



$\mathcal{O}(a)$ improvement of 2D $\mathcal{N} = (2, 2)$ lattice SYM theory

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Abstract

We perform a tree-level $\mathcal{O}(a)$ improvement of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang–Mills theory on the lattice, motivated by the fast convergence in numerical simulations. The improvement respects an exact supersymmetry Q which is needed for obtaining the correct continuum limit without a parameter fine tuning. The improved lattice action is given within a milder locality condition in which the interactions are decaying as the exponential of the distance on the lattice. We also prove that the path-integral measure is invariant under the improved Q -transformation.

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1. Introduction

Nonperturbative dynamics of supersymmetric gauge theories is of great interest for various reasons. However many interesting problems are out of reach of the current analytic understandings. For example, while properties of the vacua can be analyzed very precisely by using holomorphy (e.g. [1,2]), the dynamics of excitations is hard to study except for specific objects saturating the so-called BPS bound. In the context of the gauge/gravity duality, gauge theory dual to weakly coupled gravity is strongly coupled and furthermore the quantities of interests are not necessarily protected by supersymmetry. Numerical simulation based on lattice gauge theory is considered as a promising approach to such problems.

Historically, lattice formulation of supersymmetry brushed off many attempts. The problem was the *parameter fine tuning problem*; because a lattice breaks the infinitesimal translation, which is a part of the supersymmetry algebra, the supersymmetry cannot be preserved completely on a lattice. Then, even if the supersymmetry is restored in a naive continuum limit at the tree level, radiative corrections break it in general.

For two-dimensional theories, the parameter fine tuning problems can be circumvented by keeping a part of the supersymmetry algebra (one or two supercharges and $U(1)$ or $SU(2)$ R-symmetry) at discretized level [3–20]. Encouraged by this development, several groups have been trying lattice simulations of two-dimensional super Yang–Mills theories [21–25],¹ including the maximally supersymmetric theory relevant for the gauge/gravity duality [26–29]. Furthermore it has been pointed out that such two-dimensional theories can be used to construct four-dimensional super Yang–Mills theory [37,38] by utilizing the Myers effect with which two spatial dimensions are encoded in matrix degrees of freedom [39,40]. The two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang–Mills theory on an arbitrarily discretized Riemann surface is developed in [41–45].

At present, there is a consensus among the community of researchers that these regularization schemes can work in principle. However, in practice — especially, in order to perform precision measurements at large volume and large N , with currently available numerical resources — it is desirable to have improved regularizations which converge to the continuum limit faster. In the lattice QCD community, such improvement is known as the Symanzik improvement program [46,47]. Errors arising on discretization of a continuum system by lattice are of the order of the lattice spacing $\mathcal{O}(a)$ in general. The program makes the errors reduced to higher orders in a .

In this paper, we consider the improvement of the lattice action of two-dimensional $\mathcal{N} = (2, 2)$ super Yang–Mills theory proposed by one of the authors (F. S.) [7]. This is a technically nontrivial subject, because the improvement term must preserve the exact supercharge and R-symmetry that are relevant to realize the theory flowing to the desired continuum theory without any parameter fine tuning. As a first step, we consider $\mathcal{O}(a)$ improvement at the tree level. Because the ultraviolet divergence is mild due to the low-dimensionality and supersymmetry, it is sufficient to consider the tree and one-loop levels in this setting.² The one-loop calculation will be reported in the forthcoming publication.

¹ Another approach with a parameter fine tuning can be found in Refs. [30–32]. Also, for four-dimensional $\mathcal{N} = 1$ super Yang–Mills theory, see [33–36].

² As a comparison, the tree-level improvement is enough to achieve the significant acceleration of the simulation in the case of the $(0+1)$ -dimensional theory [56].

The previous unimproved action [7] has been constructed in the three steps:

- Write the action S in the continuum theory in a Q -exact form, $S = Q \frac{1}{2g^2} \int d^2x \Xi(x)$, by using one of the supercharges Q .
- Construct a lattice counterpart of Q -transformation, Q_{lat} , which is an exact symmetry at the regularized level.
- Define a lattice action in the Q -exact form as $S_{\text{lat}} = Q_{\text{lat}} \frac{a^2}{2g^2} \sum_x \Xi_{\text{lat}}(x)$ where Ξ_{lat} is a lattice counterpart of $\Xi(x)$.

Note that the path-integral measure has to be taken in a Q -invariant way as well, and the natural measure is actually Q -invariant. The created lattice action reproduces the continuum one with $\mathcal{O}(a)$ corrections because the lattice supersymmetry transformation generated by Q_{lat} and the term Ξ_{lat} are different from the continuum ones at the order.

For $\mathcal{O}(a)$ improvement, we need to improve both of Q_{lat} and Ξ_{lat} , keeping the path-integral measure Q -invariant. Note also that we have to make sure that an extra $\mathcal{O}(a)$ correction does not appear from the measure. The improvements will be done by lattice operators \mathcal{R}_μ and \mathcal{R}_{12} for Q and Ξ , respectively, which include a kind of Wilson terms. We will find that the improved lattice theory satisfies a milder locality condition known as the exponential locality. Such condition is accepted in obtaining a local continuum theory within the universality hypothesis [48,49,55].

This paper is organized as follows. In Sec. 2 we review the continuum theory and introduce the unimproved lattice action. On the way we clarify the origin of $\mathcal{O}(a)$ deviation from the continuum theory to illuminate a strategy for the improvement. Sec. 3 and Sec. 4 are the main parts of this paper. In Sec. 3, clarifying existence conditions of the consistent lattice Q -transformation by a lemma, we improve the lattice action with keeping the Q -exact form. In Sec. 4, we summarize the improved theory and show that it is free from the doubling problem thanks to $\mathcal{O}(a^3)$ Wilson terms. The Q -invariant measure is then consistently defined in any finite physical volume. Sec. 5 is devoted to summarize the results and discuss future directions concerning this project. A proof of the lemma is given in Appendix A, and the factors \mathcal{R}_μ and \mathcal{R}_{12} are presented with their locality properties in Appendix B. For convenience in the numerical simulation, we present the explicit form of the improved lattice action in Appendix C. Appendix D gives a computational detail related to the Q -invariance of the path-integral measure.

2. Original lattice formulation

In this section, we briefly review two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang–Mills theory and its lattice formulation with one of the four supercharges of the theory exactly preserved. As a preparation to the $\mathcal{O}(a)$ improvement, errors arising in the discretization are also discussed.

2.1. Continuum theory

We start with $\mathcal{N} = (2, 2)$ supersymmetric Yang–Mills theory on two-dimensional Euclidean space, whose field contents are gauge fields A_μ , adjoint scalars $\phi, \bar{\phi}$, gaugino fields $\lambda, \bar{\lambda}$ and an auxiliary field D which satisfies $D^\dagger = -D$. We assume that the gauge group is $SU(N)$ and all fields are matrix-valued functions which are expanded by a basis of $N \times N$ traceless hermitian matrices T_α ($\alpha = 1, \dots, N^2 - 1$) normalized as $\text{tr}(T_\alpha T_\beta) = \delta_{\alpha\beta}$.

The action is expressed as

$$S_{\text{cont}} = \frac{1}{2g^2} \int d^2x \text{tr} \left\{ \frac{1}{2} \sum_{\mu, \nu=1}^2 F_{\mu\nu}^2 + \sum_{\mu=1}^2 D_\mu \phi D_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 - D^2 + 2\bar{\lambda}_R (D_1 - iD_2) \lambda_R + 2\bar{\lambda}_L (D_1 + iD_2) \lambda_L + 2\bar{\lambda}_R [\bar{\phi}, \lambda_L] + 2\bar{\lambda}_L [\phi, \lambda_R] \right\}, \quad (2.1)$$

where g is a coupling constant, $\mu, \nu = 1, 2$, and the subscripts L and R denote the spinor indices of λ . The theory is obtained from the four-dimensional $\mathcal{N} = 1$ supersymmetric Yang–Mills theory by the dimensional reduction. In fact, the gaugino fields with/without bars signify four-dimensional chirality and the indices L and R two-dimensional chirality [17]. The covariant derivatives and the field strengths are defined by

$$D_\mu = \partial_\mu + i[A_\mu, \cdot], \quad (2.2)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \quad (2.3)$$

respectively. The action is invariant under an infinitesimal gauge transformation with a function $\omega(x) = \sum_\alpha \omega_\alpha(x) T_\alpha$:

$$\delta_\omega A_\mu(x) = -D_\mu \omega(x), \quad \delta_\omega F(x) = i[\omega(x), F(x)], \quad (2.4)$$

where the adjoint scalar and fermion fields are represented as F . The theory possesses four supersymmetries corresponding to the four spinor components (with/without bars, and L/R). In addition, there are two types (vector and axial) of the $U(1)_R$ symmetries in the theory. Among them, the lattice formulation given in section 2.2 preserves the latter, which transforms the fields according to the charge assignment: $+2$ (-2) to ϕ ($\bar{\phi}$), $+1$ (-1) to $\lambda_L, \bar{\lambda}_R$ ($\lambda_R, \bar{\lambda}_L$), and 0 to the others.

Naive lattice discretization breaks supersymmetry completely, due to the lack of Leibniz rule on the lattice. This difficulty can be partly avoided by expressing the action as an exact form with respect to a nilpotent supercharge Q . Namely, one of the four supercharges, Q , can be kept on a lattice. To find the exact form of the action, it is convenient to use new fermion variables ψ_μ, χ, η defined as

$$\begin{aligned} \psi_1 &\equiv \frac{1}{\sqrt{2}}(\lambda_L + \bar{\lambda}_R), & \psi_2 &\equiv \frac{i}{\sqrt{2}}(\lambda_L - \bar{\lambda}_R), \\ \chi &\equiv \frac{1}{\sqrt{2}}(\lambda_R - \bar{\lambda}_L), & \eta &\equiv -i\sqrt{2}(\lambda_R + \bar{\lambda}_L), \end{aligned} \quad (2.5)$$

instead of the original gaugino fields $\lambda, \bar{\lambda}$.³ The action (2.1) can be expressed in the variables as

$$S_{\text{cont}} = \frac{1}{2g^2} \int d^2x \text{tr} \left\{ H^2 - 2iHF_{12} + \sum_{\mu=1}^2 D_\mu \phi D_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 + 2i\chi(D_1\psi_2 - D_2\psi_1) + i \sum_{\mu=1}^2 \psi_\mu D_\mu \eta - \frac{1}{4} \eta[\phi, \eta] - \chi[\phi, \chi] + \sum_{\mu=1}^2 \psi_\mu [\bar{\phi}, \psi_\mu] \right\}. \quad (2.6)$$

³ These variables are used in topological field theory, and here we employ them to define the lattice action with an exact supercharge. The new auxiliary field H is given by $H = iD + iF_{12}$. Note D should be taken as anti-hermitian in (2.1). Among the four supercharges of the theory ($Q_L, Q_R, \bar{Q}_L, \bar{Q}_R$), a combination $Q \equiv -(Q_L + \bar{Q}_R)/\sqrt{2}$ transforms fields as (2.7)–(2.11).

We choose one of the four supercharges Q defined by

$$QA_\mu = \psi_\mu, \quad (2.7)$$

$$Q\psi_\mu = iD_\mu\phi, \quad (2.8)$$

$$Q\phi = 0, \quad (2.9)$$

$$Q\bar{\phi} = \eta, \quad Q\eta = [\phi, \bar{\phi}], \quad (2.10)$$

$$Q\chi = H, \quad QH = [\phi, \chi]. \quad (2.11)$$

From (2.4) and (2.7)–(2.11), we can see the nilpotency of Q up to an infinitesimal gauge transformation with the parameter ϕ ,

$$Q^2 = -i\delta_\phi. \quad (2.12)$$

By using this Q , the action can also be recast as a Q -exact form [50,51]:

$$S_{\text{cont}} = Q \frac{1}{2g^2} \int d^2x \Xi_{\text{cont}}(x), \quad (2.13)$$

where

$$\Xi_{\text{cont}}(x) \equiv \text{tr} \left[\frac{1}{4} \eta [\phi, \bar{\phi}] - i \sum_{\mu=1}^2 \psi_\mu D_\mu \bar{\phi} + \chi (H - 2iF_{12}) \right]. \quad (2.14)$$

We emphasize that the action (2.13) is just a rewriting of (2.1) with a different notation, and hence it is invariant under the full of the original supersymmetry transformations. The important point is that the Q -invariance is manifest in this form because Q satisfies (2.12) and Ξ_{cont} (2.14) is gauge invariant. It will be crucial for the lattice construction with keeping Q -symmetry discussed below.

