数理解析研究所講究録
第2002巻 2016年 25-33

THE 1ST HOMOLOGY GROUP IN MODEL THEORY AS A QUOTIENT GROUP OF THE LASCAR GROUP

BYUNGHAN KIM

This note is based on the papers [9] and [1].[1] In [2] and [3], a homology theory for model theory is developed. In particular, given a strong type $p(x)$ over $A = \mathrm{acl}^{\mathrm{eq}}(A)$ in a rosy theory T, the notion of the nth homology group $H_n(p)$ depending on an independence relation is introduced. Although the homology groups are defined analogously as in singular homology theory, the $(n+1)$th homology group for $n > 0$ in the context is to do with the nth homology group in algebraic topology. For example as in [2],[7], $H_2(p)$ is to do with (the abelianization of) the fundamental group in topology. This implies that $H_1(p)$ is detecting somewhat endemic properties of p existing only in model theory context.

Indeed, in every known example, $H_n(p)$ for $n \geq 2$ is a profinite abelian group. In [4], it is proved to be so when T is stable under a canonical condition, and conversely every profinite abelian group can arise in this form. On the other hand, in [9], it is shown that $|H_1(p)| \geq 2^{\omega}$ unless trivial, and non-profinite examples are exhibited. Moreover, the canonical epimorphism from the Lascar group of T to $H_1(p)$ is constructed. This motives our current work and in the paper [1], we (Jan Dobrowolski, Byunghan Kim and Junguk Lee) show that $H_1(p)$ is to do with the abelianization of the Lascar group of $\bar{p}(\bar{x})$, where $\bar{p}(\bar{x}) = \mathrm{tp}(\mathrm{acl}^{\mathrm{eq}}(aA)/A)$ with $a \models p$. More precisely, $H_1(p) = G/K$ where G is the group of automorphisms of $\bar{p}(\mathcal{M}^{\mathrm{eq}})$, and K is the group of automorphisms of $\bar{p}$ fixing each orbit in $\bar{p}(\mathcal{M}^{\mathrm{eq}})$ under the action of the derived subgroup of G. Surprisingly this conclusion is independent from the choice of an independence relation satisfying finite character, symmetry, transitivity and extension. Hence in fact $H_1(p)$ perfectly makes sense in any theory with the full independence (i.e. any two sets are assumed to be independent over any set), and is again G/K. An appropriate notion of the localized Lascar group $\mathrm{Gal}_{\mathrm{L}}(p)$ is also

The author was supported by Samsung Science Technology Foundation under Project Number SSTF-BA1301-03.

[1]A version of this note is also submitted as an extended abstract of the talk that the author will address in Mathematical Society of Japan Spring meeting at the University of Tsukuba in March 2016.

suggested in [1], which is independent from the choice of a monster model, and by the same manner as in [9] mentioned above, the canonical epimorphism from $\mathrm{Gal_L}(\bar{p})$ to $H_1(p)$ is constructed, so K can be considered as the kernel of this epimorphism.

1. Introduction

Throughout this note we work in a large saturated model $\mathcal{M}(=\mathcal{M}^{\mathrm{eq}})$ of a complete theory T, and use standard notations. For example, unless said otherwise, $a, b, \ldots, A, B, \ldots$ are small but possibly infinite tuples and sets from $\mathcal{M}$, and $a \equiv_A b$, $a \equiv^s_A b$, $a \equiv^L_A b$ mean $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$, $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$, $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$, respectively. For general theory of model theory or the Lascar groups, we refer to [5] or [10]. For the homology theory for model theory, see [2],[3]. In particular, $H_1(p)$ is studied in [6],[8]. In this section, we summarize some of those below.

