q'^{\,i}$ that we have not written here explicitly], the boundary value of $\bar{p}_0$ is completely specified by t and q: $\bar{p}_0(r=t)=p_0(t,q)$. By plugging the classical solution into the action $\boldsymbol{S}$, the classical action is obtained as a function of the boundary value q^i and t:

$$S(t,q)=\boldsymbol{S}[\bar{q}(r),\bar{p}_0(r)]. \tag{A·7}$$

In order to derive a differential equation that determines $S(t,q)$, we then take the variation of $S(t,q)$. Using (A·4) and (A·5), this is easily evaluated to be

$$\begin{aligned}\delta S=\delta t&\Big[p_{0i}\dot{q}^i-H_0(q,p_0)+c\,L_1(q,\dot{q},\ddot{q})\Big]\\&+\delta\bar{q}^i(t)\left[p_{0i}+c\left(\frac{\partial L_1}{\partial \dot{q}^i}(q,\dot{q},\ddot{q})-\frac{d}{dr}\left(\frac{\partial L_1}{\partial \ddot{q}^i}(\bar{q},\dot{\bar{q}},\ddot{\bar{q}})\right)\Big|_{r=t}\right)\right]\\&+c\,\delta\dot{\bar{q}}^i(t)\,\frac{\partial L_1}{\partial \ddot{q}^i}(q,\dot{q},\ddot{q}),\end{aligned} \tag{A·8}$$

where

$$\dot{q}^i\equiv\frac{d\bar{q}^i}{dr}(r\!=\!t),\qquad \ddot{q}^i\equiv\frac{d^2\bar{q}^i}{dr^2}(r\!=\!t)\,, \tag{A·9}$$

and $\delta\bar{q}^i(t)$ and $\delta\dot{\bar{q}}^i(t)$ are understood to be $\delta\bar{q}^i(r)|_{r=t}$ and $d\,\delta\bar{q}^i(r)/dr|_{r=t}$, respectively. By expanding the classical solution $\bar{q}^i(r)$ around $r=t$, we find that the variations $\delta\bar{q}^i(t)$ and $\delta\dot{\bar{q}}^i(t)$ are given by

$$\delta\bar{q}^i(t)=\delta q^i-\dot{q}^i\,\delta t,\qquad \delta\dot{\bar{q}}^i(t)=\delta\dot{q}^i-\ddot{q}^i\,\delta t. \tag{A·10}$$

Here it is important to note that $\dot{q}$ can be written in terms of q and t, since the classical solution is determined uniquely by the boundary value q. Actually it can be shown that

$$\begin{aligned}\delta\dot{q}^i &= \frac{\partial^2 H_0}{\partial q^j \partial p_{0i}}\,\delta q^j + \frac{\partial^2 H_0}{\partial p_{0i} p_{0j}}\,\delta p_{0j} \\ &= \frac{\partial^2 H_0}{\partial q^j \partial p_{0i}}\,\delta q^j + \frac{\partial^2 H_0}{\partial p_{0i} p_{0j}}\left(\frac{\partial p_{0j}}{\partial t}\delta t + \frac{\partial p_{0j}}{\partial q^k}\delta q^k\right), \end{aligned} \tag{A·11}$$

where we have used (A·4) as well as the fact that $p_0 = p_0(t,q)$. From these relations, the variation (A·8) is found to be

$$\delta S = p_i\,\delta q^i - \widetilde{H}(q,p)\,\delta t, \tag{A·12}$$

with

$$\begin{aligned} p_i &= p_{0i} + c\left[\frac{\partial L_1}{\partial \dot{q}^i}(q,\dot{q},\ddot{q}) - \frac{d}{dr}\left(\frac{\partial L_1}{\partial \ddot{q}^i}(\bar{q},\dot{\bar{q}},\ddot{\bar{q}})\right)\Big|_{r=t} \right. \\ &\qquad \left. + \frac{\partial L_1}{\partial \ddot{q}^j}\left(\frac{\partial^2 H_0}{\partial q^i \partial p_{0j}} + \frac{\partial^2 H_0}{\partial p_{0j}\partial p_{0k}}\frac{\partial p_{0k}}{\partial q^i}\right)\right], \end{aligned} \tag{A·13}$$

