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ON $\mathbf{K}_f$ IN IRRATIONAL CASES

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ABSTRACT. Consider an ab initio amalgamation class $\mathbf{K}_f$ with an unbounded increasing concave function f. We conjecture that if $\mathbf{K}_f$ has the free amalgamation property then the generic structure for $\mathbf{K}_f$ has a model complete theory. We consider the case where the predimension function has an irrational coefficient. We show some statements which seem to be useful to show our conjecture.

1. INTRODUCTION

Hrushovski constructed a seudoplane which is a counter example to a conjecture by Lachlan [6] using amalgamation classes of the form $\mathbf{K}_f$ which will be defined below. In his case, the predimension function has an irrational coefficient. The author proved that a generic graph for $\mathbf{K}_f$ has a model complete theory if the predimension function has a rational coefficient under some mild assumption on the function f [10].

We show some propositions towards the model completeness of the generic graph for $\mathbf{K}_f$ in the case that the predimension function has an irrational coefficient.

We essentially use notation and terminology from Wagner [11]. We also use terminology from graph theory [4].

For a set X, $[X]^n$ denotes the set of all n-element subsets of X, and $|X|$ the cardinality of X.

For a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ the set of edges of G. $E(G)$ is a subset of $[V(G)]^2$. For $a, b \in V(G)$, ab denotes $\{a, b\}$. For $a \in V(G)$, the number of edges of G containing a is called a *degree* of a in G. $|G|$ denotes $|V(G)|$.

To see a graph G as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where E is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of E.

Suppose A is a graph. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is same as $A[X]$ in Diestel's book [4]. We say that X is *connected* in A if $A|X$ is a connected graph in graph theoretical sense [4].

If A, B, C are graphs such that $A \subseteq C$ and $B \subseteq C$, then AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$.

Definition 1.1. Let α be a real number such that $0 < \alpha < 1$. For a finite graph A, we define a predimension function δ_α as follows:

$$\delta_\alpha(A) = |A| - \alpha|E(A)|.$$

Suppose A and B are substructures of a common graph. Put

$$\delta_\alpha(A/B) = \delta_\alpha(AB) - \delta_\alpha(B).$$

Definition 1.2. Assume that A, B are graphs such that $A \subseteq B$ and A is finite.

$A \leq_\alpha B$ if whenever $A \subseteq X \subseteq B$ with X finite then $\delta_\alpha(A) \leq \delta_\alpha(X)$.

$A <_\alpha B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta_\alpha(A) < \delta_\alpha(X)$.

We say that A is *closed* in B if $A <_\alpha B$. We also say that B is a *strong extension* of A.

Note that $\leq_\alpha$ and $<_\alpha$ are order relations. In particular, $A <_\alpha A$ for any graph A.

With this notation, put

$$\mathbf{K}_\alpha = \{A : \text{finite} \mid \emptyset <_\alpha A\}.$$

We usually fix the value of the parameter α. Therefore, we often write δ for δ_α, $<$ for $<_\alpha$, and $\leq$ for $\leq_\alpha$.

Suppose $A \subseteq B$ and $A \subseteq C$. A graph embedding $g : B \to C$ is called a *closed* embedding of B into C over A if $g(B) < C$ and $g(x) = x$ for any $x \in A$.

Definition 1.3. Let $\mathbf{K} \subseteq \mathbf{K}_\alpha$ be an infinite class. $\mathbf{K}$ has the *amalgamation property* if for any $A, B, C \in \mathbf{K}$, whenever $A < B$ and $A < C$ then there is $D \in \mathbf{K}$ such that there is a closed embedding of B into D over A and a closed embedding of C into D over A.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is called an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

Definition 1.4. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. A countable graph M is a *generic graph* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.

(2) If $A \subseteq M$ then $A \in \mathbf{K}$.

(3) For any $A, B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of B into M over A.

If $\mathbf{K}$ is an amalgamation class then there is a generic graph of $(\mathbf{K}, <)$.

There is a smallest B satisfying (1), written $\mathrm{cl}(A)$. We have $A \subseteq \mathrm{cl}(A) < M$ and if $A \subseteq B < M$ then $\mathrm{cl}(A) \subseteq B$. The set $\mathrm{cl}(A)$ is called a *closure* of A in M. Apparently, $\mathrm{cl}(A)$ is unique for a given finite set A. In general, if A and D are graphs and $A \subseteq D$, we write $\mathrm{cl}_D(A)$ for the smallest substructure B of D such that $A \subseteq B < D$.

Definition 1.5. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing concave (convex upward) unbounded function. Assume that $f(0) \leq 0$, and $f(1) \leq 1$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic graph of $(\mathbf{K}_f, <_\alpha)$ has a countably categorical theory.

