



# A note on Sachdev–Ye–Kitaev like model without random coupling

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## Abstract

We study a description of the large  $N$  limit of the Sachdev–Ye–Kitaev (SYK) model in terms of quantum mechanics without quenched disorder. Instead of random couplings, we introduce massive scalar fields coupled to fermions, and study a small mass limit of the theory. We show that, under a certain condition, the correlation functions of fermions reproduce those of the SYK model with a temperature dependent coupling constant in the large  $N$  limit. We also discuss a supersymmetric generalization of our quantum mechanical model. As a byproduct, we develop an efficient way of estimating the large  $N$  behavior of correlators in the SYK model.

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## 1. Introduction and summary

The Sachdev–Ye–Kitaev (SYK) model [1,2] is a quantum mechanical model with random coupling constants, and has recently attracted much attention for its possible connection to black holes [3–29]. In particular, the model saturates the chaos bound proposed in [30] and therefore is expected to be dual to a black hole.

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The random couplings are called “quenched disorders,” and follow the Gaussian distribution. Then the disordered correlation function of operators  $\mathcal{O}_1, \dots, \mathcal{O}_n$  is defined by

$$\overline{\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle} \equiv \int dJ e^{-\alpha J^2} \frac{\int [d\psi] \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S[\psi, J]}}{\int [d\psi] e^{-S[\psi, J]}}, \quad (1)$$

where  $J$  is the disorder and  $\psi$  stands for quantum fields. The constant  $\alpha$  is a parameter characterizing the Gaussian distribution of the disorder.<sup>1</sup> In the above definition, we first compute the correlation function for fixed  $J$  and then take its average under the Gaussian distribution of  $J$ . Therefore it is not clear whether (1) plays the same role as usual correlators of quantum field theories in various contexts such as the gauge/gravity correspondence. In particular, if the SYK model describes a black hole, we need to understand the physical origin of the disorder. Therefore, it may be desired to promote the SYK model to an ordinary quantum mechanical model without disorders.

In this paper, we propose a quantum mechanical model without quenched disorders. In the model, we just promote the variables  $J$  into dynamical fields, which are free bosonic fields, i.e. harmonic oscillator, with small mass. We show that our model reproduces essentially the large  $N$  limit of the SYK model, however,  $\mathcal{J}$  should be replaced by a temperature dependent coupling  $\mathcal{J}_{eff}$ .<sup>2</sup> Indeed, the large  $N$  behavior of the two and four point correlators of the SYK model are reproduced by our model. This means that our model also saturates the chaos bound proposed in [30], and is consistent with the effective action with the Schwarzian derivative discussed in [13]. We note that the thermodynamic quantities including the free energy of our model are completely different from ones in the SYK model because of the almost massless bosons. The correlators with larger numbers of external fields are also reproduced in a sense explained below. The simple replacement by the harmonic oscillators behaves as the Gaussian quenched disorder due to some special properties of the large  $N$  limit of the SYK model.

An important previous work in this direction is [7] where harmonic oscillators are introduced and the random couplings  $\mathcal{J}$  are replaced by the momenta of the harmonic oscillators. We discuss the path-integral formulation of this model is essentially described by the action of our model.

We also propose a supersymmetric generalization of our model which reproduces the correlators of the supersymmetric generalization of the SYK model [24]. Here the super-partners of the almost massless bosons are decoupled from others in the large  $N$  limit.

One of the motivations for the study of the SYK model is its relevance to the black hole physics. If our model indeed has some application to the black hole physics, the almost massless bosons should be interpreted appropriately. Although it is unclear that our model describes black hole physics or not, we hope that our field theory analogue of the SYK model will be important for further study of the black hole physics and other areas.

The organization of this paper is the following. In section 2, we review the SYK model and also provide an efficient way of estimating the large  $N$  behavior of correlators. In section 3, we propose a quantum mechanical model without quenched disorders and argue that our model reproduces important properties of the SYK model in the large  $N$  limit. In section 4, we discuss a supersymmetric generalization.

<sup>1</sup> Here the integration measure  $dJ$  is normalized so that  $(1) = 1$ .

<sup>2</sup> Recently, in the paper [26], another interesting model which reproduces the large  $N$  limit of the SYK model was given.