$$\begin{aligned} \widetilde{H}(q,p) &= H_0(q,p_0) \\ &\quad + c\left[-L_1(q,\dot{q},\ddot{q}) + \dot{q}^i\left(\frac{\partial L_1}{\partial \dot{q}^i}(q,\dot{q},\ddot{q}) - \frac{d}{dr}\left(\frac{\partial L_1}{\partial \ddot{q}^i}(\bar{q},\dot{\bar{q}},\ddot{\bar{q}})\right)\Big|_{r=t}\right)\right. \\ &\qquad \left. + \frac{\partial L_1}{\partial \ddot{q}^i}\left(\ddot{q}^i - \frac{\partial^2 H_0}{\partial p_{0i}\partial p_{0j}}\frac{\partial p_{0j}}{\partial t}\right)\right]. \end{aligned} \tag{A·14}$$

In order to compute $\widetilde{H}(q,p)$, we first note that the Hamilton equation appearing in (A·4) and (A·5) gives the relation

$$\ddot{q}^i = \frac{\partial^2 H_0}{\partial p_{0i}\partial q^j}\frac{\partial H_0}{\partial p_{0j}} + \frac{\partial^2 H_0}{\partial p_{0i}\partial p_{0j}}\left(\frac{\partial p_{0j}}{\partial q^k}\frac{\partial H_0}{\partial p_{0k}} + \frac{\partial p_{0k}}{\partial t}\right). \tag{A·15}$$

It is then easy to verify that $\widetilde{H}(q,p)$ takes the form

$$\widetilde{H}(q,p) = H_0(q,p) - c\,L_1(q,\dot{q},\ddot{q}) + \mathcal{O}(c^2). \tag{A·16}$$

Here $\dot{q}^i$ and $\ddot{q}^i$ in L_1 can be replaced by

$$f_1^i(q,p) \equiv \left\{H_0(q,p), q^i\right\} = \frac{\partial H_0}{\partial p_i}(q,p) \tag{A·17}$$

and

$$\begin{aligned} f_2^i(q,p) &\equiv \left\{H_0(q,p), \left\{H_0(q,p), q^i\right\}\right\} \\ &= \frac{\partial^2 H_0}{\partial p_i \partial q^j}(q,p)\frac{\partial H_0}{\partial p_j}(q,p) - \frac{\partial^2 H_0}{\partial p_i \partial p_j}(q,p)\frac{\partial H_0}{\partial q^j}(q,p), \end{aligned} \tag{A·18}$$

respectively, up to $\mathcal{O}(c^2)$. This completes the proof of (2·35) and (2·36).

Appendix B
—— ADM Decomposition ——

In this appendix, we summarize the components of the Riemann tensor, Ricci tensor and scalar curvature written in terms of the ADM decomposition.

In the ADM decomposition, the metric takes the form

$$\begin{aligned} ds^2 &= \widehat{g}_{\mu\nu}\, dX^\mu dX^\nu \\ &= N(x,r)^2 dr^2 + g_{ij}(x,r)\Big(dx^i + \lambda^i(x,r)dr\Big)\Big(dx^j + \lambda^j(x,r)dr\Big). \end{aligned} \tag{B·1}$$

Here we use the following basis instead of the coordinate basis ∂_μ:

$$\widehat{e}_{\widehat{n}} = \frac{1}{N}(\partial_r - \lambda^i \partial_i,), \qquad \widehat{e}_i = \partial_i. \tag{B·2}$$

