数理解析研究所講究録
第2002巻 2016年 61-75

AUTOMORPHISM GROUPS OF GENERIC STRUCTURES: A REVIEW

ZANIAR GHADERNEZHAD

ABSTRACT. We will review the main results and questions concerning the automorphism groups of generic structures. The main themes are: the automorphism groups that are simple, the small index property, the extension property for the generic class and ample generics, amenability and extreme amenability.

Keywords: Fraïssé-Hrushovski method, Generic structures, Smooth classes, The small index property, Rasmsey property, Extension property, Group-reducts.

CONTENTS

1. INTRODUCTION

As it is properly described in Chapter 7 in [Hod93], there are two main reasons why countable structures are interesting: First, "they can be built as the union of chain of finite pieces". "Second, there are infinitely many chances to make sure that the right pieces go in". One of the most effective ways to build countable structures is via Fraïssé construction method. In this method, one builds a countable homogeneous structure (we call it *Fraïssé-limit structure*) from a countable class of finite structures which has the "joint embedding property" and the "amalgamation property". A countable class of finite structure has the amalgamation property if for every elements A, B_1, B_2 of the class which $f_i : A \to B_i$ is an embedding for $i = 1, 2$, there

This research review was made possible partially by Colciencias grant Métodos de Estabilidad en Clases No Estables.

exists an element D in the class such that B_i's embed in D and the diagram commutes.

Fraïssé's original example was to think of the class of finite linear orders as a set of approximations to the ordering in the rational numbers. As we mentioned, one nice feature of Fraïssé-method in constructing a countable structure is that we start from a countable class of finite structures (with the amalgamation property), which is called a *Fraïssé class.* This helps us to verify some basic properties of the universal object by understanding the elements of the class and the amalgamation property (see, e.g. [Hod97]). Many interesting objects have been constructed or reconstructed using this method. For example various kind of universal graphs, the random graph and more recently rational Urysohn space (see, e.g. [CV06]).

A Generic structure similar to a Fraïssé-limit structure is built out of classes of finite structures with a stronger property between elements of the class. It has been originally used by Hrushovski in [Hru93] and many interesting countable structures have been constructed using this method.

The main motivation for studying automorphism groups of countable structures comes from the classical theorem of Engeler, Ryll-Nadrzewski, Svenonius (see [Cam90]). In this theorem one can see a connection between a countable ω-categorical structure and its automorphism group; namely we obtain a full characterization of a countable ω-categorical structure in a group-theoretical terms. Later on, model theorists and group theorists studying permutation groups became very interested in these implicit features of automorphism groups of structures and the automorphism groups of first order structures have been studied in the the recent years.

The main direction of the studies are to understand how much information one can obtain about the automorphism group from knowing the structure. A separable completely metrizable topological space is called *Polish space.* A topological group which the topology is Polish is called a *Polish group.* There is a natural topology defined on the symmetric group of a countable set Ω; that is pointwise convergence. This topology makes $S_\omega := Sym(\Omega)$ into a topological group and a Polish group. Note that the Basic open sets are $\{g \in S_\omega : g(\bar{a}) = \bar{b}\}$ where $\bar{a}$ and $\bar{b}$ are tuples of distinct elements of the same length. It is also well-known that a subgroup of G of S_ω is closed if and only if $G = \mathrm{Aut}(\mathcal{M})$ for some first order structure $\mathcal{M}$ on Ω (see [Cam90]). Therefore, automorphism groups of first order countable structures are Polish.

A rich model theoretic studies has been developed for understanding the first-order theory of Fraïssé-limit structures. In particular, the automorphism groups of Fraïssé-limit structures have been recently of central attention. A good survey for various kind of questions and results in the topic can be found in [Mac11]. Similar paths for adopting these lines of research for generic structures have been followed and in this paper we try to review some recent results about them.

2. Smooth Classes

A smooth classes is a modified version of a Fraïssé classes with a stronger property between elements of the class that is called *self-sufficiency* or *closedness* and denoted by "$\leqslant$". Here, we present a general definition of a smooth class and briefly review some the main properties of it.

Definition 1. Let $\mathfrak{L}$ be a countable relational language and $\mathcal{C}$ be a class of $\mathfrak{L}$-structures which is closed under isomorphism and substructure. Let $\leqslant$ be a reflexive and transitive relation on elements of $A \subseteq B$ of $\mathcal{C}$ and moreover, invariant under $\mathfrak{L}$-embeddings such that it has the following property:

- If $A, A_1, A_2 \in \mathcal{C}$ and $A_1, A_2 \subseteq A$, then $A_1 \leqslant A$ implies $A_1 \cap A_2 \leqslant A_2$.

The class $\mathcal{C}$ together with the relation $\leqslant$ is called a *smooth class.* For $A, B \in \mathcal{C}$ if $A \leqslant B$, then we say A is $\leqslant$-*closed substructure* of B, or simply A is $\leqslant$-*closed* in B. Moreover, if $\mathcal{N}$ is an infinite $\mathfrak{L}$-structure and $A \subseteq N$, we denote $A \leqslant \mathcal{N}$ whenever $A \leqslant B$ for every finite substructure B of N that contains A. We say an embedding $f : A \to \mathcal{N}$ is $\leqslant$-*embedding* if $f[A] \leqslant \mathcal{N}$.