## 2. SYK model

The SYK model is a quantum mechanics of  $N$  Majorana fermions,  $\psi_i$  for  $i = 1, \dots, N$ , with quenched disorders. The Hamiltonian of the model<sup>3</sup> is given by

$$H = \sum_{1 \leq i < j < k < \ell \leq N} J_{ijkl} \psi_i \psi_j \psi_k \psi_\ell, \tag{2}$$

where  $J_{ijkl}$  are disorders following the Gaussian distribution such that

$$\overline{J_{ijkl}} = 0, \quad \overline{(J_{ijkl})^2} = \frac{3!}{N^3} \mathcal{J}^2. \tag{3}$$

The constant  $\mathcal{J}$  has dimension of mass, and therefore becomes large in the infrared. Below we will work in Euclidean space. The Lagrangian corresponding to the Hamiltonian (2) is written as

$$L(\psi; J) = -\frac{1}{2} \sum_{i=1}^N \psi_i \frac{d}{d\tau} \psi_i - H, \tag{4}$$

and the disordered correlation function of  $n$  fermions is given by

$$\begin{aligned} & \overline{\langle \psi_{i_1}(\tau_1) \cdots \psi_{i_n}(\tau_n) \rangle} \\ &= \int \left( \prod_{i,j,k,l} dJ_{ijkl} e^{-\alpha (J_{ijkl})^2} \right) \frac{\int [d\psi] \psi_{i_1}(\tau_1) \cdots \psi_{i_n}(\tau_n) e^{-\int d\tau L}}{\int [d\psi] e^{-\int d\tau L}}, \end{aligned} \tag{5}$$

where  $\alpha \equiv N^3/12\mathcal{J}^2$ . Explicit computations of two and four point functions of this model were discussed in [1,6,13].

Let us illustrate how to read off the large  $N$  behavior of correlation functions in the SYK model. As in [6,13], we focus on correlators of the following form<sup>4</sup>:

$$\sum_{i_1, i_2, \dots, i_n=1}^N \overline{\langle \psi_{i_1}(\tau_1) \psi_{i_1}(\tau_2) \cdots \psi_{i_n}(\tau_{2n-1}) \psi_{i_n}(\tau_{2n}) \rangle}. \tag{6}$$

Since the subscripts of the fermions are in pairs, it is useful to divide external lines in Feynman diagrams into the corresponding pairs. We then connect external lines in each such pair with a “dot” (Fig. 1). While this operation obscures the locations of the external fermions in time, it makes it easy to read off the large  $N$  behavior of the contribution from each diagram. In the rest of this paper, a “dot” always stands for two external fermions in a pair in this sense. After the integration over the disorders  $J_{ijkl}$ , the fermion fields  $\psi_i$  have only the eight-point interaction of the form

$$\begin{aligned} & \frac{3! \mathcal{J}^2}{N^3} \int d\tau_1 d\tau_2 \sum_{1 \leq i_1 < i_2 < i_3 < i_4 < N} \psi_{i_1}(\tau_1) \psi_{i_1}(\tau_2) \\ & \times \psi_{i_2}(\tau_1) \psi_{i_2}(\tau_2) \psi_{i_3}(\tau_1) \psi_{i_3}(\tau_2) \psi_{i_4}(\tau_1) \psi_{i_4}(\tau_2). \end{aligned} \tag{7}$$

<sup>3</sup> This can be considered as a special case with  $q = 4$  of the generalized SYK model [13] where the number of the fermions in the Hamiltonian is  $q$ . Our considerations in this paper can be easily generalized to other cases of  $q$  although we do not do it explicitly.

<sup>4</sup> We will not consider other types of correlators in this paper although they might be important.

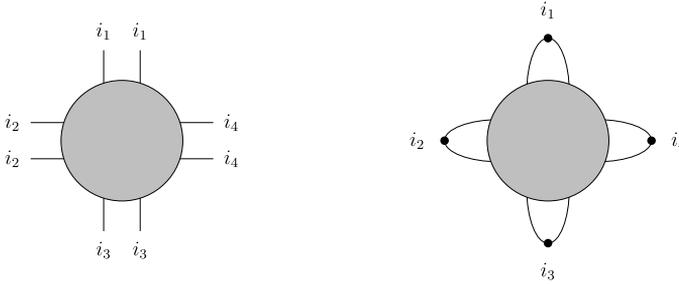


Fig. 1. Two expressions for external lines in the diagram. The shaded region is arbitrary. Left: The conventional one. Right: Our new expression for the same diagram in terms of dots. Each dot expresses two external fermions in a pair in the sense explained in the main text. Since we take the sum over  $i_k$ , we usually omit the indices in the diagram.

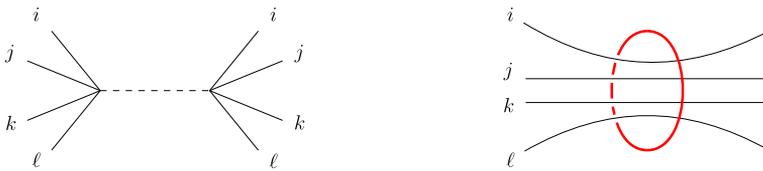


Fig. 2. Two expressions for an interaction vertex. Left: The conventional one. The dashed line stands for the disorder average associated with  $J_{ijkl}$ . Right: Another expression for the left picture in terms of a “bundle.” The red circle in the middle stands for a bundle corresponding to the disorder average in the left picture. We say two fermion lines are connected to each other when they are involved in the same bundle.