Definition 1.6. Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. A graph $A \in \mathbf{K}$ is *absolutely closed* in $\mathbf{K}$ if whenever $A \subseteq B \in \mathbf{K}$ then $A < B$.

Definition 1.7. Put $R_f = \{(x,y) \in \mathbb{R}^2 \mid f(x) \leq y \leq x\}$. A graph A is *normal to f* if $(|A|, \delta(A))$ belongs to R_f.

$A \in \mathbf{K}_f$ if and only if every substructure of A is normal to f.

Definition 1.8. Let m, n be integers. A point of the form $(n, n - m\alpha)$ is called a *lattice point* in this paper.

Proposition 1.9. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing concave unbounded function. with $f(0) \leq 0$, $f(1) \leq 1$. Suppose that whenever (x_1, y_1), (x_2, y_2), and (x_3, y_3) are lattice points in R_f with $x_1 \leq x_2 \leq x_3$ and $y_1 < y_2$ then $(x_3 + x_2 - x_1, y_3 + y_2 - y_1)$ belongs to R_f. Then $\mathbf{K}_f$ has the free amalgamation property.*

Note that Hrushovski's f in [6] satisfies the assumption of the proposition above.

In the rest of the paper, we assume that the assumption of Proposition 1.9 holds:

Assumption 1.10. (1) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing concave unbounded function.

(2) $f(0) \leq 0$, $f(1) \leq 1$.

(3) Whenever (x_1, y_1), (x_2, y_2), and (x_3, y_3) are lattice points in R_f with $x_1 \leq x_2 \leq x_3$ and $y_1 < y_2$ then $(x_3 + x_2 - x_1, y_3 + y_2 - y_1)$ belongs to R_f.

The following definition is from [10].

Definition 1.11. Suppose X, Y are sets and $\mu : X \to Y$ a map. For $Z \subseteq [X]^m$, put $\mu(Z) = \{\{\mu(x_1), \dots, \mu(x_m)\} \mid \{x_1, \dots, x_m\} \in Z\}$.

Let B, C be graphs and assume that $X \subseteq V(B) \cap V(C)$. Let D be a graph. We write $D = B \rtimes_X C$ if the following hold:

(1) There is a 1-1 map $f : V(B) \to V(D)$.
(2) There is a 1-1 map $g : V(C) \to V(D)$.
(3) $f(x) = g(x)$ for any $x \in X$.
(4) $V(D) = f(B) \cup g(C)$.
(5) $f(B) \cap g(C) = f(X) = g(X)$.
(6) $E(D) = f(E(B)) \cup g(E(C) - E(C|X))$.

f is a graph isomorphism from B to $D|f(V(B))$ but C and $D|g(V(C))$ are not necessarily isomorphic as graphs.

If $E(C|X) = \emptyset$, then $B \rtimes_X C$ is a graph obtained by attaching C to B at points in X. We have $\delta(B \rtimes_X C) = \delta(B) + \delta(C) - \delta(C|X)$.

In case that $B|X = C|X$, we write $B \otimes_X C$ for $B \rtimes_X C$. If $A = B|X = C|X$, then we also write $B \otimes_A C$ instead of $B \otimes_{V(A)} C$. We assume that operators $\rtimes_X$ and $\otimes_X$ are left associative.

When we write $B \rtimes_X C$, we assume that $X \subseteq V(B) \cap V(C)$. When $b \in V(B)$ and $c \in V(C)$, $B \otimes_{b=c} C$ denotes $B \otimes_b C$ after identifying b and c.

If $A \subseteq B$ and $q \geq 1$ is an integer, then $\bigotimes_A^q B$ is defined inductively as follows: $\bigotimes_A^1 B = B$ and $\bigotimes_A^q B = (\bigotimes_A^{q-1} B) \otimes_A B$ if $q \geq 2$.

The following lemma is immediate.

Lemma 1.12. *Suppose $D = B \rtimes_X C$.*

(1) *If $C|X < C$ then $B < D$.*
(2) *If $C|X \leq C$ then $B \leq D$.*

Definition 1.13. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. K has the *free amalgamation property* if whenever $A, B, C \in \mathbf{K}$ with $A < B$, $A < C$ then $B \otimes_A C \in \mathbf{K}$.

Fact 1.14. *If a class $\mathbf{K} \subseteq \mathbf{K}_\alpha$ has the free amalgamation property then it has the amalgamation property.*

Lemma 1.15. *Suppose A, B, C are graphs such that $A \subseteq B$, $A \subseteq C$, $\delta(A) < \delta(B)$ and $\delta(A) < \delta(C)$. If B and C are normal to f then $B \otimes_A C$ is normal to f.*

Proposition 1.16. $(\mathbf{K}_f, <)$ *has the free amalgamation property.*

2. Minimal Extensions

To prove our conjecture, given a graph $A \in \mathbf{K}_f$, we would like to construct an extension B of A such that $A < B$ and B is absolutely closed. In order to do this, we want to expand A by attaching "twigs" to make a tower of "minimal" strong extensions first. Then we want to make an alternating tower of "minimal" strong extensions and "minimal" intrinsic extensions so that the δ-rank stays around some specific value. We expect that it will eventually be absolutely closed.