2.2. Lattice formulation with an exact supersymmetry

We briefly summarize the lattice formulation given in [7]. The lattice action is defined in a Q -exact form, as (2.13) for the continuum counterpart, and possesses the exact Q supersymmetry invariance on the lattice.

Let us consider a two-dimensional square lattice with the periodic boundary conditions which is denoted by $\Lambda_L \equiv a\mathbb{Z}_L \times a\mathbb{Z}_L$, where a is the lattice spacing and L is the number of lattice sites in each direction. The result of this paper is easily extended to the case of a rectangular lattice. Hereafter, the lattice site is expressed as $x = (x_1, x_2)$, $x_\mu \in \{a, 2a, \dots, La\}$ ($\mu = 1, 2$). The fermions and scalars are defined on the sites, while the gauge fields are promoted to gauge group-valued variables $U_\mu(x) \in SU(N)$ defined on the link connecting x and $x + a\hat{\mu}$ where $\hat{\mu}$ is the unit vector in the μ -direction. For notational simplicity, we often use the same symbols for both of each continuum field and its lattice counterpart with keeping the mass dimensions. We put the subscripts “cont” and “lat” to distinguish them when needed.

The gauge transformations of the link fields U_μ and the other adjoint fields F are given in the usual manner:

$$U_\mu(x) \rightarrow \Lambda(x) U_\mu(x) \Lambda(x + a\hat{\mu})^{-1}, \quad (2.15)$$

$$F(x) \rightarrow \Lambda(x) F(x) \Lambda(x)^{-1}, \quad (2.16)$$

where $\Lambda(x) = e^{i\omega(x)} \in SU(N)$ is a gauge transformation function. For later use, we introduce the gauge covariant forward (backward) difference operator ∇_μ (∇_μ^*) as

$$\nabla_\mu F(x) \equiv \frac{1}{a} \left(U_\mu(x) F(x + a\hat{\mu}) U_\mu(x)^{-1} - F(x) \right), \tag{2.17}$$

$$\nabla_\mu^* F(x) \equiv \frac{1}{a} \left(F(x) - U_\mu(x - a\hat{\mu})^{-1} F(x - a\hat{\mu}) U_\mu(x - a\hat{\mu}) \right). \tag{2.18}$$

The covariant difference operators (2.17) and (2.18) are covariant under the lattice gauge transformations, (2.15) and (2.16), as their names suggest. The plaquette field,

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu(x + a\hat{\nu})^{-1} U_\nu(x)^{-1}, \tag{2.19}$$

is another important gauge covariant quantity.

If we use a naive relation $U_\mu(x) = e^{iaA_\mu(x)}$ and send a to zero,⁴ the difference operators (2.17) and (2.18) coincide with the correct covariant derivative (2.2), and the plaquette field reproduces the continuum field tensor (2.3) as $U_{12}(x) = 1 + ia^2 F_{12}(x) + \mathcal{O}(a^3)$. Moreover, the infinitesimal form of the lattice gauge transformations,

$$\delta_\omega U_\mu(x) = -ia \nabla_\mu \omega(x) U_\mu(x), \quad \delta_\omega F(x) = i[\omega(x), F(x)], \tag{2.20}$$

also reproduce the correct continuum limit (2.4).

The Q -transformation is realized on the lattice [7] by

$$QU_\mu(x) = ia\psi_\mu(x) U_\mu(x), \tag{2.21}$$

$$Q\psi_\mu(x) = i\nabla_\mu \phi(x) + ia\psi_\mu(x)\psi_\mu(x), \tag{2.22}$$

$$Q\phi(x) = 0, \tag{2.23}$$

$$Q\bar{\phi}(x) = \eta(x), \quad Q\eta(x) = [\phi(x), \bar{\phi}(x)], \tag{2.24}$$

$$Q\chi(x) = H(x), \quad QH(x) = [\phi(x), \chi(x)]. \tag{2.25}$$

Note that the transformation above remains nilpotent up to an infinitesimal lattice gauge transformation (2.20) with the parameter $\phi(x)$ on the lattice:

$$Q^2 = -i\delta_\phi. \tag{2.26}$$

It reproduces the transformation rule in the continuum theory, (2.7)–(2.10), after taking the continuum limit. With use of (2.21)–(2.25), the action (2.13) is transcribed to the lattice action:

$$S_{\text{lat}} \equiv Q \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \Xi_{\text{lat}}(x) \tag{2.27}$$

with

$$\Xi_{\text{lat}}(x) \equiv \text{tr} \left\{ \frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i \sum_{\mu=1}^2 \psi_\mu(x) \nabla_\mu \bar{\phi}(x) + \chi(x) \left(H(x) - \frac{i}{a^2} \Phi_{\text{TL}}(x) \right) \right\}, \tag{2.28}$$

where $\Phi_{\text{TL}}(x)$ is a lattice version of the field tensor (2.3) satisfying $\Phi_{\text{TL}}(x) \rightarrow 2a^2 F_{12}(x)$ as $a \rightarrow 0$, and explicitly given in what follows.

⁴ More detailed analysis with an improved relation (2.35) is given in the next subsection.

In discussing the continuum limit, the plaquette field is usually expanded around unity. In the current case, we need to be careful about this point. After integrating the auxiliary field H , the lattice gauge action becomes

$$S_G = \frac{1}{8g^2a^2} \sum_{x \in \Lambda_L} \text{tr}(\Phi_{\text{TL}}^2(x)), \tag{2.29}$$

and one finds that

$$\Phi_{\text{TL}}(x) = 0, \quad \text{for all } x \in \Lambda_L, \tag{2.30}$$

gives the vacuum configurations corresponding to the continuum limit (a goes to zero with g fixed). If the vacuum configuration is unique and gives $U_{12}(x) = \mathbf{1}_N$, one can expand the plaquette field around unity and confirm that the lattice action reproduces the continuum one. If it is not the case, the lattice action is not guaranteed to provide the desired continuum action in general. For instance, for $SU(2)$, a naive choice $\Phi_{\text{TL}}(x) = -i(U_{12}(x) - U_{21}(x))$ does not lead to the unique vacuum $U_{12}(x) = \mathbf{1}_2$. Actually, another vacuum $U_{12}(x) = -\mathbf{1}_2$ satisfies (2.30) as well and causes an obstacle for taking the correct continuum limit. In order to reproduce the correct continuum gauge action, $\Phi_{\text{TL}}(x)$ should be chosen so that the gauge action (2.29) has no nontrivial minima other than $U_{12}(x) = \mathbf{1}_N$.

Two possibilities for desired lattice field tensor have been discussed in [8] and [52]. Since the traceless field Φ_{TL} is generally given by

$$\Phi_{\text{TL}}(x) \equiv \Phi(x) - \left(\frac{1}{N} \text{tr} \Phi(x)\right) \mathbf{1}_N, \tag{2.31}$$

we can use Φ for defining Φ_{TL} . One possibility is

$$\Phi_1(x) = \begin{cases} \frac{-i(U_{12}(x) - U_{21}(x))}{1 - \frac{1}{\epsilon^2} \|1 - U_{12}(x)\|^2} & \text{for } \|1 - U_{12}(x)\| < \epsilon, \\ \infty & \text{otherwise,} \end{cases} \tag{2.32}$$

where ϵ is a positive number chosen in the range $0 < \epsilon < 2\sqrt{2}$ for $N = 2, 3, 4$ and $0 < \epsilon < 2\sqrt{N} \sin(\frac{\pi}{N})$ for $N \geq 5$. The other possibility is

$$\Phi_2(x) = \frac{4i}{M} \cdot \frac{2 - U_{12}(x)^M - U_{21}(x)^M}{U_{12}(x)^M - U_{21}(x)^M}, \tag{2.33}$$

with M being an integer satisfying $2M \geq N$. When M is even ($M = 2m$), (2.33) can be recast as

$$\Phi_2(x) = -\frac{2i}{m} \cdot \frac{U_{12}(x)^m - U_{21}(x)^m}{U_{12}(x)^m + U_{21}(x)^m}, \tag{2.34}$$

which would be more convenient for numerical simulation.

Note that the r.h.s. in (2.33) is a hermitian matrix since the denominator and the numerator commute with each other. In both cases, only the single vacuum $U_{12} = \mathbf{1}_N$ is allowed, and by expanding the plaquette field around unity, we can verify that $\Phi_{i,\text{TL}}(x) = 2a^2 F_{12}(x) + \mathcal{O}(a^3)$.

Thus, we can construct the continuum limit of the lattice action (2.27) with (2.32) or (2.33) and verify that it does coincide with the action (2.1) via (2.13) and (2.14). The lattice action is exactly invariant under the lattice Q -transformation (2.21)–(2.25) thanks to the nilpotency of Q , (2.26). The perturbative power counting theorem tells us that any relevant supersymmetry breaking operators are forbidden by the Q -symmetry and the internal $U(1)_R$ -symmetry [7], and all the supersymmetries are shown to be restored at least in the perturbation theory.

The lattice action can be used for the numerical simulations, since the same forward difference operators employed in (2.22) and (2.28) yield a semi-positive boson action which is suitable for the Monte-Carlo method. Numerical results indicate that the restoration of the full supersymmetries does occur beyond the perturbation theory [21–24].

2.3. Classical continuum limit

The action (2.1) is correctly reproduced from the lattice action (2.27) as we take the lattice spacing to zero, as seen in the previous subsection. In this subsection, in order to find a proper $\mathcal{O}(a)$ improvement procedure, we study the classical continuum limit more precisely by expanding various quantities (difference operators, plaquette field, and Q -transformation) in the lattice spacing, and determine the order at which the first deviation terms from the continuum theory appear in the expansion.

The lattice fields are not defined on the continuum spacetime but on the lattice. To expand the fields in the lattice spacing a , we first embed the lattice in the continuum spacetime and regard the lattice functions as smooth functions defined on the continuum spacetime. In order to associate the link fields to the continuum gauge fields, we employ the midpoint prescription

$$U_\mu(x) = e^{iaA_\mu(x_c)}, \tag{2.35}$$

where $x_c = x + (a/2)\hat{\mu}$. The link field can be interpreted as the Wilson line,

$$\begin{aligned} U_\mu(x) &= \mathbf{P} e^{i \int_0^a dt A_\mu(x+t\hat{\mu})} \\ &\equiv 1 + i \int_0^a dt A_\mu(x+t\hat{\mu}) + i^2 \int_0^a dt \int_0^a ds \theta(s-t) A_\mu(x+t\hat{\mu}) A_\mu(x+s\hat{\mu}) + \dots, \end{aligned} \tag{2.36}$$

when the lattice is embedded in the continuum spacetime. (2.35) is obtained by expanding the r.h.s. in a up to $\mathcal{O}(a^2)$. Namely, the midpoint prescription reproduces the continuum gauge transformation up to this order, $U_\mu(x) = e^{iaA_\mu(x_c) + \mathcal{O}(a^3)}$. It is straightforward to increase the precision by proceeding the expansion in a .⁵ We identify the other adjoint lattice fields (scalar and fermi fields) as the fields on the embedded points where they are defined. Under these identifications, we obtain smooth functions associated with the lattice fields by smoothly interpolating those fields on entire continuum spacetime. Thus, the expansion with respect to the lattice spacing can be simply performed by the Taylor expansion. For instance, we can expand a function $f(x)$ around the midpoint $x_c = x + (a/2)\hat{\mu}$:

$$f(x) = f(x_c) - \frac{a}{2} \partial_\mu f(x_c) + \frac{a^2}{8} \partial_\mu^2 f(x_c) + \mathcal{O}(a^3). \tag{2.37}$$

By using the identities for ∇_μ and arbitrary deformation δ :

$$\nabla_\mu f(x) = D_\mu^c f(x_c) + \frac{ia}{2} [A_\mu(x_c), D_\mu^c f(x_c)] + \mathcal{O}(a^2) \tag{2.38}$$

⁵ For example, we have

$$U_\mu(x) = \exp \left[iaA_\mu(x_c) + i \frac{a^3}{24} \partial_\mu^2 A_\mu(x_c) - \frac{a^3}{12} [A_\mu(x_c), \partial_\mu A_\mu(x_c)] + \mathcal{O}(a^4) \right].$$

with $D_\mu^c = \partial_\mu + i[A_\mu(x_c), \cdot]$, and

$$\delta A = \delta e^A \cdot e^{-A} - \frac{1}{2}[A, \delta e^A \cdot e^{-A}] + \mathcal{O}(A^3), \tag{2.39}$$

we find that the lattice gauge transformation (2.20) is automatically $\mathcal{O}(a)$ -improved as

$$\delta_{\text{lat}, \omega} = \delta_{\text{cont}, \omega} + \mathcal{O}(a^2). \tag{2.40}$$

On the other hand, the $\mathcal{O}(a)$ improvement of the lattice Q -transformation (2.21)–(2.25) is not automatic. Indeed, the Q -transformation of ψ_μ (2.22) has a quadratic $\mathcal{O}(a)$ -term of the fermion and the unimproved forward difference operator that is expanded as

$$\nabla_\mu = D_\mu + \frac{a}{2}D_\mu^2 + \mathcal{O}(a^2). \tag{2.41}$$

The Q -transformation of the gauge fields also has the non-zero $\mathcal{O}(a)$ -correction. Hence, the total Q -transformation satisfies

$$Q_{\text{lat}} = Q_{\text{cont}} + \mathcal{O}(a), \tag{2.42}$$

although those of the other fields retain their continuum forms.⁶

Similarly, the integrand Ξ_{lat} (2.28) behaves as

$$\Xi_{\text{lat}}(x) = \Xi_{\text{cont}}(x) + \mathcal{O}(a), \tag{2.43}$$

since the $\mathcal{O}(a)$ -terms come from the forward difference operator (2.41) and the unimproved plaquette field,

$$U_{12}(x) = \exp\left(ia^2 F_{12}(x) + i\frac{a^3}{2}(D_1 + D_2)F_{12}(x) + \mathcal{O}(a^4)\right). \tag{2.44}$$

Note that the $\mathcal{O}(a)$ -correction does not cancel even if we take the combination $U_{12} - U_{21}$ as in Ξ .