We denote this interaction vertex by a “bundle” of four lines to which the indices  $i_1, i_2, i_3, i_4$  are assigned (Fig. 2). This expression again obscures the locations of fermions in time, but enables us to read off the large  $N$  behavior easily.

In terms of the above “dot” and “bundle,” any Feynman diagram can be expressed as a collection of circles with dots and bundles attached to them, where each circle carries  $N$  degrees of freedom. Note that there are neither branch points nor end points of lines in this expression. In the rest of this paper, we use this new expression for diagrams. We will denote the number of circles by  $L$ , and that of bundles by  $V$ . Note that  $L$  needs not coincide with the number of fermion loops in the usual expression for the diagram, while  $V$  equals to the number of vertices in the usual expression. Then, the contribution from the diagram is of  $\mathcal{O}(N^{L-3V})$ . Note that this does not depend on the number of the dots in the diagram (i.e. (half) the number of external lines in the usual expression for the diagram).

Now, we can show that for connected diagrams the leading order contribution is always of  $\mathcal{O}(N)$  irrespective to the number of dots in the diagram. Indeed, for  $V = 0$ , the only connected diagram is a circle with dots, which has a summation over only an index and therefore is of  $\mathcal{O}(N)$ . On the other hand, from  $k$  diagrams of  $\mathcal{O}(N)$ , we can generate a new connected diagram by bundling circles.<sup>5</sup> The bundle is associated with an extra factor of  $\mathcal{J}^2/N^3$ , and therefore the new diagram is of  $\mathcal{O}(N^{k-3})$ . Since  $1 \leq k \leq 4$ , the leading contribution is again of  $\mathcal{O}(N)$ . Since any connected diagram can be generated by repeating this procedure, we see that the leading contribution is always of  $\mathcal{O}(N)$  for connected diagrams. This also implies that the leading order

<sup>5</sup> This operation corresponds to inserting an additional vertex in the usual expression for the diagram.



Fig. 3. “Untying” a bundle corresponds to replacing the left picture with right.

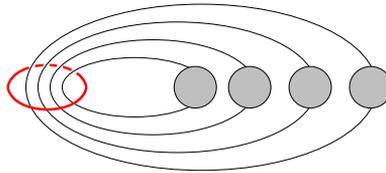


Fig. 4. A diagram whose contribution is of  $\mathcal{O}(N)$  splits into four disconnected parts when a bundle in it is untied. Here the shaded regions are arbitrary.

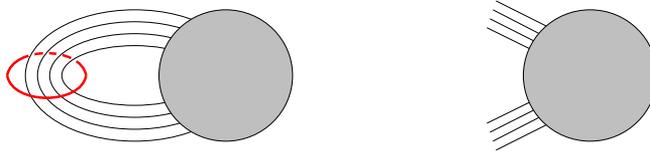


Fig. 5. “Cutting” a bundle in the diagram implies replacing the left picture with the right. This corresponds to cutting the dashed line in Fig. 2. Note that this “cutting” is different from “untying” a bundle discussed in Fig. 3.

(connected) diagram has the following property: if we “untie” any bundle from the diagram, the diagram splits into four disconnected parts (Fig. 4).

We conclude this section by comparing the disordered correlator (5) with the following quantity<sup>6</sup>:

$$\frac{\int dJ_{ijkl} [d\psi] \psi_{i_1}(\tau_1) \cdots \psi_{i_n}(\tau_n) e^{-\alpha(J_{ijkl})^2 - \int d\tau L}}{\int dJ_{ijkl} [d\psi] e^{-\alpha(J_{ijkl})^2 - \int d\tau L}}, \tag{8}$$

where  $J_{ijkl}$  is regarded as a constant auxiliary field instead of a random coupling. The only difference between (5) and (8) is the notion of vacuum bubbles. To see this, let us consider connected diagrams with the following property: if we “cut” enough number of bundles in the diagram, as in Fig. 5, then the resulting diagram has at least one connected component without dots in it (Fig. 6).<sup>7</sup> We call such a connected component without dots a “bubble” because it is regarded as a vacuum bubble in the computation of (5). Then we denote by  $n_B$  the (minimum) number of bundles we should cut to separate as many bubbles as possible from the diagram. Any diagram producing a bubble in the above cutting operation does not contribute to the disordered correlator (5) while it contributes to the quantity (8).<sup>8</sup> However, we will see below that this difference between (5) and (8) is sub-leading in the large  $N$  limit.