Remark 1.1. For the rest, we fix a ternary automorphism-invariant relation $\mathop{\smile\!\!\!\!\!\!|}^*$ among small sets of $\mathcal{M}$ satisfying

- finite character: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $a \mathop{\smile\!\!\!\!\!\!|}^*_C b$ for any finite tuples $a \in A$ and $b \in B$;
- normality: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $A \mathop{\smile\!\!\!\!\!\!|}^*_C BC$;
- symmetry: for any sets A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff $B \mathop{\smile\!\!\!\!\!\!|}^*_C A$;
- transitivity: $A \mathop{\smile\!\!\!\!\!\!|}^*_B D$ iff $A \mathop{\smile\!\!\!\!\!\!|}^*_B C$ and $A \mathop{\smile\!\!\!\!\!\!|}^*_C D$, for any sets A and $B \subseteq C \subseteq D$; and
- extension: for any sets A and $B \subseteq C$, there is $A' \equiv_B A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^*_B C$ holds.

If $A \mathop{\smile\!\!\!\!\!\!|}^*_B C$ holds then as usual we say A is $*$-independent from B over C. Notice that there is at least one such relation for any theory. Namely the full (or trivial) independence relation: For any sets A, B, C, put $A \mathop{\smile\!\!\!\!\!\!|}^*_B C$. Of course there is a non-trivial such relation when T is simple or rosy, given by forking or thorn-forking, respectively.

Now we fix a strong type $p(x)$ of possibly infinite arity over $B = \mathrm{acl}(B)$ (so $p(x)$ simply is a complete type over B with free variables in x), and recall to define the 1st homology group of p.

Notation 1.2. Let s be an arbitrary finite set of natural numbers. Given any subset $X \subseteq \mathcal{P}(s)$, we may view X as a category where for any $u, v \in X$, $\mathrm{Mor}(u, v)$ consists of a single morphism $\iota_{u,v}$ if $u \subseteq v$, and $\mathrm{Mor}(u, v) = \emptyset$ otherwise. If $f \colon X \to \mathcal{C}$ is any functor into some category $\mathcal{C}$ then for any $u, v \in X$ with $u \subseteq v$, we let f^u_v denote the morphism $f(\iota_{u,v}) \in \mathrm{Mor}_{\mathcal{C}}(f(u), f(v))$. We shall call $X \subseteq \mathcal{P}(s)$ a *primitive category* if X is non-empty and *downward closed*, i.e., for any $u, v \in \mathcal{P}(s)$, if $u \subseteq v$ and $v \in X$ then $u \in X$. (Note that all primitive categories have the empty set $\emptyset \subset \omega$ as an object.)

We use now $\mathcal{C}_B$ to denote the category whose objects are all the small subsets of $\mathcal{M}$ containing B, and whose morphisms are elementary maps over B. For a functor $f : X \to \mathcal{C}_B$ and objects $u \subseteq v$ of X, $f^u_v(u)$ denotes the set $f^u_v(f(u))(\subseteq f(v))$.

Definition 1.3. By a *$*$-independent functor in p*, we mean a functor f from some primitive category X into $\mathcal{C}_B$ satisfying the following:

(1) If $\{i\} \subset \omega$ is an object in X, then $f(\{i\})$ is of the form $\operatorname{acl}(Cb)$ where $b \models p$, $C = \operatorname{acl}(C) = f^{\emptyset}_{\{i\}}(\emptyset) \supseteq B$, and $b \mathop{\smile\!\!\!\!\!\!|}\,_B C$.

(2) Whenever $u (\neq \emptyset) \subset \omega$ is an object in X, we have
$$f(u) = \operatorname{acl}\left(\bigcup_{i\in u} f^{\{i\}}_u(\{i\})\right)$$
and $\{f^{\{i\}}_u(\{i\}) \mid i \in u\}$ is independent over $f^{\emptyset}_u(\emptyset)$.

We let $\mathcal{A}^*_p$ denote the family of all $*$-independent functors in p.

A $*$-independent functor f is called a *$*$-independent n-simplex* (or *n-$*$-simplex*) in p if $f(\emptyset) = B$ and $\operatorname{dom}(f) = \mathcal{P}(s)$ with $s \subset \omega$ and $|s| = n + 1$. We call s the *support* of f and denote it by $\operatorname{supp}(f)$.