We thus find that the total lattice action is

$$S_{\text{lat}} = S_{\text{cont}} + \mathcal{O}(a), \tag{2.45}$$

because of (2.42) and (2.43) with (2.27). In order that the lattice action coincides with the continuum action up to $\mathcal{O}(a^2)$ terms, we have to improve the covariant difference operator, the field tensor and the lattice Q -transformation with keeping the nilpotency.

3. Method of tree-level $\mathcal{O}(a)$ improvement

In the last section, we have seen that the lattice gauge transformation is already $\mathcal{O}(a)$ -improved while Q and Ξ_{lat} have the $\mathcal{O}(a)$ -corrections, (2.42) and (2.43), and thus the resultant action (2.27) produces the continuum one up to $\mathcal{O}(a)$ terms. In this section, we will explain our strategy to improve Q and Ξ_{lat} so that $S_{\text{lat}}^{\text{imp}} = S_{\text{cont}} + \mathcal{O}(a^2)$ is obtained. The explicit form of the tree-level $\mathcal{O}(a)$ -improved action will be given in the next section.

⁶ Somewhat surprisingly, the $\mathcal{O}(a)$ -term in (2.42) vanishes when applying the lattice Q -transformation twice, because both the continuum and the lattice gauge transformations with the parameter ϕ satisfy (2.40).

3.1. Locality

Before going to the detail of the strategy, we should mention on the locality condition we employ throughout this paper. Indeed, the improvement will be performed with keeping the locality of the Q -transformation in the sense that long-range interactions in the lattice unit are suppressed, at least, by the exponential of the distance.

First, let us define ultra-local operators and exponentially local ones. Suppose that the lattice size is infinite and the sites are labeled by integers, $x_\mu \in a\mathbb{Z}$, and $\varphi_\alpha(x)$ is a lattice field with finite internal index $\alpha = 1, 2, \dots, N_i$. In the present case, $N_i = N^2 - 1$ since $\varphi(x)$ is expanded by a basis of $su(N)$ generators: $\varphi(x) = \sum_{\alpha=1}^{N^2-1} \varphi_\alpha(x) T_\alpha$. An operator R acting to $\varphi(x)$ is formally represented as a kernel:

$$(R\varphi)_\alpha(x) = \sum_{y, \beta} R_{\alpha\beta}(x, y) \varphi_\beta(y). \tag{3.1}$$

The kernel is an infinite dimensional matrix with the row α, x and the column β, y . In the following and in Appendix B, $R(x, y)$ denotes the $N_i \times N_i$ matrix with respect to the internal index.

The ultra-local operator is defined by

$$R(x, y) = 0, \quad \text{for } \|x - y\|_1 > ar, \tag{3.2}$$

where the localization range r is a fixed natural number.⁷ For instance, the forward and the backward difference operators are ultra-local with the localization range one. The kernel of an ultra-local operator forms a banded diagonal matrix, which is suited for numerical applications.

On the other hand, R is referred to as an exponentially local operator if there exist positive constants C and κ such that

$$|R_{\alpha\beta}(x, y)| \leq C e^{-\frac{\kappa \|x-y\|_1}{a}}. \tag{3.3}$$

As is well-known, the overlap Dirac operator [53,54] satisfies this type of locality [55]. It is obvious that any ultra-local operator satisfies (3.3), but the converse is not true in general. The exponential locality is therefore a milder condition as the locality.

In the continuum limit, both locality conditions reproduce a local continuum field theory with finite number of derivatives contained in its classical action, because there are no contributions from y being separated from x by a finite physical length as $a \rightarrow 0$. Also in the point of view of the renormalization group and the universality hypothesis [48,49,55], the exponential locality is allowed as a locality condition of the lattice theory having the desired continuum limit. We therefore employ the exponential locality to construct the $\mathcal{O}(a)$ -improved theory in this paper.

3.2. Q -transformation

Among the Q -transformations (2.21)–(2.25), we have to improve only the two transformations QU_μ (2.21) and $Q\psi_\mu$ (2.22) since the others (2.23)–(2.25) coincide with their continuum forms and are irrelevant to the present purpose.

⁷ As a definition of the distance, we use the “taxi driver distance”: $\|x - y\|_1 \equiv \sum_\mu |x_\mu - y_\mu|$. For the Euclid distance $\|x - y\|_2 \equiv \sqrt{\sum_\mu (x_\mu - y_\mu)^2}$, the following argument will also be similar.

Here we should point out that, even if we give an improved transformation of U_μ , it is not clear *a-priori* if we can define a modified transformation of ψ_μ that is consistent with the nilpotency of Q (2.26). Furthermore, even though it is possible, it is not yet clear whether the transformation of ψ_μ satisfies the locality. The following lemma is crucial to give answers to these points:

Lemma. *Let $f_\mu(x)$ be a function that depends on $(U_\lambda, \psi_\lambda, \phi)$ with the same gauge transformation property as $U_\mu(x)$, and suppose that*

$$QU_\mu(x) = f_\mu(x). \tag{3.4}$$

Let $g_\mu(x)$ be a function that depends on $(U_\lambda, QU_\lambda, \phi)$ with the same gauge transformation property as $\psi_\mu(x)$. If (3.4) can be solved in term of $\psi_\mu(x)$ as follows:

$$\psi_\mu(x) = g_\mu(x), \tag{3.5}$$

then one can consistently define the Q -transformation by (3.4) and

$$Q\psi_\mu(x) = \sum_{y,v} \left[QU_v(y) \cdot \frac{\delta}{\delta U_v(y)} - i\delta_\phi U_v(y) \cdot \frac{\delta}{\delta(QU_v(y))} \right] g_\mu(x), \tag{3.6}$$

$$Q\phi(x) = 0, \tag{3.7}$$

so that $Q^2 = -i\delta_\phi$ for all fields.

Note that, for the original transformations (2.21) and (2.22), the r.h.s. of $QU_\mu(x)$ actually defines such a function $f_\mu(x)$. In that case, $g_\mu(x) \equiv -\frac{i}{a}(QU_\mu(x))U_\mu(x)^{-1}$ that is identical to $\psi_\mu(x)$. This lemma actually holds in general frameworks with the exact gauge invariance, including but not limited to the lattice gauge theory. We give a proof in Appendix A and simply use the result here.⁸

Let us consider a possible deformation of Q based on this lemma. Suppose that an improved transformation takes the form,

$$Q^{imp}U_\mu(x) = ia(\mathcal{R}_\mu\psi_\mu(x))U_\mu(x), \tag{3.8}$$

where \mathcal{R}_μ is an operator which improves the Q -transformation of $U_\mu(x)$. In this case, g_μ in the lemma is formally given by

$$g_\mu = \mathcal{R}_\mu^{-1} \left\{ -\frac{i}{a}(Q^{imp}U_\mu)U_\mu^{-1} \right\}, \tag{3.9}$$

to express ψ_μ as (3.5) and we have the correct transformation (3.6). The transformation of ψ_μ thus becomes

$$Q^{imp}\psi_\mu(x) = i\mathcal{R}_\mu^{-1}\nabla_\mu\phi(x) + ia\sum_{y,v}(\mathcal{R}_v\psi_v(y))U_v(y) \cdot \frac{\delta}{\delta U_v(y)}g_\mu(U_\lambda, Q^{imp}U_\lambda; x). \tag{3.10}$$

⁸ The essential point of the proof is that the r.h.s. of the expression (3.6) is identified with Qg_μ once $-i\delta_\phi U_v$ in the second term is replaced by Q^2U_v . This implies $Q^2 = -i\delta_\phi$ for U_μ . The nilpotency is also satisfied for QU_μ since Q commutes with δ_ϕ , that is, $Q^2(QU_\mu) = Q(Q^2U_\mu) = -iQ\delta_\phi U_\mu = -i\delta_\phi(QU_\mu)$. In addition, we find that $Q^2 = -i\delta_\phi$ for ψ_μ because ψ_μ is given by g_μ that is a function of U_v and QU_v for which $Q^2 = -i\delta_\phi$ was already satisfied.

Here we find that one cannot use an arbitrary \mathcal{R}_μ , because \mathcal{R}_μ must not have the zero-eigenvalue in order to have a well-defined inverse.

Keeping these points in mind, let us consider the improvement of QU_μ and $Q\psi_\mu$ in detail. In contrast to the case of the gauge transformation (2.40), the reason why the original transformation (2.21) has an $\mathcal{O}(a)$ correction is simple: The gauge field $A_\mu(x)$ is defined on the midpoint while the corresponding fermionic field $\psi_\mu(x)$ is defined on a site. Hence a heuristic choice of the improved transformation would be

$$Q_{naive}^{imp} U_\mu(x) = \frac{ia}{2} (\psi_\mu(x) U_\mu(x) + U_\mu(x) \psi_\mu(x + a\hat{\mu})), \tag{3.11}$$

which corresponds to

$$\mathcal{R}_\mu^{naive} = 1 + \frac{a}{2} \nabla_\mu. \tag{3.12}$$

Indeed, we can show that (3.11) turns out to have the desirable property

$$Q_{naive}^{imp} A_\mu(x) = \psi_\mu(x) + \mathcal{O}(a^2) \tag{3.13}$$

from (2.37), (2.38) and (2.39) with the midpoint prescription (2.35). However, this choice is too naive. In fact, since the eigenvalues of $a\nabla_\mu$ are parametrized by

$$-1 + e^{i\theta} \quad (\theta \in \mathbb{R}), \tag{3.14}$$

(3.12) has zeros in its spectrum.⁹

This suggests that

$$\mathcal{R}_\mu = 1 + \frac{a}{2} \nabla_\mu - ra^2 \nabla_\mu \nabla_\mu^* \quad (r > 0) \tag{3.15}$$

is a candidate for our desired solution improving Q -transformations up to $\mathcal{O}(a)$. In contrast to (3.12), the third term (the Wilson-term) lifts the zero-mode with keeping the small a behavior unchanged as long as $r \neq 0$. (Note that the case $r < 0$ is possible for the purpose of this section, but it will be rejected because it leads to an incorrect fermion measure as we will see in section 4.) In this case, \mathcal{R}_μ^{-1} is exponentially local, because one can show that an ultra-local operator \mathcal{R}_μ , whose spectrum is bounded by an upper and a non-zero lower bounds, has an inverse being exponentially local. Details including the definitions of the terminologies are presented in Appendix B.3.¹⁰ We thus have the $\mathcal{O}(a)$ -improved transformation for the link fields within the locality principle.

The lemma also guarantees that the Q -transformation of the fermi fields ψ_μ (3.6) are automatically improved in the case that both of the Q -transformation and the gauge transformation δ_ϕ of U_μ are improved. In fact, (3.4) and (3.5) have the same information, and thus the expansion of g_μ in a should reproduce the counterpart of the continuum theory without the error of $\mathcal{O}(a)$, from the assumption on the improvement of U_μ .

⁹ (3.14) is understood from the identity $a^2 \nabla_\mu^* \nabla_\mu = a \nabla_\mu - a \nabla_\mu^*$ and the relation $\nabla_\mu^* = -\nabla_\mu^\dagger$ with respect to the inner product of $su(N)$ -valued functions: $(f, g) \equiv \sum_{x \in \Lambda_L} \text{tr} (f(x)^\dagger g(x))$. ∇_μ and ∇_μ^* mutually commute and can be simultaneously diagonalized by a unitary transformation.

