数理解析研究所講究録
第2002巻 2016年 19-24

On Lang-type property and the superstability

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Abstract

In [3], we defined quantum 2-tori T_q^2 for q not a root of unity, and showed its first-order theory is superstable. One of key theorems in [3] is Theorem 13. In this note we give some details of the proof of superstability which is an application of Lang-type property. [1]

1 Introduction

One of typical examples of structures with superstable theorems is $(\mathbb{Z}, +, 0)$, where $\mathbb{Z}$ is the set of all integers. The structure is also a typical example of *one-based* group.

Recall first the notion of *descending chain condition* for groups which decides the stability spectrum of groups.

Definition 1 (Definition 5.1, p.92 [1]) i) A group G satisfies the *ω-stable descending chain condition* if there is no infinitely properly descending chain of definable subgroups of G.

ii) A group G satisfies the *superstable descending chain condition* if there is no infinitely properly descending chain of definable subgroups of G such that each subgroup has infinite index in its predecessor.

[1] *Mathematics Subject Classification.* Primary 03C30; Secondary 03C45, 03C50. *Keywords and phrases.* quantum 2-torus, Lang-type property, superstability

iii) A group G satisfies the *stable descending chain condition* if there is no infinitely properly descending chain of definable subgroups of G such that each defined by an instance of a single formula $\phi(x;\bar{y})$.

Recall also the key property of a *one-based* group;

Proposition 2 (Proposition 4.2.7 [8], p. 204) *A group G is one-based if and only if for every n and every definable subset $X \subset G^n$, X is a finite boolean combination of cosets of* $\mathrm{acl}(\emptyset)$*-definable subgroups of G.*

From these definitions it is easy to see that $(\mathbb{Z}, +, 0)$ is superstable, but not ω-stable, and one-based.

Now consider the following situation: let $\mathbb{F}$ be an algebraically closed field of characteristic zero and Γ be a multiplicative subgroup of $\mathbb{F}$ generated by a transcendental element of $\mathbb{F}$ such that $(\mathbb{Z}, +, 0) \simeq (\Gamma, \cdot, 1)$. Our objective in this note is to show the superstability of $\mathrm{Th}(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$ that plays a key role in [3] showing the superstability of line bundles.

2 Superstability of $\mathrm{Th}(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$

From now on, $\mathbb{F}$ is an algebraicaaly closed field of characteristic zero, $q \in \mathbb{F}$ and Γ be a multiplicative subgroup of $\mathbb{F}$ generated by q, i.e., $\Gamma = q^{\mathbb{Z}}$. Following the arguments in [3], we show that the first-order theory of $(\mathbb{F}, +, \cdot, 0, 1, \Gamma)$ is axiomatizable and superstable. Key idea is that the predicate $\Gamma(x)$ describes the property of the set $q^{\mathbb{Z}}$ as a multiplicative subgroup with the following Lang-type property.

Definition 3 (Definition 2.3 [6]) *Let K be an algebraically closed field, and A a commutative algebraic group over K and Γ a subgroup of A. We say that (K, A, Γ) is of* Lang-type *if for every $n < \omega$ and every subvariety X (over K) of $A^n = A \times \cdots \times A$ (n times), $X \cap \Gamma^n$ is a finite union of cosets of subgroups of Γ^n.*

The Lang-type property is a generalization of one-based group G; we consider the stability theoretic property of $(\mathbb{F}, +, \cdot, G)$ where G is one-based. The Lang-type property gives us

Proposition 4 (Proposition 2.6 [6]) *Let K be an algebraically closed field, A a commutative algebraic group over K, and Γ a subgroup of*

A. Then (K, A, Γ) *is of Lang-type if and only if* $\mathrm{Th}(K, +, \cdot, \Gamma, a)_{a \in K}$ *is stable and* $\Gamma(x)$ *is one-based.*

Here $\Gamma(x)$ is *one-based* means that for every n and every definable subset $X \subset \Gamma^n$, X is a finite boolean combination of cosets of definable subgroups of Γ^n.

With the above Definition 3 and Proposition 4 in mind, we axiomatize the properties of $(\mathbb{F}, +, \cdot, \Gamma)$ as follows;

Axioms for $(\mathbb{F}, +, \cdot, \Gamma)$

A. 1 Γ satisfies the first order theory of a cyclic group with generator q,

A. 2 (Lang-type) for every n and every variety X of $(\mathbb{F}^*)^n$, $X \cap \Gamma^n$ is a finite union of cosets of definable subgroups of Γ^n.