<sup>6</sup> This consideration was essentially given in [7].

<sup>7</sup> Recall here that a “dot” stands for two external fermions in a pair in the sense explained in Fig. 1, and also that “cutting” a bundle means cutting the dashed line (i.e., a propagator of  $\phi$ ) in the sense described in Fig. 2.

<sup>8</sup> Note that this bubble diagram corresponds to an effective vertex or an effective propagator of  $\phi$ .

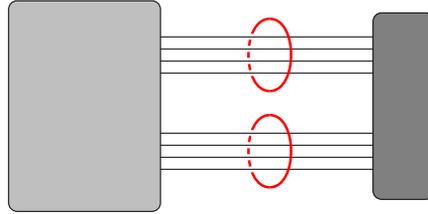


Fig. 6. A diagram which produces a “bubble” when enough number of bundles are cut. The right shaded region contains no “dot” while the left shaded region is arbitrary.

To see this, let us consider a diagram containing bubbles. Then  $n_B$  is greater than one because of the anti-commutativity of the fermions. Note that every line in one of the  $n_B$  bundles is connected, through a bubble, to a line in another one of the  $n_B$  bundles. This means that, if we “untie” all the  $n_B$  bundles, the diagram splits into at most  $2n_B$  disconnected parts. Since each such disconnected part is at most of  $\mathcal{O}(N)$ , the original diagram is at most of order  $N^{2n_B-3n_B} = N^{-n_B} \leq N^{-2}$ . Therefore, in comparison to the  $\mathcal{O}(N)$  leading contribution, contributions from diagrams containing a bubble are sub-leading at least by the factor  $N^{-3}$ . This implies that (5) and (8) are identical at the leading order of  $N$ .

Note, however, that the constant field  $J_{ijkl}$  in (8) induces non-local interactions and therefore (8) cannot be interpreted as a correlator of a one-dimensional local quantum field theory. In the next section, we propose another quantum mechanical model to overcome this difficulty.

### 3. Our model

Let us consider the quantum mechanics of  $\phi_{ijkl}(\tau)$  and  $\psi_i(\tau)$  described by the following action:

$$\tilde{S}[\psi, \phi] = \int d\tau \sum_{i < j < k < \ell} \frac{1}{2\epsilon} \left( \left( \frac{d\phi_{ijkl}}{d\tau} \right)^2 + m^2(\phi_{ijkl})^2 \right) + \int d\tau L(\psi, J = \phi(\tau)), \quad (9)$$

where  $m$  is a mass parameter,  $\epsilon/m^3$  is a dimensionless constant and  $L(\psi, J)$  is the Lagrangian defined in (4) where the constant  $J$  is replaced by the dynamical bosonic fields (harmonic oscillators)  $\phi(\tau)$ . We define the correlation function in this theory as

$$\langle \psi_{i_1}(\tau_1) \cdots \psi_{i_n}(\tau_n) \rangle \equiv \frac{\int [d\phi][d\psi] \psi_{i_1}(\tau_1) \cdots \psi_{i_n}(\tau_n) e^{-\tilde{S}[\psi, \phi]}}{\int [d\phi][d\psi] e^{-\tilde{S}[\psi, \phi]}}, \quad (10)$$

and only consider the combinations of correlation functions of the form in (6). Below we will show that, for a range of  $\epsilon$  and  $m$ , this correlation function almost agrees with (5) in the large  $N$  limit.

Here we assume the small mass limit

$$m \ll |\omega_i|, \quad (11)$$

where  $\omega_i$  is a momentum (or frequency) of an external fermion line, and the large  $N$  limit is taken before this limit.

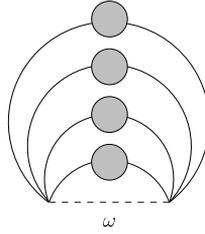


Fig. 7. Another expression for the diagram in Fig. 4. Here we explicitly show a propagator of  $\phi$  in terms of the dashed line. The shaded regions are arbitrary.

### 3.1. Correlation function

As we have explicitly seen in the previous section, the relevant diagrams in the large  $N$  limit in this model are the same as those in the SYK model.<sup>9</sup> In particular, the leading contributions from connected diagrams are of  $\mathcal{O}(N)$ , and an interaction vertex in them can always be expressed as in Fig. 7. We here denote by  $\omega$  the momentum (or frequency) of the  $\phi$  propagator shown in Fig. 7, and consider the integral over  $\omega$ . Such an integral does not appear when every fermion line involved in the bundle is connected to a dot, because in that case  $\omega$  is fixed by external momenta. We discuss such a case in the next subsection. Clearly, such a diagram without the integration over  $\omega$  contains  $2k$  external lines for  $k \geq 4$ . Thus, for two, four and six point correlators,  $\omega$  is not fixed by external momenta.