Suppose A, B, C are $\mathfrak{L}$-structures with $A, B \subseteq C$. We write AB for the $\mathfrak{L}$-substructure of C with domain $A \cup B$. For an $\mathfrak{L}$-structure $\mathcal{N}$, write $\operatorname{Age}(\mathcal{N})$ for the set of all finite substructures of $\mathcal{N}$; up to isomorphism.

Definition 2. Let $(\mathcal{C}, \leqslant)$ be a smooth class.

- Write $\overline{\mathcal{C}}$ for the class of all $\mathfrak{L}$-structures $\mathcal{N}$ such that $\operatorname{Age}(\mathcal{N}) \subseteq \mathcal{C}$.
- We say $(\mathcal{C}, \leqslant)$ has the *joint-embedding property* (JEP) if $A, B \in \mathcal{C}$ there is $C \in \mathcal{C}$ such that $A, B \leqslant C$.
- We say $(\mathcal{C}, \leqslant)$ has the $\leqslant$-*amalgamation property* (AP) if for every A, B and C elements of $\mathcal{C}$ with $A \leqslant B, C$, there is $D \in \mathcal{C}$ such that $B \leqslant D$ and $C \leqslant D$.

Let $(\mathcal{C}, \leqslant)$ be a smooth class and suppose $A \in \mathcal{C}$ and $\mathcal{N} \in \overline{\mathcal{C}}$ are $\mathfrak{L}$-structures such that $A \subseteq \mathcal{N}$. From the definition of smooth classes it follows that there is a unique smallest $\leqslant$-closed set that contains A in $\mathcal{N}$, that is called the $\leqslant$-closure of A in $\mathcal{N}$ and denoted by $\operatorname{cl}_{\mathcal{N}}(A)$ (see [KL92]).

Then the following theorem that is similar to the Fraïssé holds

Theorem 3. [KL92] *Suppose $(\mathcal{C}, \leqslant)$ is a smooth class with JEP and AP. Then there is a unique countable structure $\mathcal{M}$, up to isomorphism, satisfying:*

1. *If $A' \leqslant \mathcal{M}$ and $A' \leqslant B \in \mathcal{C}$ then there exists $B' \leqslant \mathcal{M}$ such that $B \cong_{A'} B'$ (this property is called $\leqslant$-richness);*
2. *M is a union of $\langle A_i : i \in \omega \rangle$ such that $A_i \leqslant A_{i+1}$;*
3. *Suppose B and C are finite $\leqslant$-closed subset of $\mathcal{M}$ and let α be an isomorphism of B and C. Then α extends to an automorphism of $\mathcal{M}$.*

Definition 4. Let $(\mathcal{C}, \leqslant)$ be a smooth class with JEP and AP. The structure $\mathcal{M}$ that is obtained from Theorem 3 is called the *Hrushovski-Fraïssé $(\mathcal{C}, \leqslant)$-generic structure* or simply *$(\mathcal{C}, \leqslant)$-generic structure.*

Note that every Fraïssé-class is a smooth class with JEP and AP, if one consider "$\leqslant$" to be the usual substructure relation, and then its generic structure is the Fraïssé-limit structure.

The following properties and notions are crucial in understanding the first-order theory of generic structure. The notion of "$\leqslant$" can be extended to $\overline{\mathcal{C}} \times \overline{\mathcal{C}}$. Suppose $A, B \in \mathcal{C}$. We say (A, B) is a *$\leqslant$-minimal-pair* if $A \subseteq B$, $A \nleqslant B$ but $A \leqslant B'$ for every B' with $A \subseteq B' \subsetneq B$. Suppose $\mathcal{M}, \mathcal{N} \in \overline{\mathcal{C}}$ and $\mathcal{M} \subseteq \mathcal{N}$. Write $\mathcal{M} \leqslant \mathcal{N}$ if and only if for any minimal-pair (A, B) with $A, B \subseteq \mathcal{N}$, if $A \subseteq \mathcal{M}$, then $B \subseteq \mathcal{M}$. Let $(\mathcal{C}, \leqslant)$ be a smooth class with AP and let $\mathcal{M}$ be the $(\mathcal{C}, \leqslant)$-generic structure. Suppose $A \subset \mathcal{M}$ and $A \subseteq B \in \mathcal{C}$. By a copy of B over A in $\mathcal{M}$ we mean the image of an embedding of B over A into $\mathcal{M}$. Write $\chi_{\mathcal{M}}(B; A)$ for the number of distinct copies of B over A in $\mathcal{M}$. We say $(\mathcal{C}, \leqslant)$ has the *algebraic closure* property (AC) if there is a function $\eta : \omega \times \omega \to \omega$ such that for any $A \leqslant B$ and $A \subset \mathcal{M}$, we have $\chi_{\mathcal{M}}(B; A) \leq \eta(|A|, |B|)$. It is clear that $(\mathcal{C}, \leqslant)$ has AC if and only if $\operatorname{cl}(A) \subseteq \operatorname{acl}(A)$ for any $A \subset \mathcal{M}$.

