Multiray generalization of the arcsine laws for occupation times of infinite ergodic transformations

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1 Introduction

In this report, based on the paper [3], we consider a certain distributional convergence of occupation time ratios for ergodic transformations preserving an infinite measure. We give a general limit theorem which can be regarded as a multiray extension of Thaler [6] and Thaler–Zweimüler [7]. We now explain a typical example of their results.

Example 1.1 (occupation measure of Boole's transformation)**.** Let $(B(t))_{t\geq 0}$ be a standard one-dimensional Brownian motion with $B(0) = 0$, and set

$$A_+ := \int_0^1 1\{B(t) > 0\}dt,$$
$$A_+ := \int_0^1 1\{B(t) < 0\}dt.$$

I.e., the A_+ (resp. A_-) is the amount of time which $(B(t))_{t\geq 0}$ spends on the positive (resp. negative) side up to time 1. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $Tx := x - x^{-1}$. The map T is called *Boole's transformation*. It is known that T is a conservative, ergodic, measure preserving transformation on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R}), dx\big)$. Let $[-\infty, +\infty]$ denote the two point compactification of $\mathbb{R}$. Then what Thaler proved in [6] can be stated as follows: for any probability measure $\nu(dx) \ll dx$ on $\mathbb{R}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{T^k x} \ \text{(under } \nu(dx)) \ \xrightarrow[n\to\infty]{\mathrm{d}} \ A_+\delta_{+\infty} + A_-\delta_{-\infty}, \quad \text{in } \mathcal{P}[-\infty, +\infty],$$

where δ_y denotes the Dirac measure at y, and $\mathcal{P}[-\infty, +\infty]$ denotes the space of probability measures on $[-\infty, +\infty]$, endowed with the topology of weak convergence.

2 Multiray generalization of the arcsine law

Let $d \geq 2$ be an integer. For $\alpha \in (0, 1)$ and $\beta = (\beta_i)_{i=1}^d \in [0, 1]^d$ with $\sum_{i=1}^d \beta_i = 1$, let $(Z^{(\alpha,\beta)}(t))_{t\geq 0}$ be a *skew Bessel diffusion process*, starting at 0, of dimension $2 - 2\alpha \in (0, 2)$ and with skewness parameter β on d rays which are all connected at 0. In the special case of $d = 2$ and $\alpha = \beta_1 = \beta_2 = 1/2$, this process is nothing else but a standard one-dimensional Brownian motion. Let us denote by $A_i^{(\alpha,\beta)}$ the occupation time of $(Z^{(\alpha,\beta)}(t))_{t\geq 0}$ on i-th ray up to time 1 for $i = 1, \dots, d$. Barlow–Pitman–Yor [1] showed that

$$\left(A_1^{(\alpha,\beta)}, \dots, A_d^{(\alpha,\beta)}\right) \overset{\mathrm{d}}{=} \left(\frac{\xi_1}{\xi_1 + \cdots + \xi_d}, \dots, \frac{\xi_d}{\xi_1 + \cdots + \xi_d}\right),$$

where $\xi_1, \dots, \xi_d$ are $\mathbb{R}_+$-valued independent random variables with the one-sided α-stable distributions characterized by

$$\mathbb{E}\big[\exp(-\lambda\xi_i)\big] = \exp(-\lambda^\alpha \beta_i), \quad \lambda > 0,\ i = 1, \dots, d.$$

In the special case of $\alpha = \beta_1 = 1/2$, the $A_1^{(\alpha,\beta)}$ is arcsine distributed.

3 Main results

Let $(X, \mathcal{B}, \mu)$ be a standard measurable space with a σ-finite measure such that $\mu(X) = \infty$, and let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a conservative, ergodic, measure preserving transformation (*CEMPT*), i.e., $\mu T^{-1} = \mu$ and $\sum_{k\geq 0} \mathbb{1}_A(T^k x) = \infty$, μ-a.e.x, for any $A \in \mathcal{B}$ with $\mu(A) > 0$.

Assumption 3.1 (d-ray)**.** The state space X is decomposed into $X = \sum_{i=1}^d X_i + Y$ for the *rays* $X_i \in \mathcal{B}$ with $\mu(X_i) = \infty$ $(i = 1, \dots, d)$ and the *junction* $Y \in \mathcal{B}$ with $\mu(Y) \in (0, \infty)$ such that, *when the orbit* $(T^k x)_{k\geq 0}$ *changes rays, it must visit the junction.*

We will denote by $H_n(x)$ the n-th hitting time of $(T^k x)_{k\geq 0}$ for Y. Set

$$\big(\ell_i(n)\big)(x) := \begin{cases} H_n(x) - H_{n-1}(x), & \text{if } T^{H_{n-1}(x)+1}x \in X_i, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(n) := \big(\ell_1(n), \dots, \ell_d(n)\big).$$

Note that $\ell_i(n)$ is the *n-th X_i-side excursion length of the orbit from Y*, and the sequence $(\ell(n))_{n\geq 1}$ is stationary w.r.t. the probability measure $\mu_Y := \mu(\cdot \cap Y)/\mu(Y)$. The dual operator $\widehat{T} : L^1(\mu) \to L^1(\mu)$ is characterized by $\int(\widehat{T}f)g d\mu = \int f(g \circ T)d\mu$ for any $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$.