In the rest we may call a $*$-independent n-simplex in p just as an *n-simplex* of p, as far as no confusion arises. We are ready to define the 1st homology group $H^*_1(p)$ of p depending on our choice of the independence relation $\mathop{\smile\!\!\!\!\!\!|}\,^*$.

Definition 1.4. Let $n \geq 0$. We define:
$$S_n(\mathcal{A}^*_p) := \{\, f \in \mathcal{A}^*_p \mid f \text{ is an } n\text{-simplex of } p \,\}$$
$$C_n(\mathcal{A}^*_p) := \text{the free abelian group generated by } S_n(\mathcal{A}^*_p).$$

An element of $C_n(\mathcal{A}^*_p)$ is called an *n-chain* of p. The support of a chain c, denoted by $\operatorname{supp}(c)$, is the union of the supports of all the simplices that appear in c with a non-zero coefficient. Now for $n \geq 1$ and each $i = 0, \ldots, n$, we define a group homomorphism
$$\partial^i_n \colon C_n(\mathcal{A}^*_p) \to C_{n-1}(\mathcal{A}^*_p)$$
by putting, for any n-simplex $f \colon \mathcal{P}(s) \to \mathcal{C}$ in $S_n(\mathcal{A}^*_p)$ where $s = \{s_0 < \cdots < s_n\} \subset \omega$,
$$\partial^i_n(f) := f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and then extending linearly to all n-chains in $C_n(\mathcal{A}_p^*)$. Then we define the *boundary map*

$$\partial_n \colon C_n(\mathcal{A}_p^*) \to C_{n-1}(\mathcal{A}_p^*)$$

by

$$\partial_n(c) := \sum_{0 \leq i \leq n} (-1)^i \partial_n^i(c).$$

We shall often refer to $\partial_n(c)$ as the *boundary of* c. Next, we define:

$$Z_n(\mathcal{A}_p^*) := \operatorname{Ker} \partial_n$$
$$B_n(\mathcal{A}_p^*) := \operatorname{Im} \partial_{n+1}.$$

The elements of $Z_n(\mathcal{A}_p^*)$ and $B_n(\mathcal{A}_p^*)$ are called *n-cycles* and *n-boundaries* in p, respectively. It is straightforward to check that $\partial_n \circ \partial_{n+1} = 0$. Hence we can now define the group

$$H_n^*(p) := Z_n(\mathcal{A}_p^*)/B_n(\mathcal{A}_P^*)$$

called the nth *$*$-homology group* of p.

Notation 1.5. (1) For $c \in Z_n(\mathcal{A}_p^*)$, $[c]$ denotes the homology class of c in $H_n^*(p)$.
(2) When n is clear from the context, we shall often omit n in ∂_n^i and in ∂_n, writing simply as ∂^i and ∂.

Definition 1.6. A 1-chain $c \in C_1(\mathcal{A}_p^*)$ is called a *1-$*$-shell* (or just, 1-shell) in p if it is of the form

$$c = f_0 - f_1 + f_2$$

where f_i's are 1-simplices of p satisfying

$$\partial^i f_j = \partial^{j-1} f_i \quad \text{whenever } 0 \leq i < j \leq 2.$$

Hence, for $\operatorname{supp}(c) = \{n_0 < n_1 < n_2\}$ and $k \in \{0, 1, 2\}$, it follows

$$\operatorname{supp}(f_k) = \operatorname{supp}(c) \smallsetminus \{n_k\}.$$

Notice that the boundary of any 2-simplex is a 1-shell.

Remark 1.7. If c is a 1-shell, then in $H_1^*(p)$, by the argument in [9], we have $[-c] = [c']$ where c' is another 1-shell with $\operatorname{supp}(c') = \operatorname{supp}(c)$ (See Fact 1.11 below).

Now in [3], the notion of an *amenable* collection of functors into a category is introduced, and it is transparent to see that $\mathcal{A}_p^*$ forms such a collection of functors into $\mathcal{C}_B$. Therefore the following corresponding fact holds.