2.1. **Hrushovski generic structures.** In [Hru93], Hrushovski introduced the key notion of assigning a (pre-)dimension function to finite relational structures and used it to defined a self-sufficiency "$\leqslant$". Here for simplicity, we assume the language $\mathfrak{L}$ consists of one $n_{\mathfrak{R}}$-ary relation $\mathfrak{R}$ where $\mathfrak{R}$ is irreflexive and symmetric. It should be noted that in the results that we have given here the assumption $\mathfrak{R}$ being symmetric is not essential. Moreover, similar classes can be defined when $\mathfrak{L}$ is a countable relational language (see [BS96, Wag94]).

Let $\mathbf{K}$ be the class of all finite $\mathfrak{L}$-structures. For a fixed real number $\alpha \geq 1$, define $\delta_\alpha : \mathbf{K} \longrightarrow \mathbb{R}$ as $\delta_\alpha(A) = \alpha \cdot |A| - |\mathfrak{R}(A)|$ where $\mathfrak{R}(A)$ is the set $\mathfrak{R}$-relations of A. For every $A \subseteq B \in \mathbf{K}$, define $A \leqslant_\alpha B$ if $\delta_\alpha(C/A) := \delta_\alpha(C) - \delta_\alpha(A) \geq 0$, for every C with $A \subseteq C \subseteq B$. Finally, put $\mathbf{K}_\alpha := \{A \in \mathcal{K} : \delta_\alpha(B) \geq 0, \text{ for every } B \subseteq A\}$. It is well-known that the $(\mathbf{K}_\alpha, \leqslant_\alpha)$ is a smooth class with JEP and AP. We say $(\mathbf{K}_\alpha, \leqslant_\alpha)$ is an *ab-initio class* that is constructed from δ_α. We write $\mathbf{M}_\alpha$ for the countable $(\mathbf{K}_\alpha, \leqslant_\alpha)$-generic structures that is obtained from Theorem 3 and it is called an *ab-initio generic structure* or *un-collapsed Hrushovski generic structure.* There are various ways of modifying these classes to obtain very interesting structures (for questions and variations of these structures see [Wag09, Bal15]).

When the coefficient α of the *pre-dimension* δ_α is rational, there is a way to obtain a generic structure with finite Morley rank, unlike $\mathbf{M}_\alpha$ that has infinite Morley rank. Using a finite-to-one function μ over 0-*minimally algebraic* elements (see Def. 2.2.29 in [Gha13]), one can restrict the ab-initio class $\mathbf{K}_\alpha$ to $\mathbf{K}_\alpha^\mu$ in such way that $(\mathbf{K}_\alpha^\mu, \leqslant_\alpha)$ has AP (see [Hru93] for details). We write $\mathbf{M}_\alpha^\mu$ for the $(\mathbf{K}_\alpha^\mu, \leqslant_\alpha)$-generic structure and it is called a *collapsed Hrushovski generic structure.* This method has been originally introduced by Hrushovski to provide a strongly minimal set that its geometry is not locally modular and not field-like (refuting a conjecture by Zilber). In the original paper of Hrushovski the coefficient α is one and $n_{\mathfrak{R}} = 3$. In this paper, we do not focus on the *geometric* aspects of these constructions (see [Zie13] for more details). However, in Section 3 and 5, some aspects of the geometric properties of Hrushovski construction will be briefly mentioned to address some possible connections between them and some properties of certain automorphisms.

There is also a modification of the Hrushovski's construction method to obtain ω-categorical generic structures (see [Eva02]). We are following Section 5.2 of [EGT16]. Suppose f is a continuous, increasing function with $f(x) \to \infty$ as $x \to \infty$. Let $\mathbf{K}_\alpha^f = \{A \in \mathbf{K}_\alpha : \delta(X) \geq f(|X|) \, \forall X \subseteq A\}$. If $B \subseteq A \in \mathbf{K}_\alpha^f$ and $\delta(A'/B) > 0$ for all $B \subsetneqq A' \subseteq A$, then we write $B \leqslant_f A$. For suitable choice of f which we call *good* f (see Remark 5.4 in [EGT16]), $\left(\mathbf{K}_\alpha^f, \leqslant_f\right)$ has the free-amalgamation property. In this case, we have an associated countable $\left(\mathbf{K}_\alpha^f, \leqslant_f\right)$-generic structure $\mathbf{M}_\alpha^f$ (up to isomorphism). The structure $\mathbf{M}_\alpha^f$ is ω-categorical.