Assumption 3.2 (asymptotic recursion densities)**.** For each $i = 1, \dots, d$, there exists $H_i \in L^\infty(\mu)$ such that

$$\frac{1}{\mu_Y(\ell_i(1) \geq n)} \sum_{k\geq n} \widehat{T}^{k+1} \mathbb{1}\{Y \cap \{\ell_i(1) = k\}\} \to H_i, \quad \text{in } L^\infty(\mu), \text{ as } n \to \infty,$$

and $\operatorname{essinf}_{Y\cap T^{-1}X_i} H_i > 0$ w.r.t. μ.

Set

$$A_i(n) := \sum_{k=0}^{n-1} \mathbb{1}_{X_i} \circ T^k, \quad n \geq 0,\ i = 1, \dots, d.$$

I.e., the $A_i(n)$ is the amount of time that the orbit stays on the i-th ray up to time $n - 1$. We now give our general limit theorem as follows.

Theorem 3.3 (S.–Yano [3]). *Let $\alpha \in (0,1)$ and $\beta = (\beta_i)_{i=1}^d \in [0,1]^d$ with $\sum_{i=1}^d \beta_i = 1$. Suppose that T is a CEMPT on $(X, \mathcal{B}, \mu)$ and that Assumptions 3.1 and 3.2 hold. We consider the following conditions:*

(i) *For each $\lambda > 0$ and $i = 1, \dots, d$,*

$$\lim_{r\to\infty} \frac{\mu_Y(\ell_i(1) > \lambda r)}{\mu_Y(|\ell(1)| > r)} = \lambda^{-\alpha}\beta_i. \tag{1}$$

(ii) *For any probability measure $\nu \ll \mu$ on X,*

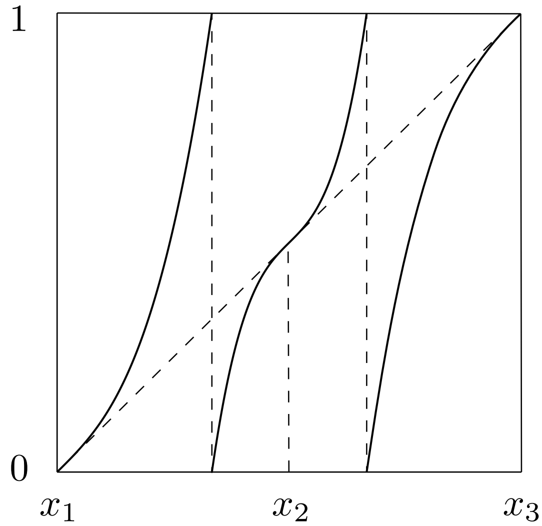
$$\left(\frac{1}{n}A_1(n), \dots, \frac{1}{n}A_d(n)\right) \text{ (under } \nu) \underset{n\to\infty}{\overset{\mathrm{d}}{\longrightarrow}} \left(A_1^{(\alpha,\beta)}, \dots, A_d^{(\alpha,\beta)}\right). \tag{2}$$

Then, (i) *implies* (ii)*. Furthermore, if $\beta \in [0,1)^d$, then* (ii) *implies* (i).

The case $d = 2$ was due to [6] and [7]. The proofs in [6] and [7] were based on the moment method, which does not seem to be suitable for our multiray case. We adopt instead the double Laplace transform method, which was utilized in the study [1] of occupation times of diffusions on multiray.

4 Application to interval map

Let $d \geq 2$ be an integer, and let $0 = a_0 = x_1 < a_1 < x_2 < \cdots < a_{d-1} < x_d = a_d = 1$. Suppose that the map $T : [0,1] \to [0,1]$ satisfies the following conditions: for each $i = 1, \dots, d$,



(1) the restriction $T|_{(a_{i-1},a_i)}$ has a C^2-extension over $[a_{i-1}, a_i]$, and $T\big((a_{i-1}, a_i)\big) = (0,1)$,

(2) the x_i satisfies

$$\begin{gathered} Tx_i = x_i, \quad T'x_i = 1, \quad \text{and} \\ (x - x_i)T''x > 0 \text{ for any } x \in (a_{i-1}, a_i) \setminus \{x_i\}. \end{gathered}$$

In particular, $T' > 1$ on $(a_{i-1}, a_i) \setminus \{x_i\}$ and consequently x_i is an indifferent fixed point of T.