¹⁰ This situation reminds us the locality of the overlap Dirac operator. Unlike the Wilson operator D that is ultra-local, the overlap operator is given by the inverse square root of $D^\dagger D$ of which the locality is not immediately obvious. In [55], it is proven that the overlap operator is indeed local with exponentially decaying tails if the eigenvalues of $D^\dagger D$ are bounded from above and below. We can apply the same logic to the present case.

Of course, we can explicitly verify that (3.10) coincides to that of the continuum theory (2.8) up to $\mathcal{O}(a)$. In the improved transformation, the forward difference operator in the original transformation (2.22) is replaced by $\mathcal{R}_\mu^{-1}\nabla_\mu$ that provides the improvement because of the expansion

$$\mathcal{R}_\mu^{-1}\nabla_\mu = D_\mu + \mathcal{O}(a^2), \tag{3.16}$$

and the fermion bilinear term in (2.22) does not appear at the order of a as is seen in the following. We may use $1 + \frac{a}{2}\nabla_\mu$ and $1 - \frac{a}{2}\nabla_\mu^*$ as \mathcal{R}_μ and \mathcal{R}_μ^{-1} respectively, within the precision of $\mathcal{O}(a)$. Then, the second term of (3.10) is shown to be

$$\frac{i}{2a} \left\{ Q^{imp} U_\mu(x) Q^{imp} U_\mu^{-1}(x) - Q^{imp} U_\mu^{-1}(x - a\hat{\mu}) Q^{imp} U_\mu(x - a\hat{\mu}) \right\} + \mathcal{O}(a^2). \tag{3.17}$$

Although this seems to be $\mathcal{O}(a)$ at the first sight since $Q U_\mu$ is of the order the lattice spacing, the leading contributions in the parenthesis cancel each other. We thus see that (3.10) is improved up to the desired order.

Therefore, for given \mathcal{R}_μ , (3.15) or any other choices listed in Appendix B, the transformations (3.8) and (3.10) with (2.23)–(2.25) are local and satisfy both (2.26) and

$$Q_{\text{lat}}^{imp} = Q_{\text{cont}} + \mathcal{O}(a^2), \tag{3.18}$$

for all the fields.

3.3. The integrand Ξ

In this section, we will improve Ξ_{lat} , in particular, the difference operator and the lattice field tensor appearing there.

As we have seen in (3.16), the forward difference operator (2.41) is improved by multiplying \mathcal{R}_μ^{-1} . Since the plaquette field is similarly expanded as (2.44), its improvement is achieved analogously:

$$U_{\mu\nu}^{imp}(x) \equiv \mathcal{R}_{12} U_{\mu\nu}(x), \tag{3.19}$$

where we adopt

$$\mathcal{R}_{12} = 1 - \frac{a}{2}(\nabla_1 + \nabla_2). \tag{3.20}$$

Note that the improvement factor \mathcal{R}_{12} is common for U_{12} and U_{21} , but (3.20) is not the unique choice. Other possible choices of \mathcal{R}_{12} are given in Appendix B.

The improved Ξ is thus given by

$$\Xi_{\text{lat}}^{imp} = \text{tr} \left\{ \frac{1}{4} \eta[\phi, \bar{\phi}] - i \sum_{\mu=1}^2 \psi_\mu \nabla_\mu^{imp} \bar{\phi} + \chi \left(H - \frac{i}{a^2} \Phi_{\text{TL}}^{imp} \right) \right\}, \tag{3.21}$$

where

$$\nabla_\mu^{imp} = \mathcal{R}_\mu^{-1} \nabla_\mu, \tag{3.22}$$

$$\Phi_{\text{TL}}^{imp}(x) = \mathcal{R}_{12} \Phi_{\text{TL}}(x), \tag{3.23}$$

for the unimproved Φ_{TL} , (2.32) or (2.33). Here, \mathcal{R}_μ appearing in (3.22) should be taken as the same as those used in the Q -transformations, (3.8) and (3.10), in order to make the bosonic part of the action semi-positive. Note that, since \mathcal{R}_{12} is invertible, the field equation $\Phi_{\text{TL}}^{imp} = 0$ is

identical with (2.30). Therefore we can use the same choice (2.32) or (2.33) to forbid the extra vacua even in this case.

Now it is easy to see that

$$\Xi_{\text{lat}}^{\text{imp}}(x) = \Xi_{\text{cont}}(x) + \mathcal{O}(a^2). \tag{3.24}$$

The improved theory is defined by both the improved Q -transformation (3.8) and (3.10) with (3.15) and the improved $\Xi(x)$ (3.21).

4. Fermion doublers and path-integral measures in improved theory

In this section, we first summarize the $\mathcal{O}(a)$ -improved lattice action explained in the previous section. After that, we show the absence of the fermion doublers and give appropriate Q^{imp} -invariant path-integral measures.

4.1. Summary of the improved theory

The improved action obtained in the previous section is given by

$$S_{\text{lat}}^{\text{imp}} = Q^{\text{imp}} \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \Xi_{\text{lat}}^{\text{imp}}(x), \tag{4.1}$$

$$\Xi_{\text{lat}}^{\text{imp}} = \text{tr} \left\{ \frac{1}{4} \eta[\phi, \bar{\phi}] - i \sum_{\mu=1}^2 \psi_{\mu} \mathcal{R}_{\mu}^{-1} \nabla_{\mu} \bar{\phi} + \chi \left(H - \frac{i}{a^2} \mathcal{R}_{12} \Phi_{\text{TL}} \right) \right\}, \tag{4.2}$$

where \mathcal{R}_{μ} and \mathcal{R}_{12} are given, for example, by

$$\mathcal{R}_{\mu} = 1 + \frac{a}{2} \nabla_{\mu} - r a^2 \nabla_{\mu} \nabla_{\mu}^* \quad (r > 0), \tag{4.3}$$

$$\mathcal{R}_{12} = 1 - \frac{a}{2} (\nabla_1 + \nabla_2). \tag{4.4}$$

Note that, as mentioned in the previous section, these factors are not unique and other possible choices are summarized in appendices B.1 and B.2.

The improved Q -transformations are expressed as

$$Q^{\text{imp}} U_{\mu}(x) = i a \psi'_{\mu}(x) U_{\mu}(x), \tag{4.5}$$

$$Q^{\text{imp}} \psi'_{\mu}(x) = i \nabla_{\mu} \phi(x) + i a \psi'_{\mu}(x) \psi'_{\mu}(x), \tag{4.6}$$

$$Q^{\text{imp}} \phi(x) = 0, \tag{4.7}$$

$$Q^{\text{imp}} \bar{\phi}(x) = \eta(x), \quad Q^{\text{imp}} \eta(x) = [\phi(x), \bar{\phi}(x)], \tag{4.8}$$

$$Q^{\text{imp}} \chi(x) = H(x), \quad Q^{\text{imp}} H(x) = [\phi(x), \chi(x)] \tag{4.9}$$

in terms of

$$\psi'_{\mu}(x) \equiv \mathcal{R}_{\mu} \psi_{\mu}(x). \tag{4.10}$$

The first two transformations are built in section 3.2, (3.8) and (3.10) with (3.9) for a given \mathcal{R}_{μ} . The others are the same with the continuum transformations (2.23)–(2.25). These transformations obey $(Q^{\text{imp}})^2 = -i \delta_{\phi}$ as in the unimproved theory, and the improved action retains the invariance

of the exact supersymmetry.¹¹ Note that the improved transformations are identical with the original transformations (2.21)–(2.25) under the replacement of ψ_μ by ψ'_μ .

The improved action (4.1) coincides with the unimproved lattice action given in [7] for $\mathcal{R}_\mu = \mathcal{R}_{12} = 1$, and the $\mathcal{O}(a)$ terms of \mathcal{R}_μ and \mathcal{R}_{12} provide the improvement. As we have already shown in the last section, Q^{imp} and Ξ^{imp} reproduce the continuum Q and Ξ with no $\mathcal{O}(a)$ error, respectively. So, the improved action indeed satisfies $S_{lat}^{imp} = S_{cont} + \mathcal{O}(a^2)$ as $a \rightarrow 0$.

As with the unimproved theory, the action (4.1) is exactly invariant under the axial $U(1)_R$ -transformation [7] (as well as Q^{imp}), because the multiplication by \mathcal{R}_μ or \mathcal{R}_{12} does not affect the transformation properties. So we can conclude that the same perturbative arguments hold in this case, and all of the supersymmetries are restored in the continuum limit at the quantum level, at least, in the perturbation theory.

In order to define the quantum theory, we need to specify not only the action but also the path-integral measure. There are two candidates; the natural measure,

$$D\varphi_{\text{natural}} \equiv \prod_x d\phi(x)d\bar{\phi}(x)dH(x)d\chi(x)d\eta(x) \prod_\mu dU_\mu(x)d\psi_\mu(x), \tag{4.11}$$

and the manifestly Q^{imp} -invariant measure,

$$\begin{aligned} D\varphi &\equiv \prod_x d\phi(x)d\bar{\phi}(x)dH(x)d\chi(x)d\eta(x) \prod_\mu dU_\mu(x)d\psi'_\mu(x) \\ &= \prod_x d\phi(x)d\bar{\phi}(x)dH(x)d\chi(x)d\eta(x) \prod_\mu dU_\mu(x)d\psi_\mu(x) \times \det(\mathcal{R}_\mu)^{-1}, \end{aligned} \tag{4.12}$$

where $dU_\mu(x)$ denotes the $SU(N)$ Haar measure, and the measure of the adjoint fields $F(x) = \sum_\alpha F_\alpha(x)T_\alpha$ is defined as $dF(x) \equiv \prod_\alpha dF_\alpha(x)$. The expression $d\phi(x)d\bar{\phi}(x)$ means that the usual measure is used for the real and imaginary parts of the complex field $\phi(x)$. The Q^{imp} -invariance of (4.12) follows from the fact that the natural measure (4.11) is invariant under (2.21)–(2.25) [12].

In conclusion, we can use any of them. In fact, the difference of the two measures is only the factor $\det(\mathcal{R}_\mu)^{-1}$ behaving as $const. \times (1 + \mathcal{O}(e^{-\ell/a}))$ ($\ell = La$ is the physical size of the system) in the continuum limit at least for $\mathcal{R}_\mu^{(\pm)}$ given in (B.1) and (B.2), as we will show in section 4.4. Therefore, even if the natural measure (4.11) breaks the Q^{imp} symmetry explicitly, it is negligible in the continuum limit. In the same way, the factor $\det(\mathcal{R}_\mu)^{-1}$ in the Q^{imp} -invariant measure (4.12) does not affect the continuum limit of the improved theory.

Here we make a comment that there is an interesting exception: For $\mathcal{R}_\mu^{(e)}$ given in (B.3), (4.11) is identical with (4.12) since $\det(\mathcal{R}_\mu^{(e)}) = 1$ as seen in the last paragraph in Appendix B.1. Then, the natural measure (4.11) is also Q^{imp} -invariant.

4.2. Convenient expressions in numerical simulations

The standard Monte-Carlo simulation is applicable for the present improved action because the boson action is semi-positive definite. Of course, it is for the case that the same \mathcal{R}_μ is chosen in (4.2) and (4.5). However the action with (4.3) has the exponentially local interactions and the actual numerical computations would demand considerable tasks.

¹¹ Conversely, once $Q^{imp}U_\mu$ is defined by (4.5), the transformation law for ψ_μ is uniquely determined under the constraint $(Q^{imp})^2 = -i\delta_\phi$.

We can avoid this difficulty by choosing such \mathcal{R}_μ that its inverse becomes ultra-local like

$$\mathcal{R}_\mu = \mathcal{R}_\mu^{(-)} \equiv \left(1 - \frac{a}{2} \nabla_\mu^* - r a^2 \nabla_\mu^* \nabla_\mu\right)^{-1} \quad (r > 0), \tag{4.13}$$

as well as the ultra-local \mathcal{R}_{12} given in (4.4).¹² Then, in addition to the ultra-local transformations (4.5)–(4.9), the integrand $\Xi_{\text{lat}}^{\text{imp}}$ becomes also ultra-local:

$$\Xi_{\text{lat}}^{\text{imp}} = \text{tr} \left\{ \frac{1}{4} \eta[\phi, \bar{\phi}] - i \sum_{\mu=1}^2 (\mathcal{R}_\mu^{-1} \psi'_\mu) (\mathcal{R}_\mu^{-1} \nabla_\mu \bar{\phi}) + \chi \left(H - \frac{i}{a^2} \mathcal{R}_{12} \Phi_{\text{TL}} \right) \right\}. \tag{4.14}$$

We may use these field variables ψ'_μ with (4.13) and (4.12) to define the improved theory used in the Monte-Carlo simulations. However, we must use ψ_μ to define observables because the tree-level $\mathcal{O}(a)$ improvement is achieved for ψ_μ rather than ψ'_μ .