Fact 1.8. [3]

$$H_1^*(p) = \{[c] \mid c \textit{ is a } 1\text{-}*\textit{-shell with } \operatorname{supp}(c) = \{0, 1, 2\}\}.$$

So if any 1-shell is the boundary of some 2-chain then $H_1^*(p) = 0$.[2]

We now begin to summarize some of important observations from [9] regarding $H_1(p)$, originally given for rosy theories with thorn-independence. But those perfectly make sense in our context of an arbitrary theory with $\mathop{\smile\!\!\!\!\!\mid}^*$. For the rest of this paper we suppress B to $\emptyset$ by naming it. In particular $\mathcal{C}$ denotes $\mathcal{C}_B$. Moreover we put $\bar{p}(\bar{x}) := \mathrm{tp}(\mathrm{acl}(a))$ where $a \models p$.

Definition 1.9. (1) We introduce some notation which will be used throughout. Let $f\colon \mathcal{P}(s) \to \mathcal{C}$ be an n-$*$-simplex in p. For $u \subset s$ with $u = \{i_0 < \ldots < i_k\}$, we shall write $f(u) = [a_0 \ldots a_k]_u$, where $a_j \models \bar{p}$, $f(u) = \mathrm{acl}(a_0 \ldots a_k)$, and $a_j = f_u^{\{i_j\}}(\{i_j\})$. So, $\{a_0, \ldots, a_k\}$ is $*$-independent.

(2) Let $s = f_{12} - f_{02} + f_{01}$ be a 1-$*$-shell in p such that $\mathrm{supp}(f_{ij}) = \{i,j\}$ for $0 \le i < j \le 2$. Clearly there is a quadruple (a_0, a_1, a_2, a_3) of realizations $\bar{p}$ such that $f_{01}(\{0,1\}) \equiv [a_0a_1]_{\{0,1\}}$, $f_{12}(\{1,2\}) \equiv [a_1a_2]_{\{1,2\}}$, and $f_{02}(\{0,2\}) \equiv [a_3a_2]_{\{0,2\}}$. We call this quadruple a *representation of s*. For any such representation of s, call a_0 an *initial point*, a_3 a *terminal point*, and (a_0, a_3) an *endpoint pair* of the representation.

In [9], it was proved (using only the finite character, symmetry, transitivity and extension of thorn-independence in a rosy theory) that each homology class and the 1st homology group structure are determined by (the types of) the endpoints pairs of representations. Therefore the same proof induces the following in our context of $H_1^*(p)$.

Fact 1.10. [9]

(1) For any pair (a, b) of realizations of $\bar{p}$, there is a 1-$*$-shell s of p with the support $\{0,1,2\}$ such that (a, b) is the endpoint pair of some representation of s.

(2) Let s_0 and s_1 be 1-$*$-shells in p with the support $\{0,1,2\}$. Let (a_0, a_0') and (a_1, a_1') be the endpoint pairs of s_0 and s_1 respectively.

(a) If $a_0a_0' \equiv a_1a_1'$, then $[s_0] = [s_1]$ (in $H_1^*(p)$).

(b) If $a_0' = a_1$, then for any 1-$*$-shell s with the support $\{0,1,2\}$ having a representation whose endpoint pair is (a_0, a_1'), it follows $[s] = [s_0] + [s_1]$ in $H_1^*(p)$.

[2]Notice that in this note, when we define '$*$-independent functor' in Definition 1.3, we take only algebraic closures in $\mathcal{M}^{\mathrm{eq}}$ (not bounded closures), thus it is not clear $H_1^*(p)$=0 with usual nonforking independence in a simple theory. Indeed we state this as an open question in Question 2.5.