In this basis, the components of the metric are given by

$$\begin{pmatrix} \widehat{g}(\widehat{e}_{\widehat{n}}, \widehat{e}_{\widehat{n}}) & \widehat{g}(\widehat{e}_{\widehat{n}}, \widehat{e}_j) \\ \widehat{g}(\widehat{e}_j, \widehat{e}_{\widehat{n}}) & \widehat{g}(\widehat{e}_i, \widehat{e}_j) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{ij} \end{pmatrix}. \tag{B·3}$$

For the purpose of computing the Riemann tensor in this basis, it is useful to start with the formula

$$\begin{aligned} \widehat{R}^\sigma{}_{\rho\mu\nu}\, \widehat{e}_\sigma &= \widehat{R}(\widehat{e}_\mu, \widehat{e}_\nu)\widehat{e}_\rho \\ &= \Big[\widehat{\nabla}_{\widehat{e}_\mu}, \widehat{\nabla}_{\widehat{e}_\nu}\Big]\, \widehat{e}_\rho - \widehat{\nabla}_{[\widehat{e}_\mu, \widehat{e}_\nu]}\, \widehat{e}_\rho. \end{aligned} \tag{B·4}$$

Each component can be calculated explicitly by using the equations

$$\begin{aligned} \widehat{\nabla}_{\widehat{e}_i} \widehat{e}_j &= -K_{ij}\widehat{e}_{\widehat{n}} + \mathit{\Gamma}^k_{ij}\, \widehat{e}_k, \\ \widehat{\nabla}_{\widehat{e}_i} \widehat{e}_{\widehat{n}} &= K_i^k\, \widehat{e}_k, \\ \widehat{\nabla}_{\widehat{e}_{\widehat{n}}} \widehat{e}_j &= \frac{1}{N}\, \partial_j N\, \widehat{e}_{\widehat{n}} + \left(K_j^k + \frac{1}{N}\, \partial_j \lambda^k\right) \widehat{e}_k, \\ \widehat{\nabla}_{\widehat{e}_{\widehat{n}}} \widehat{e}_{\widehat{n}} &= -\frac{1}{N}\, g^{kl}\, \partial_k N\, \widehat{e}_l, \\ [\widehat{e}_{\widehat{n}}, \widehat{e}_i] &= \frac{1}{N}\, \partial_i N\, \widehat{e}_{\widehat{n}} + \frac{1}{N}\, \partial_i \lambda^k\, \widehat{e}_k, \end{aligned} \tag{B·5}$$

where K_{ij} is the extrinsic curvature and $\mathit{\Gamma}^i_{jk}$ is the affine connection with respect to g_{ij}. We thus obtain

$$\begin{aligned} \widehat{R}_{ijkl} &= R_{ijkl} - K_{ik}K_{jl} + K_{il}K_{jk}, \\ \widehat{R}_{\widehat{n}jkl} &= \nabla_l K_{jk} - \nabla_k K_{jl}, \\ \widehat{R}_{\widehat{n}j\widehat{n}l} &= (K^2)_{jl} - L_{jl}, \end{aligned} \tag{B·6}$$

with

$$L_{ij} = \frac{1}{N}\left(\dot{K}_{ij} - \lambda^k\, \nabla_k K_{ij} - \nabla_i \lambda^k\, K_{kj} - \nabla_j \lambda^k\, K_{kj} + \nabla_i \nabla_j N\right). \tag{B·7}$$

The components of the Ricci tensor $\widehat{R}_{\mu\nu} \equiv \widehat{R}^{\rho}{}_{\mu\rho\nu} = \widehat{R}_{\nu\mu}$ are given by

$$\begin{aligned} \widehat{R}_{ij} &= R_{ij} + 2(K^2)_{ij} - KK_{ij} - L_{ij}, \\ \widehat{R}_{i\widehat{n}} &= \nabla^k K_{ki} - \nabla_i K, \\ \widehat{R}_{\widehat{n}\widehat{n}} &= K_{ij}^2 - g^{ij} L_{ij}\,, \end{aligned} \tag{B·8}$$

and the scalar curvature is

$$\widehat{R} = R + 3K_{ij}^2 - K^2 - 2g^{ij} L_{ij}. \tag{B·9}$$

Appendix C
—— *Boundary Terms* ——

In this appendix, we supplement the discussion of the possible boundary terms given in §3.