Let us first focus on two, four and six point correlators. Then the contribution of the diagram shown in Fig. 7 contains an integral over  $\omega$ . We first split the integrand into the  $\phi$  propagator,  $\frac{\epsilon}{\omega^2+m^2}$ , and the other part,  $A(\omega, \omega_i)$ , so that the contribution from the diagram is written as

$$\epsilon \int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2 + m^2} A(\omega, \omega_i) . \tag{12}$$

Then we introduce an auxiliary intermediate scale  $\Lambda$ ,

$$m \ll \Lambda \ll |\omega_i|, \tag{13}$$

and splits the integral (12) into two parts as

$$2\epsilon \int_0^{\Lambda} d\omega \frac{1}{\omega^2 + m^2} A(\omega, \omega_i) + 2\epsilon \int_{\Lambda}^{\infty} d\omega \frac{1}{\omega^2 + m^2} A(\omega, \omega_i) . \tag{14}$$

The first term of (14) is evaluated as<sup>10</sup>

<sup>9</sup> To be precise, the discussion in the previous section was on the large  $N$  equivalence of (5) and (8). While (8) and (10) are different, the same set of diagrams is relevant in the large  $N$  limit. The difference appears in the contribution from each diagram, as discussed below.

<sup>10</sup> The integration of the high momentum modes is not divergent, but suppressed in one dimensional system.

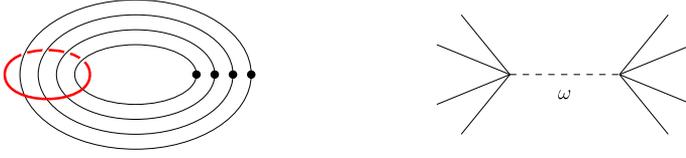


Fig. 8. A diagram for which the momentum  $\omega$  of the  $\phi$  propagator is fixed by external momenta of fermions. The left picture is in terms of dots and a bundle while the right one is an expression for the same diagram in terms of external lines and vertices.

$$\begin{aligned}
 2\epsilon \int_0^\Lambda d\omega \frac{1}{\omega^2 + m^2} A(\omega, \omega_i) &\approx 2\epsilon \left( \int_0^\infty d\omega \frac{1}{\omega^2 + m^2} \right) A(\omega = 0, \omega_i) \\
 &= \frac{\pi\epsilon}{m} A(\omega = 0, \omega_i),
 \end{aligned}
 \tag{15}$$

where we neglect  $\mathcal{O}(\frac{m}{\Lambda})$  and  $\mathcal{O}(\frac{m}{|\omega_i|})$  suppressed terms, and the second term is also negligible because of an  $\mathcal{O}(\frac{m}{\Lambda})$  factor.<sup>11</sup>

This result implies that, by identifying

$$\epsilon = \frac{3!}{\pi} \frac{m}{N^3} \mathcal{J}^2,
 \tag{16}$$

the two, four and six point correlators of our model reproduce those of the SYK model in the large  $N$  limit. In particular, for

$$m \ll |\omega_i| \ll \mathcal{J},
 \tag{17}$$

the two and four point functions are well-described by the effective action with the Schwarzian derivative proposed in [13].

### 3.2. Eight-point correlation function

As we have seen, for the eight and more-than-eight point correlation functions, there are connected diagrams for which  $\omega$  is fixed by external momenta. For example, let us consider the diagram depicted in (Fig. 8). For this diagram, the contribution of the  $\phi$  propagator is

$$\begin{aligned}
 &\int d\omega \frac{\epsilon}{\omega^2 + m^2} \delta(\sum_j \omega_j^L + \omega) \delta(\sum_j \omega_j^R - \omega) \\
 &\sim \frac{m}{(\sum_j \omega_j^L)^2 + m^2} \frac{1}{N^3} \mathcal{J}^2 \delta(\sum_j \omega_j^L + \sum_j \omega_j^R)
 \end{aligned}
 \tag{18}$$

where  $\sum_j \omega_j^{L(R)}$  is the sum of the momenta of the fermions attached to the left (right) edge of the  $\phi$  line. On the other hand, in the SYK model it is

$$3! \frac{\mathcal{J}^2}{N^3} \delta(\sum_j \omega_j^L) \delta(\sum_j \omega_j^R).
 \tag{19}$$

<sup>11</sup> Here, we have used that  $A(\omega, \omega_i)$  and  $\frac{\partial}{\partial \omega} A(\omega, \omega_i)|_{\omega=0}$  do not diverge. This follows from the fact that our model is one-dimensional system.