Using Fact 1.10, we define an equivalence relation $\sim$ on the set of pairs of realizations $\bar{p}$ as follows: For $a, a', b, b' \models \bar{p}$, $(a,b) \sim (a',b')$ if two pairs (a,b) and (a',b') are endpoint pairs of 1-shells s and s' such that $[s]=[s'] \in H_1^*(p)$. We write $\mathcal{E}^* = \bar{p}(\mathcal{M}) \times \bar{p}(\mathcal{M})/\sim$. We denote the class of $(a,b) \in \bar{p}(\mathcal{M}) \times \bar{p}(\mathcal{M})$ by $[a,b]$. By 1.10, if $ab \equiv a'b'$, then $[a,b]=[a',b']$. Now define a binary operation $+_{\mathcal{E}^*}$ on $\mathcal{E}^*$ as follows: For $[a,b],[b',c'] \in \mathcal{E}^*$, $[a,b] +_{\mathcal{E}^*} [b',c'] = [a,c]$ where $bc \equiv b'c'$.

Fact 1.11. [9] *The pair $(\mathcal{E}^*, +_{\mathcal{E}^*})$ forms a commutative group which is isomorphic to $H_1^*(p)$. More specifically, for $a,b,c \models p$ and $\sigma \in \operatorname{Aut}(\mathcal{M})$, it follows that*

- $[a,b]+[b,c]=[a,c]$;
- $[a,a]$ *is the identity element;*
- $-[a,b]=[b,a]$;
- $\sigma([a,b]) := [\sigma(a),\sigma(b)] = [a,b]$; *and*
- $f : \mathcal{E}^* \to H_1^*(p)$ *sending* $[a,b] \mapsto [s]$, *where* (a,b) *is an endpoint pair of* s, *is a group isomorphism.*

From now on, we identify $\mathcal{E}^*$ and $H_1^*(p)$. Notice that indeed the group structure of $\mathcal{E}^*$ depends only on the types of (a,b)'s with $[a,b] \in \mathcal{E}^*$.

Now by exactly the same proof of [6, Theorem 2.4], which only uses the finite character, symmetry, transitivity, and extension of thorn-forking, we can obtain the following fact for our independence $\mathop{\smile\!\!\!\!\!\!\;|}^*$ in an arbitrary theory T.

Fact 1.12. *For $a, a' \models \bar{p}$, if $a \equiv^L a'$, then $[a,a']=0$ in $\mathcal{E}^* = H_1^*(p)$.*

Using Fact 1.11 and 1.12, we obtain the following canonical epimorphism by the same manner as described in [9].

Fact 1.13. *There is a canonical epimorphism*

$$\psi_{\bar{p}}^* : \operatorname{Aut}(\bar{p}(\mathcal{M})) \to H_1^*(p)$$

sending each $\sigma \in \operatorname{Aut}(\bar{p}(\mathcal{M}))$ to $[a,\sigma(a)]$ for some/any realization a of $\bar{p}$.

Remark 1.14. Note that $\operatorname{Aut}(\bar{p}(\mathcal{M}))/\operatorname{Ker}(\psi_{\bar{p}}^*)$ is isomorphic to $H_1^*(p)$, which is independent from the choice of the monster model. Since $H_1^*(p)$ is abelian, $\operatorname{Ker}(\psi_{\bar{p}}^*)$ contains the derived subgroup of $\operatorname{Aut}(\bar{p}(\mathcal{M}))$. We shall figure out what $\operatorname{Ker}(\psi_{\bar{p}}^*)$ is, and it will turn out that even the kernel (so $H_1^*(p)$ too) is independent from the choice of $\mathop{\smile\!\!\!\!\!\!\;|}^*$.

In [1], the notions of certain localized Lascar Galois groups are introduced as follows.

Definition 1.15. (1) For a cardinal $\lambda > 0$, $\mathrm{Autf}^{\lambda}_{\mathrm{fix}}(p(\mathcal{M})) :=$
$\{\sigma \restriction p(\mathcal{M}) : \sigma \in \mathrm{Aut}(\mathcal{M})$ such that for any $a_i \models p$ and $\bar{a} = (a_i)_{i<\lambda}$, $\bar{a} \equiv^L \sigma(\bar{a})\}$;
and
(2) $\mathrm{Autf}_{\mathrm{fix}}(p(\mathcal{M})) := \{\sigma \restriction p(\mathcal{M}) : \sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\bar{a} \equiv^L \sigma(\bar{a})$ where $\bar{a}$ is some enumeration of $p(\mathcal{M})\}$.