We first consider the infinitesimal transformation

$$x^i \to x'^i = x^i + \epsilon^i(x,r), \qquad r \to r' = r + \epsilon(x,r). \tag{C·1}$$

Under this transformation, N, λ_i and g_{ij} are found to transform as

$$\begin{aligned} \frac{1}{N'} &= \frac{1}{N}(1 + \dot{\epsilon} - \lambda^i \partial_i \epsilon), \\ \lambda'_i &= \lambda_i - \partial_i \epsilon^j \lambda_j - \dot{\epsilon}\lambda_i - \partial_i \epsilon\,(N^2 + \lambda^2) - g_{ij}\dot{\epsilon}^j, \\ g'_{ij} &= g_{ij} - \partial_i \epsilon^k g_{kj} - \partial_j \epsilon^k g_{ik} - \partial_i \epsilon\, \lambda_j - \partial_j \epsilon\, \lambda_i. \end{aligned} \tag{C·2}$$

Furthermore, Γ^i_{jk}, the affine connection defined by g_{ij}, transforms under the diffeomorphism (C·1) as

$$\Gamma'^i_{jk} = \Gamma^i_{jk} - \partial_j\,\partial_k \epsilon^i + \Gamma^m_{jk}\,\partial_m \epsilon^i - \Gamma^i_{mk}\,\partial_j \epsilon^m - \Gamma^i_{jm} \partial_k \epsilon^m + \tilde{\delta}\Gamma^i_{jk}, \tag{C·3}$$

with

$$\tilde{\delta}\Gamma^i_{jk} = -\lambda^i \nabla_j \nabla_k \epsilon - \partial_j \epsilon \nabla_k \lambda^i - \partial_k \epsilon \nabla_j \lambda^i - N g^{il}(\partial_j \epsilon\, K_{lk} + \partial_k \epsilon\, K_{lj} - \partial_l \epsilon\, K_{jk}). \tag{C·4}$$

Note that $\tilde{\delta}\Gamma^i_{jk}$ does not contain ϵ^i. From these relations, it is straightforward to verify that the extrinsic curvature transforms as

$$\begin{aligned} K'_{ij} = {} & K_{ij} - \partial_i \epsilon^l\, K_{lj} - \partial_k \epsilon^l\, K_{jl} \\ & + N \nabla_i \nabla_j \epsilon + \partial_i \epsilon\,(\partial_j N - \lambda^l K_{jl}) + \partial_j \epsilon\,(\partial_i N - \lambda^l K_{lj}). \end{aligned} \tag{C·5}$$

We can also show that the Riemann curvature $R^i{}_{jkl}$ transforms under (C·1) as

$$\begin{aligned} R'^i{}_{jkl} = {} & R^i{}_{jkl} + \partial_m \epsilon^i\, R^m_{jkl} - \partial_j \epsilon^m\, R^i{}_{mkl} - \partial_k \epsilon^m\, R^i{}_{jml} - \partial_l \epsilon^m\, R^i{}_{jkm} \\ & - \partial_k \epsilon\, \dot{\Gamma}^i_{lj} + \partial_l \epsilon\, \dot{\Gamma}^i_{kj} + \nabla_k \tilde{\delta}\Gamma^i_{lj} - \nabla_l \tilde{\delta}\Gamma^i_{kj}. \end{aligned} \tag{C·6}$$

As argued in §3, we focus on the diffeomorphism that obeys the condition (3·6). This is equivalent to the following relation in an infinitesimal form:

$$\partial_i \epsilon(r = r_0) = 0. \tag{C·7}$$

Therefore, we find that the boundary action (3·7) is invariant under this diffeomorphism.