Before finishing this part we need to define the following dimension function that will appear in the statements of the theorems and it is crucial for understanding the model theory of Hrushovski generic structures. From the pre-dimension δ_α, one can define the following *dimension* function:

$$\mathrm{d}_{\mathcal{M}}^{\delta_\alpha}(A) := \inf\left\{\delta\left(A'\right) : A \subseteq A' \subset_{fin} \mathcal{M}\right\};$$

where $\mathcal{M} \in \left\{\mathbf{M}_\alpha, \mathbf{M}_\alpha^\mu, \mathbf{M}_\alpha^f\right\}$. We simply write d for $\mathrm{d}_{\mathcal{M}}^{\delta_\alpha}$ when δ_α and $\mathcal{M}$ is clear from the context and write $\mathrm{d}(A/B)$ for $\mathrm{d}(AB) - \mathrm{d}(B)$ when $A, B \subset_{fin} M$. An interesting fact is that the forking-independence is $\mathrm{Th}(\mathcal{M})$ where $\mathcal{M} \in \left\{\mathbf{M}_\alpha, \mathbf{M}_\alpha^\mu, \mathbf{M}_\alpha^f\right\}$ is characterizable using $\mathrm{d}_{\mathcal{M}}^{\delta_\alpha}$ (see [BS96]). If we choose α to be a rational number then we can consider *min* instead of inf. If X is an infinite subset of $\mathcal{M}$, then we define: $\mathrm{d}(X) = \max\{\mathrm{d}(X_0) : X_0 \subseteq_{fin} X\}$. and for infinite B, let $\mathrm{d}(A/B) := \min\ \{\mathrm{d}(A/B') : B' \subset_{fin} B\}$. The following closure operator can also be defined and it seems to play in important role $\mathrm{cl}^{\mathrm{d}}(A) := \{x \in \mathcal{M} : \mathrm{d}(x/A) = 0\}$ for $A \subseteq \mathcal{M}$. The notion of d in the generic structure provides a nice technical tool and most of the time it is the essential part that is missing when want to state or to prove a general theorem about generic structures of some arbitrary smooth class. One final remark is that the classes $\mathbf{K}_\alpha, \mathbf{K}_\alpha^\mu$ and $\mathbf{K}_\alpha^f$ with $\leqslant_\alpha$ all have AC. The smooth classes without AC has been studied in [Pou02].

3. Simple Groups

One initial and standard question with a group theoretical interest is to determine the normal and maximal normal subgroups of automorphism groups of countable structures. Suppose $\mathcal{M}$ is any countable $\mathfrak{L}$-structure.

For $A \subseteq \mathcal{M}$, the subgroup of the *A-strong automorphisms* is: $\mathrm{Autf}_A(\mathcal{M}) = \{f \in \mathrm{Aut}_A(\mathcal{M}) : \mathrm{stp}(\bar{m}/A) = \mathrm{stp}(f(\bar{m})/A) \text{ for all } \bar{m} \in \mathcal{M}\}$.
If A is the empty set, $\mathrm{Autf}_\emptyset(\mathcal{M})$ becomes the group of all strong automorphisms of $\mathcal{M}$, and it is denoted by $\mathrm{Autf}(\mathcal{M})$. Note that $\mathrm{Autf}(\mathcal{M})$ is a normal subgroup $\mathrm{Aut}(\mathcal{M})$. An automorphism $\beta \in \mathrm{Aut}(\mathcal{M})$ is called *bounded* if there exists a finite set $A \subset \mathcal{M}$ such that $\beta(m) \in \mathrm{acl}(mA)$ for all $m \in \mathcal{M}$. Let $\mathrm{Bdd}(\mathcal{M})$ be the set of all bounded automorphisms of $\mathcal{M}$.

In a remarkable result by Lascar in [Las92], it has been shown that that if $\mathcal{M}$ is a countable saturated structure which is in the algebraic closure of a $\emptyset$-definable strongly minimal set, the group $\mathrm{Autf}(\mathcal{M}) / (\mathrm{Bdd}(\mathcal{M}) \cap \mathrm{Autf}(\mathcal{M}))$ is simple. It worth mentioning when $\mathcal{M}$ is a strongly minimal structure and "Dim" denotes the dimension in the strongly minimal structures, $g \in \mathrm{Aut}(\mathcal{M})$ is bounded if there exists an entirety n such that for any set $X \subseteq_{<\omega} \mathcal{M}$, $\mathrm{Dim}(g(X)/X) \leq n$; this is the original definition of a bounded automorphism in [Las92].

The mentioned result by Lascar implies that if F and K are algebraically closed field of characteristic zero such that $K \subseteq F$ and the transcendental degree of F over K is strictly bigger than $\aleph_0$, then the automorphism group of F that fixes K point-wise is a simple group. Especially it implies that the automorphism group of the complex numbers that fixes algebraic points is a simple group. Lascar's result has been directly used in [GT14] to prove the existence of simple groups with BN-pairs which do not arise from algebraic groups.