Somewhat surprisingly, instead of (4.13), if we take

$$\mathcal{R}_\mu = \mathcal{R}_\mu^{(e)} \equiv \exp\left(\frac{a}{2} \nabla_\mu^{(s)}\right), \tag{4.15}$$

where $\nabla_\mu^{(s)} = \frac{1}{2}(\nabla_\mu + \nabla_\mu^*)$, the situation becomes much simpler thanks to the property $\mathcal{R}_\mu^T = \mathcal{R}_\mu^{-1}$. The integrand $\Xi_{\text{lat}}^{\text{imp}}$ becomes

$$\Xi_{\text{lat}}^{\text{imp}} = \text{tr} \left\{ \frac{1}{4} \eta[\phi, \bar{\phi}] - i \sum_{\mu=1}^2 \psi'_\mu \nabla_\mu \bar{\phi} + \chi \left(H - \frac{i}{a^2} \mathcal{R}_{12} \Phi_{\text{TL}} \right) \right\} \tag{4.16}$$

with (4.4). Of course, also in this case, the improved lattice action is given by an ultra-local form. The Q^{imp} -transformations are the same with those of the unimproved theory and the integrand (4.16) is also mostly the same with the original one. The $\mathcal{O}(a)$ improvement is then encoded only in (4.10) and the definition of the improved lattice field tensor $\Phi_{\text{TL}}^{\text{imp}} = \mathcal{R}_{12} \Phi_{\text{TL}}$.

In Appendix C, we present the explicit form of the lattice action obtained by acting Q^{imp} to $\Xi_{\text{lat}}^{\text{imp}}$.

4.3. Absence of the fermion doublers

The doubler modes do not exist in the bosonic sector of the improved theory because the kinetic terms for the scalar and the gauge fields are defined by the forward difference operator with invertible \mathcal{R}_μ and \mathcal{R}_{12} . The exact supersymmetry implies that the fermionic sector should also be free from the doubling problem. We will show it by examining the free lattice Dirac operator, explicitly.

The fermion kinetic terms of the improved free theory for Φ_1 (2.32) and Φ_2 (2.33) are given by

$$S_{\text{lat, F}}^{\text{imp}} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left[i \sum_{\mu=1}^2 \psi_\mu(x) \tilde{\Delta}_\mu \eta(x) + 2i \tilde{\chi}(x) \left(\tilde{\Delta}_1 \psi_2(x) - \tilde{\Delta}_2 \psi_1(x) \right) \right], \tag{4.17}$$

¹² Here and in (4.16), we can also use more general (B.10) with $r > \frac{1}{4}$ as \mathcal{R}_{12} .

where

$$\tilde{\Delta}_\mu \equiv \left(\mathcal{R}_\mu^{(0)}\right)^{-1} \Delta_\mu, \quad \tilde{\chi}(x) \equiv (\mathcal{R}_1^{(0)}\mathcal{R}_2^{(0)}\mathcal{R}_{12}^{(0)})^T \chi(x), \tag{4.18}$$

with the forward and backward difference operators,

$$\Delta_\mu f(x) \equiv \frac{1}{a} (f(x + a\hat{\mu}) - f(x)), \tag{4.19}$$

$$\Delta_\mu^* f(x) \equiv \frac{1}{a} (f(x) - f(x - a\hat{\mu})). \tag{4.20}$$

$\mathcal{R}_\mu^{(0)}$ and $\mathcal{R}_{12}^{(0)}$ denote \mathcal{R}_μ and \mathcal{R}_{12} with the gauge field turned off, respectively. Namely, for simple examples of \mathcal{R}_μ (B.1)–(B.3) and \mathcal{R}_{12} (B.9)–(B.11), the covariant operators ∇_μ and ∇_μ^* there are replaced by (4.19) and (4.20), respectively.¹³ Note that $\mathcal{R}_\mu^{(0)}$ and $\mathcal{R}_{12}^{(0)}$ commute with each other since they do not depend on the gauge field.

The form of (4.17) is the same as that of the original unimproved lattice model [7,8] which has no doubler modes, except $\mathcal{R}_\mu^{(0)}$ and $\mathcal{R}_{12}^{(0)}$ are included in the redefinition (4.18). It is clear that the improved theory also has no doublers since the improvement factors in (4.18) are invertible and do not affect the conclusion in the original model.

Explicitly, in terms of the four-component spinor $\tilde{\Psi} \equiv (\psi_1, \psi_2, \tilde{\chi}, \frac{1}{2}\eta)^T$, (4.17) is expressed as¹⁴

$$S_{\text{lat, F}}^{\text{imp}} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left[\tilde{\Psi}(x)^T \tilde{D} \tilde{\Psi}(x) \right], \tag{4.21}$$

where

$$\tilde{D} \equiv \sum_{\mu=1}^2 \left[-\frac{1}{2} \gamma_\mu (\tilde{\Delta}_\mu - \tilde{\Delta}_\mu^T) - \frac{1}{2} P_\mu (\tilde{\Delta}_\mu + \tilde{\Delta}_\mu^T) \right], \tag{4.22}$$

with

$$\begin{aligned} \gamma_1 &= -i\sigma_1 \otimes \sigma_1, & \gamma_2 &= i\sigma_1 \otimes \sigma_3, \\ P_1 &= \sigma_1 \otimes \sigma_2, & P_2 &= \sigma_2 \otimes \mathbb{1}_2 \end{aligned} \tag{4.23}$$

satisfying

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, P_\nu\} = 0. \tag{4.24}$$

Since the Dirac operator (4.22) is Hermitian, we may consider the zero of \tilde{D}^2 in order to see that of \tilde{D} . From (4.24), $\tilde{D}^2 = \sum_{\mu=1}^2 \tilde{\Delta}_\mu \tilde{\Delta}_\mu^T$, which means that \tilde{D} vanishes only at the zero of $\tilde{\Delta}_\mu$ that is nothing but the zero of Δ_μ (the origin in the momentum space) since \mathcal{R}_μ are invertible.

In the ordinary Wilson–Dirac operator, fermion doublers are lift by the Wilson term of the order of $\mathcal{O}(a)$. Since our lattice action is improved and has no $\mathcal{O}(a)$ term, the Wilson term

¹³ In $\mathcal{R}_\mu^{(0)}$ and $\mathcal{R}_{12}^{(0)}$, the transpose operation denoted by the superscript T maps Δ_μ and Δ_μ^* to $-\Delta_\mu^*$ and $-\Delta_\mu$, respectively.

¹⁴ The transpose operation of $\Psi(x)$ acts only to the spinor indices.

appearing in (4.22) should be higher order in a .¹⁵ Let us see this in the case of $\mathcal{R}_\mu^{(-)} \Big|_{r=\frac{1}{2}}$ in (B.2). The Dirac operator can be written as

$$\begin{aligned} \tilde{D} = & -\frac{1}{2} \sum_\mu \gamma_\mu (\Delta_\mu + \Delta_\mu^*) - \frac{a}{2} \sum_\mu P_\mu (\Delta_\mu \Delta_\mu^*) \\ & + \frac{a^2}{4} \sum_\mu \gamma_\mu (\Delta_\mu + \Delta_\mu^*) (\Delta_\mu \Delta_\mu^*) + \frac{a}{4} \sum_\mu P_\mu \left((\Delta_\mu)^2 + (\Delta_\mu^*)^2 \right). \end{aligned} \tag{4.25}$$

The first and the second terms are the standard kinetic term and the Wilson term, respectively, in the original unimproved model, while the third and fourth terms are generated by the $\mathcal{O}(a)$ improvement. The first term reproduces the naive kinetic term in the continuum limit up to $\mathcal{O}(a)$, and the third term has the same zeros with those of the first term. Interestingly, the second and fourth terms combine to yield the $\mathcal{O}(a^3)$ Wilson term $\frac{a^3}{4} \sum_\mu P_\mu (\Delta_\mu \Delta_\mu^*)^2$. Thus, we have

$$\tilde{D} = -\frac{1}{2} \sum_\mu \gamma_\mu (\Delta_\mu + \Delta_\mu^*) \left(1 - \frac{a^2}{2} \Delta_\mu \Delta_\mu^* \right) + \frac{a^3}{4} \sum_\mu P_\mu (\Delta_\mu \Delta_\mu^*)^2, \tag{4.26}$$

with the second term being the Wilson term in the improved action actually of the order of a^3 .

4.4. Path-integral measure

In section 4.1, we have defined the path-integral measure (4.12) which is invariant under the improved transformations (4.5)–(4.9). In what follows, we show that the factor $\det \mathcal{R}_\mu$ is irrelevant in the continuum limit $a \rightarrow 0$ with keeping the physical length $\ell \equiv La$ fixed, and the natural measure is therefore reproduced without breaking the tree-level $\mathcal{O}(a)$ improvement, at least, for $\mathcal{R}_\mu^{(+)}$ and $\mathcal{R}_\mu^{(-)}$ given in (B.1) and (B.2) as announced.

We only have to evaluate the determinant of $R_\mu^{(+)}$ because one can show that

$$\det \mathcal{R}_\mu^{(-)} = \left(\det \mathcal{R}_\mu^{(+)} \right)^{-1}, \tag{4.27}$$

as mentioned in Appendix B.1. For simplicity of the explanation, we focus on the case of $\det \mathcal{R}_1^{(+)}$. Of course, the same result is obtained for $\mathcal{R}_2^{(+)}$ by interchanging the role of the directions 1 and 2.

The matrix representation of $\mathcal{R}_\mu^{(+)}$ can be extracted as (3.1) for $x, y \in \Lambda_L$ and $\alpha, \beta = 1, 2, \dots, N^2 - 1$:

$$\begin{aligned} (\mathcal{R}_\mu^{(+)})_{\alpha\beta}(x, y) = & \delta_{x_\rho, y_\rho} \left[\left(\frac{1}{2} + 2r \right) \delta_{x_\mu, y_\mu} \delta_{\alpha\beta} + \left(\frac{1}{2} - r \right) \delta_{x_\mu+a, y_\mu} \hat{U}_{\mu, \alpha\beta}(x) \right. \\ & \left. - r \delta_{x_\mu-a, y_\mu} \hat{U}_{\mu, \beta\alpha}(y) \right]. \end{aligned} \tag{4.28}$$

¹⁵ The factor appearing in the redefinition of χ (4.18) does not affect the $\mathcal{O}(a)$ contribution since it behaves as $1 + \mathcal{O}(a^2)$ for any \mathcal{R}_μ and \mathcal{R}_{12} . Also, it does not lift the doublers in the lattice Dirac operator D which is defined by $S_{\text{lat}, F}^{\text{imp}} = \frac{a^2}{2g^2} \sum_{x \in \Gamma_L} \text{tr} (\Psi^T D \Psi)$ where $\Psi = (\psi_1, \psi_2, \chi, \frac{1}{2}\eta)^T$.

Here ρ denotes the direction other than μ ($\rho = 1$ and 2 , when $\mu = 2$ and 1 , respectively), and $\hat{U}_{\mu,\alpha\beta}(x)$ is the adjoint representation of $U_\mu(x)$:

$$\hat{U}_{\mu,\alpha\beta}(x) \equiv \text{tr} \left[T_\alpha U_\mu(x) T_\beta U_\mu(x)^{-1} \right]. \tag{4.29}$$

Likewise, we express the adjoint representation of an $SU(N)$ matrix A by putting a hat as

$$\hat{A}_{\alpha\beta} \equiv \text{tr} \left[T_\alpha A T_\beta A^{-1} \right]. \tag{4.30}$$

Note that $(\hat{A}^{-1})_{\alpha\beta} = \hat{A}_{\beta\alpha}$ is always satisfied.

We should note that (4.28) is an $\mathcal{M} \times \mathcal{M}$ matrix ($\mathcal{M} \equiv L^2(N^2 - 1)$) which is diagonal with respect to the ρ direction. In order to evaluate $\det \mathcal{R}_1^{(+)}$ explicitly, we first diagonalize (4.28) in the color space. To this end, let us consider a gauge function,

$$g_1(x_1, x_2) \equiv \begin{cases} 1, & (x_1 = La) \\ (P_1(x_2))^{-\frac{x_1}{La}} U_1(0, x_2) \cdots U_1(x_1 - a, x_2), & (x_1 = a, 2a, \dots, (L - 1)a) \end{cases} \tag{4.31}$$

where $P_1(x_2)$ is the Polyakov line along the x_1 -direction:

$$P_1(x_2) \equiv U_1(0, x_2) U_1(a, x_2) \cdots U_1((L - 1)a, x_2). \tag{4.32}$$

Then we can eliminate the x_1 dependence of $U_1(x_1, x_2)$ by the gauge transformation with $g_1(x_1, x_2)$ as

$$g_1(x_1, x_2) U_1(x_1, x_2) g_1(x_1 + a, x_2)^{-1} = P_1(x_2)^{1/L}. \tag{4.33}$$

This means that $\hat{U}_1(x_1, x_2)$ in (4.28) is given by a gauge transformation of (the L -th root of) the Polyakov line in the adjoint representation:

$$\hat{U}_1(x_1, x_2) = \hat{g}_1(x_1, x_2)^{-1} \hat{P}_1(x_2)^{1/L} \hat{g}_1(x_1 + a, x_2). \tag{4.34}$$

Recall that the hatted variables mean the adjoint representation of the corresponding unitary matrices as mentioned around (4.30).