We remark that in the above, we have discarded boundary terms of the form

$$\boldsymbol{S}'_b = \int_{\Sigma_d} d^d x \sqrt{g} \left(K^{ij} L_{ij} + K g^{ij} L_{ij} \right), \tag{C·8}$$

although these are allowed by the diffeomorphism.*) The reason is that if there were such boundary terms, they would require us to further introduce an extra boundary condition, since

$$\delta \boldsymbol{S}'_b = \int_{\Sigma_d} d^d x \sqrt{g} \left[\cdots + \delta \dot{K}_{ij} P_2^{ij}(g_{kl}, K_{kl}) \right]. \tag{C·9}$$

Appendix D
—— *Coefficients in Eq. (3·18)* ——

We have the following values for the coefficients in Eq. (3·18):

$$\begin{aligned}
A_1 &= \frac{2b - x_2}{2(b+4c)}, \qquad A_2 = \frac{4b + 8c - 3x_5}{2(b+4c)}, \qquad A_3 = -\frac{b + x_4}{b + 4c},\\
A_4 &= -\frac{4ab - 16ac + bx_1 + 4cx_1 - 4ax_2 + 2b^2 - bx_2}{2(b+4c)(4da + (d+1)b + 4c)},\\
A_5 &= -\frac{4ab - 16ac + 2b^2 + bx_4 + 4cx_4 - 12ax_5 - 3bx_5}{2(b+4c)(4da + (d+1)b + 4c)},\\
A_6 &= \frac{4ab - 16ac - 3bx_3 - 12cx_3 + 8ax_4 + 2b^2 + 2bx_4}{2(b+4c)(4da + (d+1)b + 4c)},
\end{aligned} \tag{D·1}$$

$$\begin{aligned}
B_1 &= \frac{1}{4(b+4c)(4da + (d+1)b + 4c)}\\
&\quad \times \Big[4b^3 + 4(d+1)ab^2 + 4a{x_2}^2 - 4b^2 x_2 + b{x_2}^2 + 64ac^2 - 8abx_2\\
&\quad + 16(d-2)abc - 4dc{x_1}^2 - db{x_1}^2 + 4b^2 x_1 + 16bcx_1 - 8cx_1x_2\\
&\quad + 32acx_2 - 2bx_1x_2 + 8dabx_1 + 32dacx_1 \Big],\\
B_2 &= \frac{16bc + 4bx_2 - {x_2}^2}{4(b+4c)}, \qquad B_3 = c,
\end{aligned} \tag{D·2}$$

$$C_1 = \frac{1}{4(b+4c)(4da + (d+1)b + 4c)}$$

*) By definition, the $(d+1)$-dimensional scalar curvature $\widehat{R}$ is a scalar. It thus follows from (B·9) that $L_{ij}(r = r_0)$ transforms as a tensor under the diffeomorphism with (C·7).

$$\times\Big[8b^3 - 8abx_2 - 16(d+1)bcx_1 - 64c^2x_1 - 32dacx_1$$
$$-4db^2x_1 + 8dabx_1 - 4b^2x_2 + 32acx_2 + 8\,dabx_4 - 24abx_5$$
$$+24ax_2x_5 + 6bx_2x_5 - 12b^2x_5 + 32(d-2)abc - 2dbx_1x_4$$
$$+8(d+1)ab^2 + 16bcx_4 + 4b^2x_4 - 2bx_2x_4 - 6bx_1x_5$$
$$-8cx_2x_4 - 8dcx_1x_4 + 32dacx_4 - 24cx_1x_5 + 96acx_5 + 128ac^2\Big],$$