Then T has the unique (up to multiplication of positive constants) σ-finite invariant measure μ equivalent to the Lebesgue measure and T is a CEMPT on $\big([0,1], \mathcal{B}([0,1]), \mu\big)$. By the assumptions (1) and (2), we see that each $T|_{(a_{i-1},a_i)}$ is invertible, and its inverse has a C^2-extension over $[0,1]$, which will be denoted by f_i. We also see that

$$|x - f_i(x)| = O(|x - x_i|^2), \text{ as } x \to x_i.$$

It is known that the density of μ w.r.t. the Lebesgue measure has a version h of the form

$$h(x) = h_0(x) \prod_{i=1}^{d} \frac{x - x_i}{x - f_i(x)},$$

where h_0 is continuous and positive on $[0,1]$. Thus, for any $\varepsilon > 0$,

$$\mu\Big([0,1] \setminus \sum_{i=1}^{d} (x_i - \varepsilon, x_i + \varepsilon)\Big) < \infty, \tag{3}$$

$$\mu\big((x_i - \varepsilon, x_i + \varepsilon)\big) = \infty, \quad i = 1, \dots, d. \tag{4}$$

Let $\mathcal{R}_\rho(0+)$ be the set of regularly varying functions of index ρ at 0. The following corollary is an multiray extension of the 2-ray result [6].

Corollary 4.1. *Let $\alpha \in (0,1)$. Suppose that there exist $\Phi \in \mathcal{R}_{1+1/\alpha}(0+)$ and $c = (c_i)_{i=1}^d \in (0,\infty)^d$ such that, for each $i = 1, \dots, d$,*

$$|Tx - x| \sim c_i \Phi\big(|x - x_i|\big), \quad \text{as } x \to x_i.$$

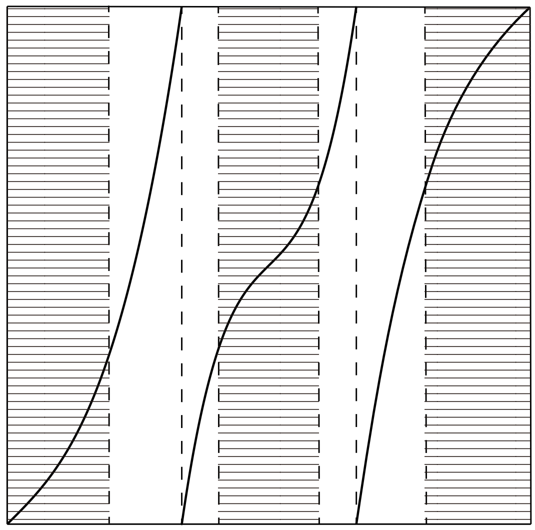
Then, for any probability measure $\nu(dx) \ll dx$ on $[0,1]$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x} \ (\text{under } \nu(dx)) \xrightarrow[n\to\infty]{\mathrm{d}} A_1^{(\alpha,\beta)} \delta_{x_1} + \dots + A_d^{(\alpha,\beta)} \delta_{x_d}, \quad \text{in } \mathcal{P}[0,1]. \tag{5}$$

for the constant $\beta = (\beta_i)_{i=1}^d \in (0,1)^d$ with $\sum_{i=1}^d \beta_i = 1$, defined by $\beta_i := \widetilde{\beta}_j / \sum_{j=1}^d \widetilde{\beta}_j$, where

$$\widetilde{\beta}_i := \begin{cases} v_i, & j = 1, d, \\ 2v_i, & j = 2, \dots, d-1, \end{cases}$$

$$v_i := c_i^{-\alpha} \sum_{j \neq i} (h \circ f_j)(x_i) f_j'(x_i).$$



Sketch of the proof. Set

$$X_i := f_i\big((a_{i-1}, a_i)\big), \quad i = 1, \ldots, d,$$
$$Y := [0,1] \setminus \sum_{i=1}^{d} X_i,$$

and let $\ell_i(n)$, $\ell(n)$ and $A_i(n)$ be as in Section 3. Then Assumptions 3.1 and 3.2 hold. Furthermore, the condition (i) of Theorem 3.3 holds. Therefore, for any probability measure $\nu(dx) \ll dx$ (and hence $\nu \ll \mu$) on $[0,1]$,

$$\left(\frac{1}{n} A_i(n)\right)_{i=1}^{d} \text{ (under } \nu) \xrightarrow[n\to\infty]{\mathrm{d}} \left(A_i^{(\alpha,\beta)}\right)_{i=1}^{d}. \tag{6}$$

By (3) and (4), we have, for any $\varepsilon > 0$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}\left\{T^k x \in \sum_{i=1}^{d} (x_i - \varepsilon, x_i + \varepsilon)\right\} \to 1, \quad \mu\text{-a.e.}x, \text{ as } n \to \infty. \tag{7}$$

Combining (6) and (7), we can easily obtain (5). □

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