His method inspired many works recently and has been generalized for a broader class of structures (see [MT11, TZ13, EGT16]). In [TZ13], Tent and Ziegler provide a criterion for the (bounded) simplicity of the automorphism groups of certain countable structures. A key feature in their paper is the use of a natural independence relation, called *stationary independence.* Then using that they show the isometry group of the Urysohn space modulo the normal subgroup of bounded isometries is a simple group.

In [EGT16], it has been shown that the automorphism groups of certain countable structures obtained using the Hrushovski's amalgamation method are simple groups. The structures that have been considered in [EGT16] are the un-collapsed structures of infinite Morley rank obtained by the ab-initio construction and the (unstable) ω-categorical pseudoplanes. The simplicity of the automorphism groups of these follows from results which generalize work of Lascar and of Tent and Ziegler.

Theorem 5. *[Theorem 4.15. in* [EGT16]*] Suppose α the coefficient of the pre-dimention δ_α is rational and $\alpha = \frac{n}{m}$. Suppose either that $n_{\mathfrak{R}} = 2$ such that $n > m$, or that $n_{\mathfrak{R}} \geq 3$ and $n \geq m$. Then $\mathrm{Aut}\left(\mathbf{M}_\alpha/\mathrm{cl}^d(\emptyset)\right)$ is a simple group. In fact, if $g \in \mathrm{Aut}\left(\mathbf{M}_\alpha/\mathrm{cl}^d(\emptyset)\right)$ is not the identity then every element of $\mathrm{Aut}\left(\mathbf{M}_\alpha/\mathrm{cl}^d(\emptyset)\right)$ can be written as a product of 96 conjugates of $g^{\pm 1}$.*

In [EGT16] the notion of *monodimensionality* for a structure with a dimension function has been defined (see Def. 3.5. in [EGT16]). Then for the ω-categorical case the following theorem holds

Theorem 6. *[Theorem 5.10. in* [EGT16]*] Suppose the assumption that has been described in Section 2.1 for the class* $\mathbf{K}_\alpha^f$ *hold. Suppose* $\mathbf{M}_\alpha^f$ *is monodimensional and* $1 \neq g \in \operatorname{Aut}\left(\mathbf{M}_\alpha^f\right)$. *Then every element of* $\operatorname{Aut}\left(\mathbf{M}_\alpha^f\right)$ *is a product of* 192 *conjugates of* $g^{\pm 1}$. *In particular,* $\operatorname{Aut}\left(\mathbf{M}_\alpha^f\right)$ *is a simple group.*

In [EGT16], it has been proven that the automorphism group of an uncollapsed ab-initio generic structure with rational coefficients which fixes pointwise every dimension-zero set is boundedly simple. However, the developed machinery, is not adequate to answer the simplicity question for two important classes of generic structures: the class is ab-initio generic structures that are obtained from pre-dimension functions with irrational coefficients and the smooth classes without AC. The generic structures of the two classes of generic structures mainly lay in a broader framework of stable and simple theories.

There is another interesting feature about the bounded automorphism groups. Indeed, Lascar proves that the bounded automorphism group of the complex numbers that are fixing point-wise the algebraic points (algebraic closure of the prime field) is trivial. He also shows that the bounded automorphism group of a *locally modular* structure is always non-trivial. In [Gha13], it has been shown that there are no non-trivial bounded automorphisms in the automorphism group of the Hrushovski's strongly minimal structures. More recently bounded automorphisms have been studied in [BHMP15] and [Wag15]. The natural question that arises is whether the bounded automorphism group can characterize the locally modularity (question in page 248 of [Las92]). It is interesting to determine whether the *ample hierarchy*, which is a combinatorial notion of geometric properties of forking, even in a more general context of stable structures can be characterized by bounded automorphisms (see Question C in Chapter 5.2. in [Gha13]).

4. The small index property and Ample Generics

Suppose $\mathcal{M}$ is a first-order countable structure and let $G := \operatorname{Aut}(\mathcal{M})$. A subgroup H of G is said to have *small index* in G if $[G : H] < 2^{\aleph_0}$. Consider the usual point-wise convergence topology on G. One can easily see that open subgroups of G has small index in G. We say G has the *small index property*, denoted by SIP, if every subgroup of G with small index in G is open.

When G has the small index property, then the topological structure of G can be recovered from its abstract group structure. This property has applications in *reconstruction* of a structure from its abstract group (see [Las02, Mac11] for more details). For instance, when G_1 has SIP, then any abstract group isomorphism $h : G_1 \to G_2$ leads to a topological isomorphism, in particular when they are automorphism groups of some ω-categorical structures M_1 and M_2; respectively, then the two structures M_1 and M_2 are *bi-interpretable* (see [AZ86]).

The small index property has been proven for the automorphism groups of various countable ω-categorical first-order structures: The countable infinite set without structure; The countable dense linear ordering $(\mathbb{Q}, <)$; A vector space of dimension ω over a finite or countable division ring; The random graph; Countable ω-stable ω-categorical structures (see [Las02, Mac11] for references). There are few known examples outside the ω-categorical case.