It is straightforward to see that the groups

$$\mathrm{Autf}_{\mathrm{fix}}(p(\mathcal{M})) \leq \mathrm{Autf}^{\lambda}_{\mathrm{fix}}(p(\mathcal{M}))$$

are normal subgroups of $\mathrm{Aut}(p(\mathcal{M}))$.

Definition 1.16. (1) $\mathrm{Gal}^{\mathrm{fix},\lambda}_{\mathrm{L}}(p(\mathcal{M})) := \mathrm{Aut}(p(\mathcal{M}))/\mathrm{Autf}^{\lambda}_{\mathrm{fix}}(p(\mathcal{M}))$;
and
(2) $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p(\mathcal{M})) := \mathrm{Aut}(p(\mathcal{M}))/\mathrm{Autf}_{\mathrm{fix}}(p(\mathcal{M}))$.

Remark 1.17. In [1], it is observed that $\mathrm{Autf}_{\mathrm{fix}}(p(\mathcal{M})) = \mathrm{Autf}^{\omega}_{\mathrm{fix}}(p(\mathcal{M}))$. So $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p(\mathcal{M})) = \mathrm{Gal}^{\mathrm{fix},\omega}_{\mathrm{L}}(p(\mathcal{M}))$. In addition, $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(p)$ on p is shown to be independent from the choice of a monster model of T (only depending on p). Then due to Fact 1.12, $\mathrm{Ker}(\psi^*_{\bar{p}})$ contains $\mathrm{Autf}_{\mathrm{fix}}(\bar{p}(\mathcal{M}))$. Hence this induces a canonical epimorphism $\Psi^*_{\bar{p}} : \mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(\bar{p}) \to H^*_1(p)$ as well. Therefore $H^*_1(p)$ can be considered as a quotient group of the Lascar group $\mathrm{Gal}^{\mathrm{fix}}_{L}(\bar{p})$.

2. The first homology groups of strong types in arbitrary theories

The goal of this section is to identify what $\mathrm{Ker}(\psi^*_{\bar{p}})$ is. In [6][8], the 2-chains in p with 1-shell boundaries are classified when T is rosy with thorn-independence. However again the only properties used for thorn-forking there are finite character, symmetry, transitivity, and extension. Therefore the following same conclusion can be obtained in our context of $*$-independence in any T.

Fact 2.1. *A 1-$*$-shell s in p is the boundary of a 2-chain if and only if there is a representation (a, b, c, a') of s such that for some $n \geq 0$ there is a finite sequence $(d_i)_{0 \leq i \leq 2n+2}$ of realizations of $\bar{p}$ satisfying the following conditions:*

(1) $d_0 = a$, $d_{2n+1} = c$ and $d_{2n+2} = a'$;
(2) $\{d_j, d_{j+1}, b\}$ is $*$-independent for each $0 \leq j \leq 2n+1$; and
(3) there is a bijection

$$\sigma : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$$

such that $d_{2i}d_{2i+1} \equiv d_{2\sigma(i)+2}d_{2\sigma(i)+1}$ for $0 \leq i \leq n$.

Using Fact 2.1, we can identify $\mathrm{Ker}(\psi^*_{\bar{p}})$ as follows.

Theorem 2.2. *For each $h \in K := \mathrm{Ker}(\psi^*_{\bar{p}})$ and $a \models \bar{p}$, there is an automorphism h' in the derived subgroup G' of $G := \mathrm{Aut}(\bar{p}(\mathcal{M}))$ such that $h(a) = h'(a)$. Thus $K(\geq G')$ is the normal subgroup of G of automorphisms fixing all orbits of realizations in $\bar{p}(\mathcal{M})$ under the action of G', and $H^*_1(p) = G/K$.*

Remark 2.3. Due to above Theorem 2.2, $H^*_1(p)$, which of course does not depend on the choice of a monster model, is all the same regardless of our choice of independence $\mathop{\smile\!\!\!\!\!\!\;|}^*$ satisfying finite character, symmetry, transitivity and extension. Hence we can write it simply as $H_1(p)$.