Let $T_{\mathbb{F},\Gamma}$ denote the set of all logical consequences of the axioms for Γ and ACF_0 the axioms for the algebraically closed fields of characteristic zero.

Lemma 5 *The* Lang-type *property of* $(\mathbb{F}, +, \cdot, \Gamma)$ *is witnessed by its first-order theory.*

Proof: We may suppose X is irreducible. Each such variety $X \subset (\mathbb{F}^*)^n$ is definable by an irreducible polynomial $f(x_1, \cdots, x_n)$ over $\mathbb{F}^*$. Definable cosets of Γ^n are of the form $\overline{\gamma}\Gamma^n = \gamma_1\Gamma \times \cdots \times \gamma_n\Gamma$ where $\gamma_1, \cdots, \gamma_n \in \Gamma(\mathbb{F})$. Hence the sentence "*$X \cap \Gamma^n$ is a finite union of cosets of definable subgroups*" is expressed as

$$(f(x_1, \cdots, x_n) = 0) \wedge \Gamma(x_1) \wedge \cdots \wedge \Gamma(x_n) \quad \longleftrightarrow \quad \bigvee_{i=1}^{N_f} \varphi_i(x_1, \cdots, x_n).$$

Where each $\varphi_i(x_1, \cdots, x_n)$ defines a coset. Crucial point here is that the number N_f of the bound of cosets is computable for each polynomial f. For this note first that for any $k \in \mathbb{N}$ the number of cosets of $q^{k\mathbb{Z}}$ in $q^{\mathbb{Z}}$ is k. Suppose

$$f(x_1, \cdots, x_n) = \sum_{i=0}^{\deg(f)} \overline{a}_i \overline{x}_i^{m_1},$$

where each m_i is a multi index. Let M_i be the sum of multi index m_i. Then the bound N_f of number of cosets is $\deg(f) \cdot \sum_{i=0}^{\deg(f)} M_i$. Therefore the Lang-type property is first-order. ■

Proposition 6 *$T_{\mathbb{F},\Gamma}$ is complete. Hence $T_{\mathbb{F},\Gamma} = \mathrm{Th}(\mathbb{F}, +, \cdot, \Gamma)$.*

Proof: Consider a saturated model $(\mathbb{F}, +, \cdot, \Gamma, q)$ of $T_{\mathbb{F},\Gamma}$. Set $\Gamma(\mathbb{F}) = \{x \in \mathbb{F} : \mathbb{F} \models \Gamma(x)\}$. Let q be an element of $\mathbb{F}$ interpreting the constant. By the axioms for Γ, $q^{\mathbb{Z}} \subset \Gamma(\mathbb{F}) \subset \mathbb{F}$.

Consider a complete type $t_0(x)$ generated by the following set of formulas,

$$t(x) = \{\Gamma(x),\ \exists y(x = qy),\ \exists y(x = q^2 y),\ \cdots\}.$$

By saturation there exists $\gamma_0 \in \Gamma(\mathbb{F})$ realizing $t_0(x)$. Clearly, $\gamma_0 \notin q^{\mathbb{Z}}$. Suppose elements $\gamma_0, \cdots, \gamma_i \in \Gamma(\mathbb{F})$ have been defined. Let $t_{i+1}(x)$ be a complete type generated by the type $t(x)$ and the set

$$\{x \neq \gamma_0^{n_0}, \cdots, x \neq \gamma_i^{n_i} \ : \ n_0, \cdots, n_i \in \mathbb{Z}\}.$$

From saturation, we have $\gamma_{i+1} \in \Gamma(\mathbb{F})$ such that

$$\gamma_{i+1} \notin \bigcup_{l=0}^{i} \gamma_l^{\mathbb{Z}}.$$

In this way by saturation as before we see that there exist $\gamma_0, \gamma_1, \cdots, \gamma_i, \cdots \in \Gamma(\mathbb{F})$ $(i < |\mathbb{F}|)$ such that

$$\Gamma(\mathbb{F}) = q^{\mathbb{Z}} \cup \bigcup_{i<|\mathbb{F}|} \gamma_i^{\mathbb{Z}}.$$

Now take two saturated models $(\mathbb{F}, +, \cdot, \Gamma, q)$ and $(\mathbb{F}', +, \cdot, \Gamma', q')$ of $T_{\mathbb{F},\Gamma}$ of the same cardinality. There is an isomorphism i from $\mathbb{F}$ to $\mathbb{F}'$ sending q to q'. By the above formula for $\Gamma(\mathbb{F})$ and the back-and-forth argument we can extend i to have that $\Gamma(\mathbb{F}) \simeq \Gamma'(\mathbb{F}')$. Hence $(\mathbb{F}, +, \cdot, \Gamma, q)$ and $(\mathbb{F}', +, \cdot, \Gamma', q')$ are isomorphic as saturated models of $T_{\mathbb{F},\Gamma}$. This completes the proof of the completeness of the theory $T_{\mathbb{F},\Gamma}$. ■