There are few known methods for proving the small index property (see Chaper 5.2. in [Mac11] for more details). Since generic structures, similar to the Fraïssé-limit structures are built from a class of finite structures the following procedure is a suggestive way of proving the automorphism group of a $(\mathcal{C}, \leqslant)$-generic has SIP. First, prove the class $(\mathcal{C}, \leqslant)$ has the *extension property*. Then similar to Theorem 6.2 in [KPT05] conclude that G has *ample generics* (see Def. 2.7 in [HHLS93a]) and then using Theorem 5.3 in [HHLS93b] show G has SIP.

Hrushovski in [Hru92], proved that the class of all finite graphs has extension property. His result have been generalized by Herwig in [Her95] to a broader class of structures. The extension property can be modified for smooth classes as follows.

Definition 7. We say a smooth class $(\mathcal{C}, \leqslant)$ has the *extension property*, denoted by EP, if for every $A \in \mathcal{C}$ and $e_0, \cdots, e_n$, finite elementary maps of $\leqslant$-closed subsets of A, there exist $B \in \mathcal{C}$ and $f_i \in \mathrm{Aut}(B)$'s such that $A \subseteq B$ and f_i's extend e_i's; respectively for $0 \leq i \leq n$.

In the case of Hrushovski generic structures it turns out that EP fails. In Chapter 5.1 in [Gha13] it has been shown that in the un-collapsed ab-initio classes that are obtained from pre-dimensions with rational coefficients and in the collpased ab-initio classes, EP fails. Recently in [GKP15] a connection between having substructures that form a *tree-pair* (see Def. 12) and failure of EP has been observed. It is interesting to comment that for the classes that are obtained from pre-dimensions with irrational coefficients (or simple ω-categorical generic structures with rational coefficients see [Gha13, EGT16]) one can still show EP fails: with a slightly different argument and with at least two partial isomorphisms. David M. Evans in an email correspondence has also noted that using a different proof, he can show EP does not hold for both classes that are obtained from pre-dimensions with rational and irrational coefficients. However, the following holds

Theorem 8. *[Theorem 6. in* [Gha15]*] Suppose α the coefficient of the pre-dimention δ_α is rational. Let d be the dimension function that is defined from δ_α. Let $\mathbf{M}_{\alpha,0} := \mathrm{cl}^d(\emptyset)$ and $\mathbf{K}_{\alpha,0} := \{A \subset_{fin} \mathbf{M}_{\alpha,0} : \delta_\alpha(A) = 0\}$, up to isomorphism. Then the class $(\mathbf{K}_{\alpha,0}, \leqslant_\alpha)$ has the extension property. Moreover,* $\mathrm{Aut}(M_0)$ *has ample generics and hence* $\mathrm{Aut}(M_0)$ *has SIP.*

Moreover, using that the following theorem has been proven

Theorem 9. *[Theorem 8. in* [Gha15]*] Suppose α the coefficient of the pre-dimension δ_α is rational. Let $\mathbf{M}_\alpha$ be the $(\mathbf{K}_\alpha, \leqslant_\alpha)$-generic structure. Then $G = \mathrm{Aut}(\mathbf{M}_\alpha)$ has SIP.*

The proof of Theorem 9 is using a technique in Théoremè 2 by Lascar in [Las92]. Lascar in [Las92] proves the following: Suppose that $\mathcal{M}$ is a countable saturated structure and it is almost strongly minimal then if H is a subgroup of a small index in $G = \mathrm{Aut}(\mathcal{M})$, then there exists a finite set A in $\mathcal{M}$ such that $\mathrm{Autf}_A(\mathcal{M}) \leq H$. We call this *almost SIP*. Note that the topology that is generated by $\mathrm{Autf}_A(\mathcal{M})$ where A a finite subset in $\mathcal{M}$, is indeed a finer topology that the usual pointwise convergence topology on G and it is not necessarily Polish. A similar theorem has been suggested in [Gha13, EGT16] and has been proven in [Gha15] (see Theorem 4. in [Gha15]). Then from the results in [Gha15] follows that the automorphism groups of un-collapsed ab-initio generic structures that are obtained from pre-dimension function with rational coefficients have indeed *almost* SIP and moreover, SIP. However, their ab-initio classes fail to have EP.

The small index property and almost SIP for the automorphism groups of the following generic structures still remain unanswered: ab-initio generic structures which are obtained from pre-dimension functions with irrational coefficients, and simple ω-categorical generic structures. Moreover, still the question whether the automorphism groups of Hrushvski generic structures in have ample generics remains open. Another related interesting property also remains unanswered: it is not known whether or not for the automorphism groups of Hrushovski generic structures in both cases of collpased and un-collapsed Bergman property (see Def. 5.5.3. in [Mac11]) holds. Holding this property for the automorphism group of Hrushovski's generic structures and then modifying it for the generalized n-gons constructed by Tent in [Ten00], would give us some interesting family of algebraic/geometric objects which the automorphism group has the Bergman property. In Section 5.4. in [Mac11] Rubin's approach to reconstruction for countable ω-categorical have been discussed. It is interesting to verify whether Rubin's approach can be adopted for some generic structures and some versions of *weak* $\forall\exists$-*interpretation* (see Def. 5.4.1 in [Mac11]) can be proven for Hrushovski generic structures.