Thus, the connected eight point function in the large  $N$  limit in our model is different from that in the SYK model. However, the difference exists only for  $(\sum_j \omega_j^L)^2 = \mathcal{O}(m)$ , which is realized by tuning the external momenta. Note that a Euclidean time correlator can be obtained by the Fourier transform of the momentum correlator, which includes the integration over  $\sum_j \omega_j^L$ . Therefore, in a generic Euclidean time correlator, (18) is approximated by

$$\int d\omega \frac{\epsilon}{\omega^2 + m^2} \delta(\sum_j \omega_j^L) \delta(\sum_j \omega_j^R), \tag{20}$$

which coincides with the one in the SYK model. In this sense, the correlation functions of our model is almost the same as those of the SYK model.

We can use the Feynman rules in position space, instead of momentum space. Then, the propagator for  $\phi$  in the  $m \rightarrow 0$  limit becomes a constant:

$$\frac{\epsilon}{m} e^{-m|\tau|} \rightarrow \frac{3!}{\pi} \frac{1}{N^3} \mathcal{J}^2. \tag{21}$$

With this propagator, we can easily see that the path-integral of the  $\phi$  is equivalent to the disorder average. However, this propagator (21) is not valid if the integration over  $\tau$  for  $|\tau| = \mathcal{O}(1/m)$  does not vanish in the limit. As we have seen by using the propagator in momentum space, we expect that the naive limit (21) is valid for the correlation function of the form in (6).

So far, we have considered the zero temperature result. As we will see below, at finite temperature, our model is different from the SYK model with the constant random coupling  $\mathcal{J}$ . On the other hand, one could think that only fermions are excited by finite temperature while bosons are at zero temperature, because the bosons and fermions are weakly coupled in the large  $N$  limit. Moreover, the number of the bosons is of  $\mathcal{O}(N^4)$  and much larger than that of the fermions, and therefore one could neglect the finite temperature effects on the bosons for a sufficiently long time. If this is the case, the state corresponding to such a situation in our model will be the same as the thermal state at the temperature in the SYK model with  $\mathcal{J}$ . This possibility would be interesting although we will not study it further in this paper.

### 3.3. Finite temperature

Let us consider our model at a finite temperature  $1/\beta$ . Here we take the small  $m$  limit, and therefore  $\beta m \ll 1$  and  $m \ll |\omega_i|$ . Since the Euclidean time direction is compactified, the momentum integration in (12) is now replaced to a discrete sum:

$$\epsilon \frac{2\pi}{\beta} \sum_n \frac{1}{(\omega_n)^2 + m^2} A(\omega_n, \omega_i), \tag{22}$$

where  $\omega_n = 2\pi n/\beta$ . The dominant contribution in the small mass limit is

$$\epsilon \frac{2\pi}{\beta} \sum_n \frac{1}{(\omega_n)^2 + m^2} A(\omega_n, \omega_i) \approx \frac{3!}{N^3} (\mathcal{J}_{eff})^2 A(0, \omega_i), \tag{23}$$

where  $\mathcal{J}_{eff}$  is a function of  $\beta m$ . Here we note that the fermions have half-integer momenta at finite temperature, hence there will be no singular behavior in  $A(\omega, \omega_i)$  near  $\omega = 0$  even when there is no external lines. For  $\beta m \ll 1$ ,  $\mathcal{J}_{eff}$  is easily determined as

$$\mathcal{J}_{eff} = \sqrt{\frac{2}{\beta m}} \mathcal{J}, \tag{24}$$

where only the zero mode, i.e.  $n = 0$ , contributes in (22).

Thus, our model at the temperature  $1/\beta$  reproduces the SYK model at the same temperature with the coupling constant  $\mathcal{J}$  replaced by  $\mathcal{J}_{eff}$ . Note that, unlike  $\mathcal{J}$ , the effective coupling  $\mathcal{J}_{eff}$  depends on the temperature. This difference between our model and the SYK model is originated from the almost massless bosons which are highly excited at finite temperature. The conformal limit,  $\beta\mathcal{J} \gg 1$ , in the SYK model corresponds to  $\beta\mathcal{J}_{eff} \gg 1$  in our model at the finite temperature. On the other hand, as we have ignored the  $\mathcal{O}(m)$  suppressed terms, we need another constraint

$$(1 \gg) \frac{1}{\beta\mathcal{J}_{eff}} \gg m\beta, \quad (25)$$

where  $\beta$  can be replaced by any other typical finite scale in the correlator. These two inequalities can be satisfied simultaneously by choosing  $\mathcal{J}$  appropriately. Therefore, four point correlators of our model at finite temperature reproduce those of the SYK model with the temperature-dependent coupling  $\mathcal{J}_{eff}$ , in the large  $N$  and small  $m$  limit. In particular, with (25), the two and four point functions are well-described by the effective action with the Schwarzian derivative proposed in [13].