In particular if we choose $\mathop{\smile\!\!\!\!\!\!\;|}^*$ to be the full independence, then obviously that $\{x_1, \ldots, x_n\}$ is $*$-independent over B is B-type-definable in $\bar{p}(x_1) \wedge \ldots \wedge \bar{p}(x_n)$. This is the only property (in addition to the four independence axioms) used in [9] to conclude that $|H_1(p)| = 1$ or $\geq 2^\omega$ (for rosy theories). Hence we get the same conclusion in the context of arbitrary theories. By the same token, the orbit equivalence relation $\equiv^{H_1}$ on $\bar{p}(\mathcal{M})$ under the action of K (equivalently G') in Theorem 2.2 (i.e., for $a, b \models \bar{p}$, $a \equiv^{H_1} b$ iff there is $f \in K$ (or $\in G'$) such that $b = f(a)$ iff $[a, b] = 0 \in H_1(p)$) is an F_σ-relation, as pointed out in [9], i.e., there are countably many B-type-definable reflexive, symmetric relations $R_i(x, y)$ such that

$$\bar{p}(x) \wedge \bar{p}(y) \models x \equiv^{H_1} y \leftrightarrow \bigvee_{i<\omega} R_i(x, y).$$

Now the following corollary says that, in any theory, $H_1(p)$ being non-trivial or $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(\bar{p})$ (or, $\mathrm{Gal}^{\mathrm{fix},1}_{\mathrm{L}}(\bar{p})$) being non-abelian are two cirteria for $\bar{p}$ not being Lascar type.

Corollary 2.4. *(T any theory.) The following are equivalent.*

(1) $\bar{p}(\bar{x})$ is a Lascar type.

(2) $H_1(p) = 0$ and $\mathrm{Gal}^{\mathrm{fix},1}_{\mathrm{L}}(\bar{p})$ is abelian.

(3) Both $H_1(p)$ and $\mathrm{Gal}^{\mathrm{fix},1}_{\mathrm{L}}(\bar{p})$ are trivial.

In particular if $H_1(p)$ is trivial and $\mathrm{Gal}^{\mathrm{fix}}_{\mathrm{L}}(\bar{p})$ is abelian, then $\bar{p}$ is a Lascar type.

Question 2.5. In a simple theory T, is always $H_1(p) = 0$?

References

[1] Jan Dobrowolski, Byunghan Kim, and Junguk Lee. The Lascar groups and the first homology groups in model theory. Submitted.

[2] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Homology groups of types in model theory and the computation of $H_2(p)$. *Journal of Symbolic Logic*, 78, (2013) 1086-1114.

[3] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Amalgamation functors and homology groups in model theory. *Proceedings of ICM 2014*, II, (2014) 41-58.

[4] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Homology groups of types in stable theories and the Hurewicz correspondence. Submitted.

[5] Byunghan Kim. *Simplicity theory.* Oxford Logic Guides, 53, (Oxford University Press 2014).

[6] Byunghan Kim, SunYoung Kim, and Junguk Lee. A classification of 2-chains having 1-shell boundaries in rosy theories. *Journal of Symbolic Logic*, 80, (2015), 322-340.

[7] Byunghan Kim, SunYoung Kim, and Junguk Lee. Non-cummutative groupoids obtained from the failure of 3-uniqueness in stable theories. Submitted.

[8] SunYoung Kim, and Junguk Lee. More on 2-chains with 1-shell boundaries in rosy theories. To appear in *Journal of the Mathematical Society of Japan.*

[9] Junguk Lee. The Lascar groups and the first homology groups of strong types in rosy theories. Submitted.

[10] Martin Ziegler. Introduction to the Lascar group, in *Tits buildings and the model theory of groups.* London Math. Soc. Lecture Note Series, 291, (Cambridge University Press 2002) 279-298.

DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY
50 YONSEI-RO, SEODAEMUN-GU, SEOUL 03722, KOREA
E-mail address: bkim@yonsei.ac.kr