5. Ramsey Property

Let G be a topological group. A continuous action Γ of G on a compact Hausdorff space X is called a *G-flow*. Group G is called *extremely amenable* if every G-flow (G, Γ, X) has a fix point in X. A Hausdorff topological group G is *amenable* if every G-flow (G, Γ, X) supports an G-invariant Borel probability measure on X.

In the seminal paper [KPT05] of Kechris, Pestov and Todorcevic a correspondence between extreme amenability of $\mathrm{Aut}(\mathcal{M})$, where $\mathcal{M}$ is the

Fraïssé-limit of a Fraïssé class $\mathcal{K}$ and the *Ramsey property* for the class $\mathcal{K}$ has been discovered. Since then an extensive research has been devoted to studying dynamical properties of automorphism groups of Fraïssé-limit structures (see [Hod97, KPT05] for more information). In [KPT05], they have shown that the automorphism group of an ordered Fraïssé-limit structure $\mathcal{M}$ is extremely amenable if and only if its ordered Fraïssé-class has the Ramsey property. Later in [Moo13], it has been shown that Aut $(\mathcal{M})$ is amenable if and only if $\mathcal{K}$ has the *convex Ramsey property.*

In [GKP15], similar correspondences of [KPT05] and [Moo13] between extreme amenability and amenability of the automorphism group of a $(\mathcal{C}, \leqslant)$-generic model, and the modified definition of the Ramsey property and the convex Ramsey property; respectively for $(\mathcal{C}, \leqslant)$ have been proven. Suppose $A \in \mathcal{C}$ and let $\mathcal{N}$ is any $\mathfrak{L}$-structure. We denote $\binom{\mathcal{N}}{A}$ for the set of all $\leqslant$-embeddings of A into $\mathcal{N}$. For $k \in \mathbb{N} \setminus \{0\}$, we call a function $c : \binom{N}{A} \to \{0, 1, \cdots, k-1\}$ a *k-coloring function.* Suppose $A \in \mathcal{C}$. The group G acts naturally on $\binom{\mathcal{M}}{A}$ in the following way: When $\Gamma \in \binom{\mathcal{M}}{A}$

$$g \cdot \Gamma := \Gamma'$$

if $\Gamma'(A) = g[\Gamma(A)]$. It is worth noting, since elements of G sends $\leqslant$-closed sets to $\leqslant$-closed sets, this action is well-defined. We say G *preserves a linear ordering* $\preceq$ on $\mathcal{M}$ if $a \preceq b$, implies $g(a) \preceq g(b)$, for every $a, b \in \mathcal{M}$ and $g \in G$. Assume $A \leqslant B \leqslant C \in \mathcal{K}$ and $k \geq 1$. We write

$$C \longrightarrow (B)^{A}_{k},$$

if for every k-coloring $c : \binom{C}{A} \longrightarrow \{0, 1, \cdots, k-1\}$, there exists $\lambda \in \binom{C}{B}$ such that $c(\lambda \circ \gamma)$ is constant for all $\gamma \in \binom{\lambda(B)}{A}$. We say that the class $(\mathcal{C}, \leqslant)$ has the *$\leqslant$-Ramsey property* if for every $A \leqslant B \in \mathcal{C}$ and $k \geq 2$, there exists $C \in \mathcal{C}$ with $B \leqslant C$ such that $C \longrightarrow (B)^{A}_{k}$. Similar to the classical Ramsey theory it is enough to show $\leqslant$-Ramsey property when $k = 2$. Here is the correspondence similar to Proposition 4.3. in [KPT05]

Theorem 10. *[Theorem 18. in* [GKP15]*] The followings are equivalent:*

(1) *G is extremely amenable;*
(2) (a) *G preserves a linear ordering;*
    (b) *$(\mathcal{C}, \leqslant)$ has the $\leqslant$-Ramsey property.*

Suppose $A \in \mathcal{K}$ and $\mathcal{N}$ is a substructure of $\mathcal{M}$. Denote $\left\langle \mathcal{N} \atop A \right\rangle$ for the set of all finitely supported probability measures on $\binom{\mathcal{N}}{A}$. If $f : \binom{\mathcal{N}}{A} \to \{0, 1\}$ is a 2-coloring map, then f extends to a linear function defined on the vector space generated by $\binom{\mathcal{N}}{A}$; with abuse of notation this extension will also be denoted by f . Suppose $X \leqslant Y \leqslant Z$ are $\leqslant$-closed substructures of $\mathcal{M}$ and let $\epsilon \in \left\langle {Z \atop Y} \right\rangle$. We define $\left\langle {\epsilon \atop X} \right\rangle$ to be the set

$$\left\{ \delta \in \left\langle {Z \atop X} \right\rangle : \exists \theta \in \left\langle {Y \atop X} \right\rangle, \delta(\Lambda \circ \Gamma) = \epsilon(\Lambda) \cdot \theta(\Gamma) \, \forall \Gamma \in \binom{Y}{X} \, \forall \Lambda \in \binom{Z}{Y} \right\}.$$