The eigenvalues of the adjoint Polyakov line $\hat{P}_1(x_2)$ are given by

$$p_1(x_2, \alpha) \in S^1, \quad \text{for } \alpha = 1, \dots, N^2 - 1. \tag{4.35}$$

Then, $\hat{P}_1(x_2)$ can be diagonalized as

$$\hat{P}_1(x_2) = W_1(x_2)^{-1} \text{diag} \left\{ p_1(x_2, 1), p_1(x_2, 2) \cdots, p_1(x_2, N^2 - 1) \right\} W_1(x_2), \tag{4.36}$$

where $W_1(x_2)$ is a unitary matrix with the size of $N^2 - 1$.

Combining (4.34) and (4.36) into $V_{\alpha\beta}(x, y) = \delta_{x_1, y_1} \delta_{x_2, y_2} (W_1(x_2) \hat{g}_1(x_1, x_2))_{\alpha\beta}$, we can diagonalize $\mathcal{R}_1^{(+)}$ in (4.28) with respect to the color index α :

$$(V \mathcal{R}_1^{(+)} V^{-1})_{\alpha\beta}(x, y) = \delta_{x_2, y_2} \delta_{\alpha\beta} \mathcal{D}_{x_2, \alpha}(x_1, y_1), \tag{4.37}$$

where

$$\mathcal{D}_{x_2, \alpha}(x_1, y_1) = A \delta_{x_1, y_1} + B \delta_{x_1 + a, y_1} + C \delta_{x_1 - a, y_1}, \tag{4.38}$$

with

$$A = \frac{1}{2} + 2r, \quad B = \left(\frac{1}{2} - r\right) (p_1(x_2, \alpha))^{\frac{1}{L}}, \quad C = -r(p_1(x_2, \alpha))^{-\frac{1}{L}}. \tag{4.39}$$

Now, the computation of $\det \mathcal{R}_1^{(+)}$ reduces to evaluating the determinant of $L \times L$ matrix $\mathcal{D}_{x_2, \alpha}$:

$$\det \mathcal{R}_1^{(+)} = \prod_{x_2=a}^{La} \prod_{\alpha=1}^{N^2-1} \det(\mathcal{D}_{x_2, \alpha}), \tag{4.40}$$

where $\mathcal{D}_{x_2, \alpha}$ in (4.38) is expressed in the form of

$$\mathcal{D}_{x_2, \alpha} = \begin{pmatrix} A & B & & C \\ C & \ddots & \ddots & \\ & \ddots & \ddots & B \\ B & & C & A \end{pmatrix}. \tag{4.41}$$

The determinant of the circulant matrix (4.41) can be evaluated straightforwardly (see Appendix D) and the result is

$$\det(\mathcal{D}_{x_2, \alpha}) = \xi_+^L + \xi_-^L - (-B)^L - (-C)^L, \tag{4.42}$$

where $\xi_{\pm} = \frac{1}{2} \left(A \pm \sqrt{A^2 - 4BC} \right)$.

We thus find that

$$\det \mathcal{R}_1^{(+)} = \prod_{x_2=a}^{La} \prod_{\alpha=1}^{N^2-1} (F(r, L) - G(r, L, p_1(x_2, \alpha))), \tag{4.43}$$

where

$$F(r, L) \equiv \left(\frac{1 + 4r + \sqrt{1 + 16r}}{4} \right)^L + \left(\frac{1 + 4r - \sqrt{1 + 16r}}{4} \right)^L, \tag{4.44}$$

$$G(r, L, p) \equiv \left(r - \frac{1}{2} \right)^L p + r^L p^{-1}. \tag{4.45}$$

$G(r, L, p_1(x_2, \alpha))$ carry information of the gauge fields via the phase factor $p_1(x_2, \alpha)$ (4.35) whereas $F(r, L)$ does not. We can say that $\det \mathcal{R}_1^{(+)}$ is irrelevant in the continuum limit if G becomes negligible compared with F as $a \rightarrow 0$ (keeping $\ell = La$ fixed). It is straightforward to see that $G/F = \mathcal{O}(e^{-\ell/a})$ as $a \rightarrow 0$ for $r > 0$. We can repeat the same argument for $\det \mathcal{R}_2^{(+)}$ and obtain the same conclusion.

Thus, we reach the final point: *For $r > 0$, the Jacobian factor $\prod_{\mu} \det \mathcal{R}_{\mu}^{(\pm)}$ is irrelevant in the continuum limit, and both of the measures (4.11) and (4.12) can be used to define the $O(a)$ -improved theory.*¹⁶

¹⁶ If radiative corrections are taken into account, the speed of the convergence to the continuum limit could be different.

5. Conclusion and discussions

In this paper we have discussed tree-level $\mathcal{O}(a)$ improvement of a lattice formulation of 2d $\mathcal{N} = (2, 2)$ super Yang–Mills theory introduced by one of the authors (F. S.) [7]. Technically important ingredient is the improvement of the Q -transformation, the term Ξ in (2.27) and the path-integral measure. The problem of zero-modes arising in improving the Q -transformation is resolved by the use of a kind of the Wilson terms, which leads to the action containing exponentially local terms in general.

In the framework of the Symanzik $\mathcal{O}(a)$ improvement [46,47], we should also consider the improvement of the effective action at the loop level. Namely, we have to determine $\mathcal{O}(a)$ counterterms to be added to the lattice action so that they cancel all of $\mathcal{O}(a)$ radiative corrections. Thanks to the superrenormalizable property of the theory, investigation at the one-loop level is sufficient. This will be reported in the forthcoming publication.

Needless to say, the most important application of this method is the actual numerical simulations. It is interesting to see how the tree-level improvement discussed here accelerates the simulation and how the addition of the one-loop improvement changes the situation.

The same idea can be applied to other theories, with different amount of supersymmetries and/or with various matter fields. In particular, the application to a similar lattice formulation of 2d $\mathcal{N} = (4, 4)$, $(8, 8)$ super Yang–Mills theories that preserves two supercharges [7,8] should be straightforward. It would be interesting to consider if the improvement can be generalized to other types of supersymmetric lattice formulations [3–5,9]. To a plane wave deformation of 2d $\mathcal{N} = (8, 8)$ super Yang–Mills theory on lattice [37,38], from which 4d $\mathcal{N} = 4$ SYM can be obtained without parameter fine tunings, the application of the improvement is worth being investigated.

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Appendix A. Proof of Lemma

In this appendix, we prove the lemma in section 3.2. Since the essential points are not lost by suppressing Lorentz and sites indices, we show it without them:

Lemma 1. *Let f be a function that depends on (U, ψ, ϕ) with the same gauge transformation property as U , and suppose that*

$$QU = f(U, \psi, \phi). \tag{A.1}$$

Let g be a function that depends on (U, QU, ϕ) with the same gauge transformation property as ψ . If (A.1) can be solved in term of ψ as follows:

$$\psi = g(U, QU, \phi), \tag{A.2}$$

then one can consistently define the Q -transformation by (A.1) and

$$Q\psi = \left[QU \cdot \frac{\delta}{\delta U} - i\delta_\phi U \cdot \frac{\delta}{\delta(QU)} \right] g(U, QU, \phi), \tag{A.3}$$

$$Q\phi = 0, \tag{A.4}$$

so that $Q^2 = -i\delta_\phi$ for all fields.

Proof. We immediately confirm that $Q^2 = -i\delta_\phi$ for ϕ since the both sides vanish from (A.4) and the gauge transformation $\delta_\phi\phi = 0$. The main task is to show the lemma for U and ψ .

Acting Q to (A.1) leads to

$$Q^2U = \left[QU \cdot \frac{\delta}{\delta U} + Q\psi \cdot \frac{\delta}{\delta\psi} \right] f(U, \psi, \phi). \tag{A.5}$$

Plugging (A.3) into (A.5), we have

$$Q^2U = -i\delta_\phi U \cdot \left\{ \frac{\delta}{\delta(QU)} g \cdot \frac{\delta}{\delta\psi} f \right\} + QU \cdot \left\{ \frac{\delta}{\delta U} f + \frac{\delta}{\delta U} g \cdot \frac{\delta}{\delta\psi} f \right\}. \tag{A.6}$$

Once the variable ψ in f is eliminated by using (A.2), f is a function of U, QU, ϕ as $f(U, g(U, QU, \phi), \phi)$ which does not actually depend on U and ϕ because of (A.1). It means that, in the r.h.s. of (A.6), the expression inside of the first curly bracket is one and the second one vanishes. Thus, we obtain

$$Q^2U = -i\delta_\phi U. \tag{A.7}$$

By operating a bosonic transformation Q^2 to (A.2), we find

$$Q^2\psi = \left[Q^2U \cdot \frac{\delta}{\delta U} + Q^3U \cdot \frac{\delta}{\delta(QU)} \right] g(U, QU, \phi). \tag{A.8}$$

Note that $Q^3U = -iQ\delta_\phi U = -i\delta_\phi(QU)$ from (A.7). It is found that

$$Q^2\psi = -i\delta_\phi\psi. \tag{A.9}$$

(A.7) and (A.9) establish the statement. \square

The remarkable point of the proof above is that it relies only on the algebraic structure of Q -transformation. This kind of argument is, therefore, applicable beyond the framework of the lattice gauge theory. In particular, U is not needed to be a unitary variable, and x -space is not limited to the lattice. If the gauge symmetry is realized in a framework, this lemma allows us to construct the Q -transformation satisfying $Q^2 = -i\delta_\phi$ in it.

Appendix B. \mathcal{R}_μ and \mathcal{R}_{12}

We present several \mathcal{R}_μ and \mathcal{R}_{12} which can be used for improving the lattice Q -transformation and the lattice field tensor, respectively.

B.1. \mathcal{R}_μ

In sections 3.2 and 4.4, we have explained that \mathcal{R}_μ should obey several conditions: it behaves as $1 + \frac{a}{2}D_\mu$ near the continuum limit, and both of \mathcal{R}_μ and its inverse \mathcal{R}_μ^{-1} are local. In addition, the gauge-field dependence of $\det(\mathcal{R}_\mu)$ rapidly decays when taking the continuum limit.

There are actually infinite solutions for \mathcal{R}_μ . We list a few types of \mathcal{R}_μ that satisfy the conditions above:

$$\mathcal{R}_\mu^{(+)} \equiv 1 + \frac{a}{2}\nabla_\mu - ra^2\nabla_\mu^*\nabla_\mu \quad (r > 0), \tag{B.1}$$

$$\mathcal{R}_\mu^{(-)} \equiv \left(1 - \frac{a}{2}\nabla_\mu^* - ra^2\nabla_\mu^*\nabla_\mu\right)^{-1} \quad (r > 0), \tag{B.2}$$

$$\mathcal{R}_\mu^{(e)} \equiv \exp\left(\frac{a}{2}\nabla_\mu^{(s)}\right), \tag{B.3}$$

where $\nabla_\mu^{(s)}$ is the covariant symmetric difference operator,

$$\nabla_\mu^{(s)} \equiv \frac{1}{2}(\nabla_\mu + \nabla_\mu^*). \tag{B.4}$$

These operators have different properties as explained below.

The first one (B.1) is the simplest ultra-local solution which is given in section 3.2. For $r = \frac{1}{4}$ and $r = \frac{1}{2}$, (B.1) becomes further simple depending only on the symmetric and the backward difference operators linearly, respectively. From the fact that the eigenvalue of $a\nabla_\mu$ is expressed in the form (3.14), $\mathcal{R}_\mu^{(+)}$ is bounded in the sense that the absolute value of the singular values has the upper and non-zero lower bounds. Namely,

$$u \leq \left(\mathcal{R}_\mu^{(+)}\right)^\dagger \mathcal{R}_\mu^{(+)} \leq v, \quad \text{for } u, v > 0, \tag{B.5}$$

where the inequality is understood as for the eigenvalues of $\mathcal{R}^\dagger\mathcal{R}$. Lemma 2 given in section B.3 tells us that the ultra-local and bounded operator has a local inversion with an exponentially decaying tail:

$$\left|(\mathcal{R}_\mu^{(+)}\right)^{-1}(x, y)\left| \leq C e^{-\rho|x_\mu-y_\mu|/a}, \tag{B.6}$$

where C and ρ are positive constants. So, we can make use of this operator with keeping the principle of locality. As shown in section 4.4, $\det(\mathcal{R}_\mu^{(+)})$ is Q -invariant up to a strongly suppressed term in the continuum limit $\exp(-l/a)$ (with the physical size l is fixed in the limit).