For the final step of proof we need:

Theorem 7 (Theorem 1, [12], Sec. 5) *Let Ω be a multiplicative subgroup of $\mathbb{F}\backslash\{0\}$ generated by a transcendental element q. Then $\mathrm{Th}(\mathbb{F}, +, \cdot, \Omega)$ is superstable. Here $\mathbb{F}$ is an algebraically closed field of characteristic zero.*

Outline of proof: We know that $\mathrm{Th}(\mathbb{F},+,\cdot)$ is strongly-minimal, hence ω-stable. We need to show that the cardinality of the complete types in the language $\{+,\cdot,\Gamma\}$ is the same as in the language $\{+,\Gamma\}$. A key lemma for this is the following theorem of H. Mann [5].

Definition 8 (Definition 1, [9]) *Let $c_1x_1+\cdots+c_nx_n=1$ be a linear equation with integer co-efficients. Call its solution $\langle u_1,\cdots,u_n\rangle$ primitive if there is no equation with integer co-efficients $c'_1,\cdots,c_n$ with $|c'|<|c|$, compairing lexicographically, having the solution $\langle u_1,\cdots,u_n\rangle$.*

Theorem 9 (Theorem 1 (H. Mann), [9]) *For any linear equation with integer co-efficients there is no more than finitely many primitive solutions in complex roots of unity.*

Mann's theorem gives us that for any elementary extensions $\mathbb{F}_1$ of $(\mathbb{F},+,\cdot,\Omega(\mathbb{F}))$ and a subset X of $\mathbb{F}_1$ there is a subset X' of $\Omega(\mathbb{F}_1)$ such that $|X'|=|X|+\aleph_0$ and any quantifier-free relation with parameters X between elements of $\Omega(\mathbb{F}_1)$ is equivalent to a quantifier-free relation in the group language with parameters X'.

It follows that the cardinality of the complete types in the language $\{+,\cdot,\Gamma\}$ is the same as in the language $\{+,\Gamma\}$. Hence $\mathrm{Th}(\mathbb{F},+,\cdot,\Omega)$ is superstable. ■ (Theorem 7)

We now proceed to the proof of:

Theorem 10 *$T_{\mathbb{F},\Gamma}$ is superstable.*

Proof: Notice first that the multiplication of $\mathbb{F}$ is an algebraic group and $(\mathbb{F},\cdot,\Gamma)$ is of the Lang-type by A. 2 above. Thus by Proposition 2.6 of [6], we see that $T_{\mathbb{F},\Gamma}$ is at least stable. $T_{\mathbb{F},\Gamma}$ is in fact superstable since

1. the stability spectrum of $T_{\mathbb{F},\Gamma}$ is the same as that of $T_{\Gamma(\mathbb{F})}$, the theory of restriction of $(\mathbb{F},+,\cdot,\Gamma)$ to $\Gamma(\mathbb{F})$. Let $C\subset\mathbb{F}$. Observe first that there is only one complete 1-type over C in $T_{\mathbb{F},\Gamma}$, which is realized by elements in $\mathbb{F}-\mathrm{acl}_{\mathrm{F}}(\Gamma(\mathbb{F})\cup C)$ where $\mathrm{acl}_{\mathrm{F}}$ is the field-theoretic algebraic closure. Hence the cardinality of complete 1-types in $T_{\mathbb{F},\Gamma}$ is bounded by the cardinality of the complete 1-types in $T_{\Gamma(\mathbb{F})}$. Thus they have the same stability spectrum.

2. $T_{\Gamma(\mathbb{F})}$ is superstable. For q transcendental, this is Theorem 7 above. If q is not a root of unity, we have the same conclusion since $q^{\mathbb{Z}}$ is an infinite cyclic group. If q is a root of unity, combined with Proposition 4, we still have the same conclusion. ■

Remark 11 *Let Γ be the group of complex roots of unity, i.e., $\Gamma = \{x \in \mathbb{C} \mid \exists n\ x^n = 1\}$. Then $\mathrm{Th}(\mathbb{C}, +, \cdot, \Gamma)$ is ω-stable. See Theorem 2 in [9] and concluding remarks on p. 105 of [12]. Note that in this case $(\Gamma, \cdot, 1)$ is ω-stable.*

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