Let us comment on the chaos bound proposed in [30], which is known to be saturated by the SYK model [1,6,13]. This saturation of the chaos bound can be seen by studying analytic continuations of the Euclidean correlators at finite temperature. The Euclidean correlators at finite temperature in our model is the same, in the small mass limit, as the ones in the SYK model with  $\mathcal{J}$  replaced by  $\mathcal{J}_{eff}$ . Thus their analytic continuations are also the same for these two models. This means that, at the finite temperature (such that  $\beta m \ll 1$ ), the saturation of the chaos bound can be seen in our model for a sufficiently long time.

### 3.4. Relations to other models

Finally, let us comment on relations to other models without disorder [7].<sup>12</sup> First, schematically, the Lagrangian in our model is

$$\frac{1}{2\epsilon}((\dot{\phi})^2 + m^2\phi^2) + \phi\psi^4, \quad (26)$$

which is equivalent to

$$\frac{1}{2\epsilon}(-C^2 + 2\dot{\phi}C + m^2\phi^2) + \phi\psi^4, \quad (27)$$

where  $C$  is an auxiliary field. Now we integrate out  $\phi$  first, then we obtain the following Lagrangian:

$$\frac{1}{2\epsilon} \left( -C^2 - \frac{1}{m^2} \dot{C}^2 \right) - \frac{1}{m^2} \dot{C} \psi^4, \quad (28)$$

where we have neglected  $(\psi)^4(\psi)^4$  term because of the fermion anti-symmetry. In terms of  $\tilde{\phi} = iC/m$ , this Lagrangian is written as

$$\frac{1}{2\epsilon}((\tilde{\phi})^2 + m^2\tilde{\phi}^2) - i\frac{1}{m}\tilde{\phi}\psi^4, \quad (29)$$

<sup>12</sup> Note that we use a notation for fields and parameters which is different from the one used in [7].

where the factor  $i$  is necessary for the corresponding Lorentzian theory to be unitary. Note that we have ignored the zero modes of  $\phi$  and  $C$  in the Lagrangians although they are important at finite temperature. Thus, the derivative coupling model (29) is equivalent to our model except the absence of the zero mode. Indeed, the propagator of  $\dot{\phi}$  is

$$-\frac{1}{m^2} \langle \dot{\phi}(\tau) \dot{\phi}(0) \rangle = \frac{1}{m^2} \frac{\partial^2}{\partial^2 \tau} \langle \phi(\tau) \phi(0) \rangle = \frac{\epsilon}{m} e^{-m|\tau|} \left( 1 - 2 \frac{1}{m} \delta(\tau) \right), \tag{30}$$

which is the same as the propagator of  $\phi$  except an extra delta-function term. When computing correlators of  $\psi$ , this extra term vanishes because it is accompanied with  $(\psi^4(\tau))(\psi^4(\tau))$  which is vanishing because of the anti-commutativity of the fermions. At finite temperature, the model described by the Lagrangian (29) is not equivalent to the SYK model because there is no zero mode contributions in the propagator of  $\dot{\phi}$ .

In [7], two quantum mechanical models without quenched disorders are briefly discussed.<sup>13</sup> In particular, the second model is obtained by replacing the random coupling  $\mathcal{J}$  of the SYK model with the momenta of harmonic oscillators. This model is essentially equivalent to the model described by the above Lagrangian (29). Here we note that the path-integral formulation of a theory is not straightforwardly obtained when the Hamiltonian has interaction terms including canonical momenta.<sup>14</sup> On the other hand, in the operator formalism, an interchange between  $\hat{x}$  and  $\hat{p}$  does not change the theory except  $[\hat{x}, \hat{p}] = i \rightarrow [\hat{x}, \hat{p}] = -i$ . Therefore, the second model in [7] is essentially equivalent to our model.

Let us also consider the first model discussed in Sec. 2.3 of [7]. The model is described by the following action:

$$\int_0^\beta d\tau \left( \dot{\phi} C + \frac{\alpha}{\beta} \phi^2 + \phi \psi^4 \right), \tag{32}$$

where we have introduced  $\beta$  as an IR regulator or an inverse temperature. Because only the zero mode of  $\phi$  remains after integrating out  $C$ , this model is equivalent to the SYK model. However, the Lagrangian explicitly depends on  $\beta$ , and moreover  $\alpha/\beta$  vanishes in the zero temperature limit since  $\alpha = N^3/12\mathcal{J}^2$  should be fixed. Thus, this action is not suitable for an action of a quantum theory.