In this way, the operator (B.1) correctly improves the Q -transformation and is useful to explain the method of the tree-level $\mathcal{O}(a)$ improvement. However, the numerical application will not be easy since the lattice action has an exponentially local interactions because $Q\psi_\mu$ yields the factor $(\mathcal{R}_\mu^{(+)})^{-1}$ that spreads all over the lattice.

Instead, the second one (B.2) provides an ultra-local action which is suitable for numerical applications. As discussed in section 4.2, with the change of variables in the fermion fields ψ_μ , the lattice action is actually expressed as an ultra-local expression. This is because $(\mathcal{R}_\mu^{(-)})^{-1}$ is taken to be ultra-local. Since $(\mathcal{R}_\mu^{(-)})^{-1} = (\mathcal{R}_\mu^{(+)})^T$ is derived from $\nabla_\mu^T = -\nabla_\mu^*$, $\mathcal{R}_\mu^{(-)}$ satisfies the exponentially locality condition such as (B.6). As seen in section 4.4, the fermion measure is also consistently defined as long as $r > 0$. In this case, we also obtain simple expressions for $r = \frac{1}{4}$ and $r = \frac{1}{2}$. The latter is convenient in writing the improved action explicitly.

The third choice $\mathcal{R}_\mu^{(e)}$ looks complicated for numerical applications but is worth noting since it has a trivial determinant:

$$\det \mathcal{R}_\mu^{(e)} = 1, \tag{B.7}$$

that is, the measures (4.11) and (4.12) are identical and Q^{imp} -invariant. (B.7) is easily shown from $(\nabla_\mu^{(s)})^T = -\nabla_\mu^{(s)}$, or equivalently, from the fact that $\mathcal{R}_\mu^{(e)}$ is an orthogonal matrix whose determinant is +1 or -1. The sign is fixed because (B.7) holds in the free limit and the sign does not change under the continuous deformation of the link fields. Lemma 3 in Appendix B.3 allows us to conclude that

$$\left| \mathcal{R}_\mu^{(e)}(x, y) \right| \leq \frac{C e^{-\rho |x_\mu - y_\mu|/a}}{(|x_\mu - y_\mu|/a)!}, \tag{B.8}$$

with positive constants C and $\rho = \ln 2$. Due to the factorial growth of the denominator, $\mathcal{R}_\mu^{(e)}$ decays much faster than general exponentially local operators as $|x_\mu - y_\mu|/a \rightarrow \infty$.

B.2. \mathcal{R}_{12}

The lattice field tensor is improved by multiplying \mathcal{R}_{12} which obeys the several conditions as with \mathcal{R}_μ . As seen in section 3.3, \mathcal{R}_{12} should behaves as $1 - \frac{a}{2}(D_1 + D_2)$ near the continuum limit, and should be local and invertible to lead to the correct continuum theory around the unique vacuum. There are also infinite possibilities satisfying these conditions.

We give a few \mathcal{R}_{12} which are similar to \mathcal{R}_μ (B.1)–(B.3):

$$\mathcal{R}_{12}^{(+)} \equiv \left[1 + \frac{a}{2} \sum_{\mu=1}^2 \nabla_\mu - r a^2 \sum_{\mu=1}^2 \nabla_\mu^* \nabla_\mu \right]^{-1}, \tag{B.9}$$

$$\mathcal{R}_{12}^{(-)} \equiv 1 - \frac{a}{2} \sum_{\mu=1}^2 \nabla_\mu^* - r a^2 \sum_{\mu=1}^2 \nabla_\mu^* \nabla_\mu, \tag{B.10}$$

$$\mathcal{R}_{12}^{(e)} \equiv \exp \left(-\frac{a}{2} \sum_{\mu=1}^2 \nabla_\mu^{(s)} \right). \tag{B.11}$$

These operators coincide with each other up to $\mathcal{O}(a)$, that is, they improve Φ_{TL} to the order as shown in section 3.3.¹⁷ We can show that (B.9) and the inversion of (B.10) exist for $r > 1/4$ as follows. For an operator Z_μ defined as

¹⁷ These operators maintain the reflection symmetry of the original lattice action (2.27): $x = (x_1, x_2) \rightarrow \tilde{x} \equiv (x_2, x_1)$ with

$$\begin{aligned} (U_1(x), U_2(x)) &\rightarrow (U_2(\tilde{x}), U_1(\tilde{x})) \\ (\psi_1(x), \psi_2(x)) &\rightarrow (\psi_2(\tilde{x}), \psi_1(\tilde{x})) \\ (H(x), \chi(x)) &\rightarrow (-H(\tilde{x}), -\chi(\tilde{x})) \\ (\phi(x), \bar{\phi}(x), \eta(x)) &\rightarrow (\phi(\tilde{x}), \bar{\phi}(\tilde{x}), \eta(\tilde{x})). \end{aligned} \tag{B.12}$$

$$Z_\mu \varphi(x) = r U_\mu(x) \varphi(x + a\hat{\mu}) U_\mu(x)^{-1} - \left(\frac{1}{2} - r\right) U_\mu(x - a\hat{\mu})^{-1} \varphi(x - a\hat{\mu}) U_\mu(x - a\hat{\mu}) \tag{B.13}$$

for any adjoint field $\varphi(x)$, whose induced norm¹⁸ is less than or equal to $|r| + |\frac{1}{2} - r|$. Then,

$$\mathcal{R}_{12}^{(-)} = 4r - Z_1 - Z_2 \tag{B.14}$$

are shown to have no zeros for $r > 1/4$. The existence of (B.9) obeys $(\mathcal{R}_{12}^{(+)})^{-1} = (\mathcal{R}_{12}^{(-)})^T$. Note that (B.9) and (B.11) are local operators from Lemmas 2 and 3 given in the next subsection.

For $r = \frac{1}{2}$, $\mathcal{R}_{12}^{(\pm)}$ reduce to simpler expressions given with the backward (or the forward) operators, respectively. In particular, $\mathcal{R}_{12}^{(-)}$ is convenient to write the ultra-local action with $\mathcal{R}_\mu^{(-)}$ (B.2) as given in Appendix C.

B.3. Locality of \mathcal{R}

The locality of operators is discussed in detail. We present the locality conditions for the inverse (the exponential) of an ultra-local operator in Lemma 2 (Lemma 3). These give the solid theoretical grounds of the tree-level $\mathcal{O}(a)$ improvement with \mathcal{R}_μ and \mathcal{R}_{12} in this paper.

For an ultra-local operator R , it is easy to show that R^\dagger and $R^\dagger R$ are also ultra-local. In Lemma 2, we show that if $R^\dagger R$ has the upper and non-zero lower bounds, R^{-1} satisfies the exponential locality condition. Although this can be shown by applying an argument in [55], we present a proof to make this paper self-contained as much as possible.

Lemma 2. *If R is an ultra-local operator that satisfies, for $u, v > 0$,*

$$u \leq R^\dagger R \leq v, \tag{B.15}$$

where the inequality stands for the eigenvalues, R^{-1} is exponentially local.

Proof. The locality of R^{-1} follows from that of $(R^\dagger R)^{-1}$ because $R^{-1} = (R^\dagger R)^{-1} R^\dagger$. To study the locality of $(R^\dagger R)^{-1}$, let us set

$$Z \equiv \frac{2}{u-v} \left(R^\dagger R - \frac{u+v}{2} \right), \tag{B.16}$$

whose eigenvalues have the absolute value not exceeding one. Then, with $e^{-\theta} \equiv \frac{v-u}{u+v} < 1$, we can show

$$(R^\dagger R)^{-1} = \frac{2}{u+v} \sum_{n=0}^{\infty} e^{-n\theta} Z^n, \tag{B.17}$$

by expanding $(R^\dagger R)^{-1} = \frac{2}{u+v} (1 - e^{-\theta} Z)^{-1}$ with respect to Z .

¹⁸ Z_μ and φ can be regarded as a matrix of the size $\mathcal{M} = (N^2 - 1)L^2$ and an \mathcal{M} -dimensional vector, respectively. For any non-zero \mathcal{M} -dimensional vector \vec{u} with a norm $\|\vec{u}\| = \sqrt{\vec{u}^\dagger \vec{u}}$, the induced norm of an $\mathcal{M} \times \mathcal{M}$ matrix A is defined by $\|A\|_{\text{ind}} \equiv \max_{\vec{u}} \frac{\|A\vec{u}\|}{\|\vec{u}\|}$.

Since Z is ultra-local from the assumption, there is a positive constant M such that $(Z^n)(x, y) = 0$ for $n < M\|x - y\|_1/a$.¹⁹ Thus,

$$(R^\dagger R)^{-1}(x, y) = \frac{2}{u + v} \sum_{n=M\|x-y\|_1/a}^{\infty} e^{-n\theta} Z^n(x, y). \tag{B.18}$$

We can easily show that²⁰

$$|Z_{\alpha\beta}(x, y)| \leq 1, \tag{B.20}$$

and finally obtain

$$|(R^\dagger R)^{-1}_{\alpha\beta}(x, y)| \leq C e^{-\rho\|x-y\|_1/a}, \tag{B.21}$$

where C and $\rho = M\theta$ are constants independent of the lattice spacing. Namely, $(R^\dagger R)^{-1}$ is exponentially local. So, we can conclude that R^{-1} satisfies the locality in the same sense. \square

The operator e^R is also useful to understand our formulation. The following lemma tells us the condition on R under which e^R is local.

Lemma 3. *If R is an ultra-local operator whose singular values have the upper bound $w > 0$, e^R is local in the sense that at long distance it decays faster than the exponential.*

Proof. Instead of (B.17), we have

$$e^R = \sum_{n=0}^{\infty} \frac{1}{n!} R^n. \tag{B.22}$$

Since $(R^n)(x, y) = 0$ for $n < M\|x - y\|_1/a$ for a positive constant M , the kernel representation of (B.22) is

$$e^R(x, y) = \sum_{n=M\|x-y\|_1/a}^{\infty} \frac{1}{n!} R^n(x, y). \tag{B.23}$$

Noting that $|(R)_{\alpha\beta}(x, y)| \leq w$ from the assumption, we obtain

$$|(e^R)_{\alpha\beta}(x, y)| \leq C \frac{w^{M\|x-y\|_1/a}}{(M\|x - y\|_1/a)!}, \tag{B.24}$$

¹⁹ $M \sim 1/r$ where r is the range of the ultra-locality of Z .

²⁰ Let W be an $\mathcal{M} \times \mathcal{M}$ matrix whose singular values have the upper bound w . For a norm of complex vectors $\|\vec{u}\| \equiv \sqrt{\vec{u}^\dagger \vec{u}}$, it is found that $\|W\vec{u}\| \leq w\|\vec{u}\|$ holds for any vector $\vec{u} \in \mathbb{C}^{\mathcal{M}}$. Then, for the matrix elements of W , we can easily see

$$\sqrt{\sum_{I=1}^{\mathcal{M}} |W_{IJ}|^2} \leq w \quad \text{for all } J, \tag{B.19}$$

by taking the unit vectors $(\vec{e}_J)_I \equiv \delta_{IJ}$ as \vec{u} , and find that $|W_{IJ}| \leq w$.

where C is a positive constant. Using an identity $\log(n!) \geq \kappa n \log n$ for $n \gg 1$ with $0 < \exists \kappa < 1$, we find that

$$|(e^R)_{\alpha\beta}(x, y)| \lesssim C' e^{-\kappa \frac{\|x-y\|_1}{a} \log(\frac{\|x-y\|_1}{a})}, \tag{B.25}$$

at large distance $\|x - y\|_1/a \gg 1$, where C' and κ are positive constants. Since $e^{-n \log n}$ is larger than e^{-n} for $n \gg 1$, e^R decays faster than the exponential at large distance. \square

Appendix C. Explicit form of the lattice action

In this appendix, we present the explicit form of the lattice action (4.1) after performing Q^{imp} -transformations. Before seeing it, we also present the case of the unimproved lattice action to clarify the differences arising in the improvement.