Definition 2.1. Suppose A, B are graphs such that $A \subseteq B$. B is a *minimal strong extension of* A if $A < B$ and whenever $A \subsetneq X \subsetneq B$ then $\delta(B/A) < \delta(X/A)$.

B is a *minimal intrinsic extension of* A if $\delta(B/A) \leq 0$ but whenever $A \subsetneq X \subsetneq B$ then $0 < \delta(X/A)$.

Fact 2.2 ([10])**.** *Suppose m, d are relatively prime integers such that $0 < m < d$. Then there is a tree (a graph with no cycles) G such that $V(G) = F \cup B$, $F \cap B = \emptyset$, $|B| = m$, $G|F$ has no edges, and G is a minimal 0-extension of $G|F$ with respect to $\delta_{m/d}$. This means that $\delta_{m/d}(G/F) = 0$ and whenever $F \subsetneq X \subsetneq G$ then $\delta_{m/d}(X/F) > 0$. This G is called a twig for m/d in* [10]. *F will be called a* base *of G and G will be called a* body part *of G.*

Proposition 2.3. *Suppose α is an irrational number such that $0 < \alpha < 1$. Let $n_0 \geq 2$ be an arbitrary natural number and m, d integers such that $m - d\alpha$ is a smallest number among the positive numbers of the form $m' - d'\alpha$ with $d' \leq n_0$. Then any twig for m/d is a minimal strong extension over its base. The body part of G has a size m. We call G a* minimal strong extender.

Proof. First of all, m and d are relatively prime because the value of $m - d\alpha$ can be reduced in a positive value if m and d have a common divisor. Also, we have $\alpha < m/d$.

Let G be a twig for m/d. Let F be the base of G. For any proper substructure U of G with $U \setminus F \neq \emptyset$, $\delta_{m/d}(U/U \cap F) > 0$. Since $\delta_{m/d}(U/U \cap F) = m' - d'(m/d) > 0$ with $d' \leq n_0$ and $\alpha < m/d$, we have $\delta_\alpha(U/U \cap F) = m' - d'\alpha > 0$. Also, since $m - d(m/d) = 0$, we have $\delta_\alpha(G/F) = m - d\alpha > 0$. Therefore, G is a strong extension of $G|F$. By the choice of m and d, G is minimal strong extension over $G|F$. □

Proposition 2.4. *Suppose α is an irrational number such that $0 < \alpha < 1$. Let $n_0 \geq 2$ be an arbitrary natural number and m, d integers such that m/d is a largest number among the rational numbers of the form m'/d' with $m'/d' < \alpha$ and $d' \leq n_0$. Then any twig for m/d is a minimal intrinsic extension over its base. We call such twig a* minimal intrinsic extender.

Proof. Let G be a twig for m/d. We can assume that m and d are relatively prime. Since $m - d(m/d) = 0$ and $m/d < \alpha$, we have $\delta_\alpha(G) = m - d\alpha < 0$. Suppose U is a proper substructure of G such that $U \setminus F \neq \emptyset$. Then $\delta_{m/d}(U/U \cap F) = m' - d'(m/d) > 0$ with $d' \leq d \leq n_0$. We have $m'/d' > m/d$. If $\delta_\alpha(U/U \cap F) < 0$ then $m' - d'\alpha = \delta_\alpha(U/U \cap F) < 0$. This implies that $m'/d' < \alpha$. This contradicts the choice of m/d. Therefore, $\delta_\alpha(U/U \cap F) > 0$. $\square$

Proposition 2.5. *Any tree belongs to* $\mathbf{K}_f$ *(under Assumption 1.10). In particular, twigs in Propositions 2.3 and 2.4 belong to* $\mathbf{K}_f$.

Proof. A graph with 2 points and 1 edge belongs to $\mathbf{K}_f$. By induction, we can see that the δ_α-value of any tree is greater than 1. Hence, any point is closed in a tree. By induction, any tree belongs to $\mathbf{K}_f$ by the free amalgamation property. $\square$

By Assumption 1.10 and Proposition 2.5, we have the following.

Proposition 2.6. *Suppose* $A \in \mathbf{K}_f$.

- *If* G *is a minimal strong extender with* $|G| \leq |A|$ *then* $A \rtimes_F G$ *belongs to* $\mathbf{K}_f$ *where* F *is a base of* G.
- *If* G *is a minimal intrinsic extender with* $|G| \leq |A|$ *then any proper substructure of* $A \rtimes_F G$ *belongs to* $\mathbf{K}_f$ *where* F *is a base of* G.

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