On the other hand, the Lagrangian (27) of our model, at a finite temperature such that  $m\beta \ll 1$ , will be

$$\frac{1}{2\epsilon} \left( 2\dot{\phi} C + m^2 \phi^2 \right) + \phi \psi^4 = \dot{\phi} \tilde{C} + m\alpha \phi^2 + \phi \psi^4, \tag{33}$$

where  $\tilde{C} \equiv C/\epsilon$ . This is of the same form as (32) but does not explicitly depend on  $\beta$ . This reflects the fact that the corresponding SYK model has  $\beta$ -dependent coupling constant  $\mathcal{J}_{eff}$  (i.e.  $\alpha_{eff} = \beta m \alpha$ ).

<sup>13</sup> See the last paragraph of Sec. 2.3 of [7].

<sup>14</sup> We also note that a correlator in the path-integral formalism corresponds to a v.e.v. of a time ordered product operators where derivatives are taken outside the correlator (sometimes this is called  $\mathbf{T}^*$ -product). For example, for the harmonic oscillator with  $H = (\hat{x}^2 + \hat{p}^2)/2$  where  $\hat{p}(t) = \dot{\hat{x}}(t)$ , we see that

$$\langle \dot{\hat{x}}(t) \dot{\hat{x}}(0) \rangle = -\frac{\partial^2}{\partial^2 t} \langle 0 | \mathbf{T}(\hat{x}(t) \hat{x}(0)) | 0 \rangle \neq \langle 0 | \mathbf{T}(\hat{p}(t) \hat{p}(0)) | 0 \rangle. \tag{31}$$

### 4. SUSY generalization

In [24], a supersymmetric generalization of the SYK model is discussed. In the superspace representation, the Lagrangian of the  $\mathcal{N} = 1$  model is

$$L_{SUSY} = \int d\theta \left( \frac{1}{2} \Psi^i D_\theta \Psi^i + i C_{ijk} \Psi^i \Psi^j \Psi^k \right), \tag{34}$$

where  $\Psi^i = \psi(\tau) + \theta b(\tau)$ ,  $D_\theta = \partial_\theta + \theta \partial_\tau$  and  $C_{ijk}$  is a Gaussian random coupling with

$$\overline{C_{ijk}^2} = \frac{2J}{N^2}, \quad \overline{C_{ijk}} = 0. \tag{35}$$

We now promote  $C_{ijk}$  to a dynamical field as follows. First, we introduce the following bosonic and fermionic superfields:  $X_{ijk} = C_{ijk}(\tau) + \theta \eta_{ijk}(\tau)$  and  $\Xi_{ijk} = \xi_{ijk}(\tau) + \theta F_{ijk}(\tau)$ . Then, the Lagrangian of our model is  $L_{SUSY} + L_X$  where

$$L_X = \int d\theta \frac{1}{2\epsilon} \left( D_\theta^2 X_{ijk} D_\theta X_{ijk} + \Xi_{ijk} D_\theta \Xi_{ijk} + m \Xi_{ijk} X_{ijk} \right). \tag{36}$$

Note that we need to introduce the fermionic partners of  $C$ . In terms of the superfield  $\Psi^i$ , the action (34) is of the same form as the action of the  $q = 3$  SYK model,<sup>15</sup> a similar discussion on the large  $N$  limit as in the previous section may be available. Therefore we expect that diagrams including “bubbles” are sub-leading in the large  $N$  limit, as in the bosonic case.

Now we consider the massless limit of the propagator at zero temperature of this SUSY model. We take  $\epsilon \sim mJ/N^2$  so that the propagators of the bosonic components  $C$  become constants as we have seen before in (21). On the other hand, for the fermions  $\eta$ , the propagator in position space becomes

$$\epsilon \frac{1}{2} \text{sgn}(\tau), \tag{37}$$

which is of  $\mathcal{O}(m)$  and therefore suppressed compared with the bosonic propagator in the massless limit. This difference between bosons and fermions does not contradict with supersymmetry because the fermion propagator is related by supersymmetry to the derivative of the bosonic one, i.e.,

$$\frac{\partial}{\partial \tau} \frac{\epsilon}{m} e^{-m|\tau|} = \frac{\partial}{\partial \tau} \left( \frac{\epsilon}{m} - \epsilon|\tau| + \dots \right), \tag{38}$$

where only the sub-leading part in the massless limit remains. Therefore, in the massless limit, the fermionic partner of  $C$  decouples from the  $\Psi$ . Moreover, since  $X$  and  $\Xi$  are decoupled from each other in the massless limit, the path-integrations over the dynamical fields  $X, \Xi$  reduce to Gaussian random couplings as in the non-SUSY model discussed in the previous sections. Therefore, our SUSY model described by  $L_{SUSY} + L_X$  is essentially equivalent to the original SUSY generalization of the SYK model in the large  $N$  and massless limit.

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<sup>15</sup> Here  $q$  stands for the number fermions involved in the interaction term, as in [13].

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