$$C_2 = \frac{1}{4(b+4c)(4da+(d+1)b+4c)}$$
$$\times\Big[-16b^2c + 8bcx_2 + 64c^2x_1 + 32dacx_1 - 32(d+2)abc$$
$$-8(7d+5)ab^2 - (d-3)bx_2x_4 - 4(d-4)ax_2x_4$$
$$+8(d-2)abx_4 + 2(d-3)b^2x_4 + 3(d-1)bx_2x_3 - 6(d-1)b^2x_3$$
$$-4(d+3)b^3 + 32acx_2 + 24(d+1)abx_2 + 16(d+1)bcx_1$$
$$+64da^2x_2 - 12cx_2x_3 + 2(d+3)b^2x_2 + 8dabx_1 + 4db^2x_1$$
$$-6dbx_1x_3 + 24bcx_3 - 4bx_1x_4 + 12dax_2x_3 + 8bcx_4 + 96dacx_3$$
$$-4cx_2x_4 - 16cx_1x_4 + 64acx_4 - 24dcx_1x_3 - 128da^2b - 128ac^2\Big],$$

$$C_3 = \frac{32bc + 6bx_5 - 3x_2x_5 + 64c^2 - 8cx_2}{2(b+4c)},$$

$$C_4 = \frac{-8bc + 2bx_4 - 2bx_1 - x_2x_4 - 8cx_1 + 4cx_2}{b+4c}, \qquad C_5 = -12c - 2x_2, \tag{D$\cdot$3}$$

$$D_1 = \frac{8bc - 9{x_5}^2 - 48cx_5 - 32c^2}{4(b+4c)},$$

$$D_2 = \frac{-4bx_5 - 16bc - 16cx_4 - 6x_4x_5 + 8cx_5}{2(b+4c)},$$

$$D_3 = \frac{1}{4(b+4c)(4da+(d+1)b+4c)}$$
$$\times\Big[-6bx_4x_5 - 64c^2x_4 + 96acx_5 - 16(d+1)bcx_4 - 32dacx_4 + 128c^3$$
$$-4db^2x_4 - 24cx_4x_5 + 32(d+2)bc^2 - db{x_4}^2 - 4dc{x_4}^2 + 8(d+1)b^2c$$
$$+4b^3 + 64(2d+1)ac^2 + 4(d+1)ab^2 + 16(3d-2)abc - 24abx_5$$
$$-12b^2x_5 - 8dabx_4 + 9b{x_5}^2 + 36a{x_5}^2\Big],$$

$$D_4 = \frac{1}{4(b+4c)(4da+(d+1)b+4c)}$$
$$\times\Big[-8b^3 - 32c{x_4}^2 + 48ax_4x_5 - 16da{x_4}^2 + 24abx_5 + 12b^2x_5 + 12bx_4x_5$$
$$-24dabx_3 - 96dacx_3 + 64c^2x_4 - 96acx_5 - 192c^2x_3$$

$$-72cx_3x_5+32(d+2)abc-48(d+1)bcx_3-8(3d+2)abx_4$$
$$+16(d-1)bcx_4+32(d+2)acx_4+16(d+1)b^2c-4(d+2)b{x_4}^2$$
$$-4(d+4)b^2x_4-8(d+1)ab^2-128ac^2-6dbx_3x_4$$
$$\left.-18bx_3x_5-24dcx_3x_4-12db^2x_3+64bc^2\right],$$

$$D_5=\frac{1}{4(b+4c)(4da+(d+1)b+4c)}$$
$$\times\left[16a{x_4}^2+64c^2x_3-8dabx_3-32dacx_3-12bx_3x_4-48cx_3x_4,\right.$$
$$+4(d-2)b^2x_3-64acx_4+8b^2x_4+4b{x_4}^2+4b^3+64ac^2-9db{x_3}^2$$
$$\left.-36dc{x_3}^2+4(d+1)ab^2+16(d-2)abc+16abx_4+16(d-1)bcx_3\right],$$
$$\tag{D·4}$$

$$E_1=4b+2x_1-x_2,\qquad E_2=-2b+8c-2x_2,\qquad E_3=-8c-x_2,$$
$$E_4=2b-2x_1. \tag{D·5}$$

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