The action is divided as

$$S = S_B + S_F + S_Y, \tag{C.1}$$

where S_B is the boson action, and S_F and S_Y are the fermion actions which include kinetic terms and the Yukawa interaction terms, respectively. In the continuum theory with the twisted variables,

$$S_B = \frac{1}{2g^2} \int d^2x \operatorname{tr} \left\{ H^2 - 2i H F_{12} + \sum_{\mu=1}^2 D_\mu \phi D_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 \right\}, \tag{C.2}$$

$$S_F = \frac{1}{2g^2} \int d^2x \operatorname{tr} \left\{ i \sum_{\mu=1}^2 \psi_\mu D_\mu \eta + 2i \chi (D_1 \psi_2 - D_2 \psi_1) \right\}, \tag{C.3}$$

$$S_Y = \frac{1}{2g^2} \int d^2x \operatorname{tr} \left\{ -\frac{1}{4} \eta [\phi, \eta] - \chi [\phi, \chi] + \sum_{\mu=1}^2 \psi_\mu [\bar{\phi}, \psi_\mu] \right\}. \tag{C.4}$$

In contrast to the continuum theory, this classification would not be strict on the lattice since the lattice action has higher order terms whose types are unclear. At least, we will present one possibility for the lattice actions $S_{\text{lat}B}$, $S_{\text{lat}F}$ and $S_{\text{lat}Y}$ such that they reproduce the continuum counterparts S_B , S_F and S_Y , respectively.

In the unimproved lattice model, we have

$$S_{\text{lat}B} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \operatorname{tr} \left\{ H^2 - \frac{i}{a^2} H \Phi_{\text{TL}} + \sum_{\mu=1}^2 \nabla_\mu \phi \nabla_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 \right\}, \tag{C.5}$$

$$S_{\text{lat}F} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \operatorname{tr} \left\{ i \sum_{\mu=1}^2 \psi_\mu \nabla_\mu \eta + \frac{i}{a^2} \chi Q \Phi_{\text{TL}} \right\}, \tag{C.6}$$

$$S_{\text{lat}Y} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \operatorname{tr} \left\{ -\frac{1}{4} \eta [\phi, \eta] - \chi [\phi, \chi] + \frac{1}{2} \sum_{\mu=1}^2 \psi_\mu [U_\mu \bar{\phi} U_\mu^{-1} + \bar{\phi}, \psi_\mu] \right\}. \tag{C.7}$$

The term $Q\Phi_{\text{TL}}$ is quite complicated. Instead of giving it explicitly, let us present $Q\Phi$ since $Q\Phi_{\text{TL}}(x) = Q\Phi(x) - \{\frac{1}{N} \operatorname{tr}(Q\Phi(x))\} \mathbf{1}_N$. For Φ_1 and Φ_2 , which are given in (2.32) and (2.34), respectively, $Q\Phi$ is given by

$$Q\Phi_1(x) = \frac{a^2}{1 - \frac{1}{\epsilon^2} \|1 - U_{12}(x)\|^2} \times \left\{ \Psi_{12}(x) - \Psi_{21}(x) - \frac{i}{\epsilon^2} \text{tr} \{ \Psi_{12}(x) + \Psi_{21}(x) \} \Phi_1(x) \right\}, \tag{C.8}$$

for $\|1 - U_{12}(x)\| < \epsilon$, and

$$Q\Phi_2(x) = \frac{2a^2}{m} (U_{12}(x)^m + U_{21}(x)^m)^{-1} \times \sum_{k=0}^{m-1} \left\{ U_{12}(x)^k \Psi_{12}(x) U_{12}(x)^{m-k-1} - U_{21}(x)^k \Psi_{21}(x) U_{21}(x)^{m-k-1} - \frac{im}{2} \left(U_{12}(x)^k \Psi_{12}(x) U_{12}(x)^{m-k-1} + U_{21}(x)^k \Psi_{21}(x) U_{21}(x)^{m-k-1} \right) \Phi_2(x) \right\}, \tag{C.9}$$

where

$$\Psi_{\mu\nu}(x) \equiv \nabla_\mu \psi_\nu(x) U_{\mu\nu}(x) - U_{\mu\nu}(x) \nabla_\nu \psi_\mu(x) + \frac{1}{a} [\psi_1(x) + \psi_2(x), U_{\mu\nu}(x)]. \tag{C.10}$$

Note that $QU_{\mu\nu}(x) = ia^2\Psi_{\mu\nu}(x)$.

For the improved lattice action, we use the variable ψ'_μ in (4.10) that makes the expression as simple as possible. For some special cases, the action is written in an ultra-local form in terms of ψ'_μ . The Q^{imp} -transformations (4.5)–(4.9) applied to (4.14) leads to

$$S_{\text{lat B}}^{imp} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left\{ H^2 - \frac{i}{a^2} H \mathcal{R}_{12} \Phi_{\text{TL}} + \sum_{\mu=1}^2 \nabla_\mu \phi \mathcal{S}(\mathcal{R}_\mu) \nabla_\mu \bar{\phi} + \frac{1}{4} [\phi, \bar{\phi}]^2 \right\}, \tag{C.11}$$

$$S_{\text{lat F}}^{imp} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left\{ i \sum_{\mu=1}^2 \psi'_\mu \mathcal{S}(\mathcal{R}_\mu) \nabla_\mu \eta + \frac{i}{a^2} \chi \mathcal{R}_{12} Q^{imp} \Phi_{\text{TL}} + \frac{i}{a^2} \chi (Q^{imp} \mathcal{R}_{12}) \Phi_{\text{TL}} \right\}, \tag{C.12}$$

$$S_{\text{lat Y}}^{imp} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left\{ -\frac{1}{4} \eta [\phi, \eta] - \chi [\phi, \chi] + i \sum_{\mu=1}^2 \psi'_\mu (Q^{imp} \mathcal{S}(\mathcal{R}_\mu)) \nabla_\mu \bar{\phi} + \sum_{\mu=1}^2 \psi'_\mu [U_\mu \bar{\phi} U_\mu^{-1}, \mathcal{S}(\mathcal{R}_\mu) \psi'_\mu] - \frac{a}{2} \sum_{\mu=1}^2 \psi'_\mu [\mathcal{S}(\mathcal{R}_\mu) \nabla_\mu \bar{\phi}, \psi'_\mu] \right\}, \tag{C.13}$$

where

$$\mathcal{S}(\mathcal{R}) = (\mathcal{R} \mathcal{R}^T)^{-1}, \tag{C.14}$$

and $Q^{imp} \Phi_{\text{TL}}$ for $\Phi_i (i = 1, 2)$ are given by $Q\Phi$ in (C.8) and (C.9) with ψ_μ replaced by ψ'_μ . The Q^{imp} -transformations for \mathcal{R}_μ and \mathcal{R}_{12} remain unperformed since they are not determined in general.

The improved action becomes ultra-local for $\mathcal{R}_\mu^{(-)}$ and $\mathcal{R}_{12}^{(-)}$ given in (B.2) and (B.10). In the case of

$$\mathcal{R}_\mu^{(-)} \Big|_{r=\frac{1}{2}} = \left(1 - \frac{a}{2} \nabla_\mu\right)^{-1}, \tag{C.15}$$

$$\mathcal{R}_{12}^{(-)} \Big|_{r=\frac{1}{2}} = 1 - \frac{a}{2} \sum_{\mu=1}^2 \nabla_\mu, \tag{C.16}$$

we find that

$$S_{\text{lat B}}^{\text{imp}} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left\{ H^2 - \frac{i}{a^2} H \left(2\Phi_{\text{TL}} - \frac{1}{2} \sum_{\mu=1}^2 U_\mu \Phi_{\text{TL}} U_\mu^{-1} \right) + \frac{1}{4} \sum_{\mu=1}^2 (3\nabla_\mu \phi - \nabla_\mu^* \phi) (3\nabla_\mu \bar{\phi} - \nabla_\mu^* \bar{\phi}) + \frac{1}{4} [\phi, \bar{\phi}]^2 \right\}, \tag{C.17}$$

$$S_{\text{lat F}}^{\text{imp}} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left\{ i \sum_{\mu=1}^2 \psi'_\mu \left(1 - \frac{3a^2}{4} \nabla_\mu \nabla_\mu^* \right) \nabla_\mu \eta + \frac{i}{a^2} \chi \left(1 - \frac{a}{2} \sum_{\mu=1}^2 \nabla_\mu \right) Q^{\text{imp}} \Phi_{\text{TL}} - \frac{1}{2a} \sum_{\mu=1}^2 \chi \left[U_\mu \Phi_{\text{TL}} U_\mu^{-1}, \psi'_\mu \right] \right\}, \tag{C.18}$$

$$S_{\text{lat Y}}^{\text{imp}} = \frac{a^2}{2g^2} \sum_{x \in \Lambda_L} \text{tr} \left\{ -\frac{1}{4} \eta [\phi, \eta] - \chi [\phi, \chi] + \frac{1}{8} \sum_{\mu=1}^2 \psi'_\mu \left[13\bar{\phi} + 13U_\mu \bar{\phi} U_\mu^{-1} - 3U_\mu^{-1} \bar{\phi} U_\mu - 3U_\mu U_\mu \bar{\phi} U_\mu^{-1} U_\mu^{-1}, \psi'_\mu \right] - \frac{3}{4} \sum_{\mu=1}^2 \psi'_\mu \left[U_\mu \bar{\phi} U_\mu^{-1} + U_\mu^{-1} \bar{\phi} U_\mu, U_\mu^{-1} \psi'_\mu U_\mu \right] \right\}, \tag{C.19}$$

where $Q^{\text{imp}} \Phi_{\text{TL}}$ for $\Phi_i (i = 1, 2)$ are again given by $Q\Phi$ in (C.8) and (C.9) with the replacement of ψ_μ by ψ'_μ . These actions are clearly ultra-local and suitable for numerical simulations.

We can show that the Yukawa interactions of the improved lattice action coincide with those of the unimproved one for (B.3) since $\mathcal{S}(\mathcal{R}_\mu^{(e)}) = 1$. Then we can show that

$$S_{\text{lat B}}^{\text{imp}} = S_{\text{lat B}} |_{\Phi \rightarrow \Phi^{\text{imp}}}, \tag{C.20}$$

$$S_{\text{lat F}}^{\text{imp}} = S_{\text{lat F}} |_{\psi_\mu \rightarrow \psi'_\mu, Q\Phi \rightarrow Q^{\text{imp}} \Phi^{\text{imp}}}, \tag{C.21}$$

$$S_{\text{lat Y}}^{\text{imp}} = S_{\text{lat Y}} |_{\psi_\mu \rightarrow \psi'_\mu}, \tag{C.22}$$

where

$$\Phi^{\text{imp}} = \mathcal{R}_{12} \Phi. \tag{C.23}$$

The difference is only from the definition of the improved lattice field tensor Φ^{imp} given via \mathcal{R}_{12} . Furthermore, if we use (B.11) for \mathcal{R}_{12} and integrate the auxiliary field H , the factor \mathcal{R}_{12} disappears in the boson action since $\mathcal{S}(\mathcal{R}_{12}^{(e)}) = 1$. In other words, the improved actions are the same

with those of the unimproved theory for the boson and the Yukawa interactions. The difference remains only in the fermion kinetic term relevant to χ . This would have some theoretical importance and should be studied further.

Appendix D. Evaluation of the determinant in (4.42)

In this appendix, we calculate the determinant of the $L \times L$ circulant matrix,

$$R_L = \begin{pmatrix} A & B & & C \\ C & \ddots & \ddots & \\ & \ddots & \ddots & B \\ B & & C & A \end{pmatrix}, \tag{D.1}$$

for constant A, B and C . To this end, it is convenient to introduce a purely tridiagonal matrix:

$$Q_L \equiv \begin{pmatrix} A & B & & \\ C & \ddots & \ddots & \\ & \ddots & \ddots & B \\ & & C & A \end{pmatrix}. \tag{D.2}$$

In computing $\det R_L$, the cofactor expansion with respect to the first and second rows or columns gives

$$\det R_L = A \det Q_{L-1} - 2BC \det Q_{L-2} - (-B)^L - (-C)^L. \tag{D.3}$$

Similarly,

$$\det Q_L = A \det Q_{L-1} - BC \det Q_{L-2}. \tag{D.4}$$

Defining the solutions of the quadratic equation $x^2 - Ax + BC = 0$ as

$$\xi_{\pm} \equiv \frac{1}{2} \left(A \pm \sqrt{A^2 - 4BC} \right), \tag{D.5}$$

the recursion equation (D.4) is solved as

$$\det Q_L = \begin{cases} (L+1)\xi_+^L & \text{if } A^2 = 4BC \\ \frac{\xi_+^{L+1} - \xi_-^{L+1}}{\xi_+ - \xi_-} & \text{otherwise.} \end{cases} \tag{D.6}$$

Plugging this to (D.3) leads to the simple expression

$$\det R_L = \xi_+^L + \xi_-^L - (-B)^L - (-C)^L. \tag{D.7}$$

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