We say $\mathrm{Aut}(\mathcal{M})$ has the *convex $\leqslant$-Ramsey property with respect to* $(\mathcal{C}, \leqslant)$ if for every $A, B \in \mathcal{C}$ with $A \leqslant B$ and every 2-coloring function $f : \binom{\mathcal{M}}{A} \to \{0,1\}$, there exists $\beta \in \left\langle {\mathcal{M} \atop B} \right\rangle$ such that $|f(\alpha_1) - f(\alpha_2)| \leq \frac{1}{2}$ for every $\alpha_1, \alpha_2 \in \left\langle {\beta \atop A} \right\rangle$. Similar to Theorem 6.1. in [Moo13] the following holds.

Theorem 11. *[Theorem 31. in* [GKP15]*] Suppose $\mathcal{M}$ is the $(\mathcal{C}, \leqslant)$-generic structure of a smooth class $(\mathcal{C}, \leqslant)$ with JEP and AP. Then, the followings are equivalent:*

(1) $\mathrm{Aut}(\mathcal{M})$ *has the convex $\leqslant$-Ramsey property with respect to $(\mathcal{C}, \leqslant)$.*
(2) *For every $A, B \in \mathcal{C}$ with $A \leqslant B$, there is $C \in \mathcal{C}$ such that $B \leqslant C$ and for every $f : \binom{C}{A} \to \{0,1\}$ there is $\beta \in \left\langle {C \atop B} \right\rangle$ such that for every $\alpha, \alpha' \in \left\langle {\beta \atop A} \right\rangle$,*
$$\left| f(\alpha) - f(\alpha') \right| \leq \frac{1}{2}.$$
(3) *For every $A, B \in \mathcal{C}$ with $A \leqslant B$ and every $\epsilon > 0$, there is $C \in \mathcal{C}$ such that $B \leqslant C$ and for every $f : \binom{C}{A} \to [0,1]$ there is $\beta \in \left\langle {C \atop B} \right\rangle$ such that for every $\alpha, \alpha' \in \left\langle {\beta \atop A} \right\rangle$,*
$$\left| f(\alpha) - f(\alpha') \right| \leq \epsilon.$$
(4) *For every $A, B \in \mathcal{K}$ with $A \leqslant B$ and every $\epsilon > 0$ and $n \in \mathbb{N}$, there is $C \in \mathcal{C}$ such that $B \leqslant C$ for every sequence of functions $f_i : \binom{C}{A} \to [0,1]$ with $i < n$, there is $\beta \in \left\langle {C \atop B} \right\rangle$ such that for every $\alpha, \alpha' \in \left\langle {\beta \atop A} \right\rangle$ and $i < n$,*
$$\left| f_i(\alpha) - f_i(\alpha') \right| \leq \epsilon.$$
(5) $\mathrm{Aut}(\mathcal{M})$ *is amenable.*

Definition 12. Let $(\mathcal{C}, \leqslant)$ be a smooth class and $\mathcal{M}$ the $(\mathcal{C}, \leqslant)$-generic model. Let $A, B \in \mathcal{C}$ with $A \leqslant B$. Define an indirected graph on the set $\binom{\mathcal{M}}{B}$ as follows. For $\Lambda, \Lambda' \in \binom{\mathcal{M}}{B}$ we define an edge between Λ and Λ' if $\Lambda(B) \cap \Lambda'(B)$ contains at least one $\leqslant$-closed copy of A. Call this graph $(A; B)$-graph. We say $(A; B)$ is a *tree-pair* if the following conditions hold:

(1) $\Lambda(B) \cap \Lambda'(B)$ contains at most one $\leqslant$-closed copy of A for any two distinct $\Lambda, \Lambda' \in \binom{\mathcal{M}}{B}$;

(2) Every connected component of the corresponding $(A;B)$-graph forms a tree.

Then the following general theorem has been proven

Theorem 13. *[Theorem 39. in* [GKP15]*] Suppose $(\mathcal{C}, \leqslant)$ is a smooth class with AP and HP, and $\mathcal{M}$ the $(\mathcal{C}, \leqslant)$-generic structure. Suppose there are $A, B \in \mathcal{C}$ and $A \leqslant B$ such that $(A;B)$ is a tree-pair with $\left|\binom{B}{A}\right| = 6$. Then,* $\mathrm{Aut}(\mathcal{M})$ *does not have the convex $\leqslant$-Ramsey property with respect to $(\mathcal{C}, \leqslant)$.*

Theorems 10,11 and Theorem 13 have been the used to determine whether the automorphism groups of Hrushovski ab-initio generic structures are extremely amenable and amenable.

Theorem 14. *[Theorem 42. in* [GKP15]*] Suppose α the coefficient of the pre-dimention δ_α is rational. Let $\mathbf{M}_\alpha$ be the $(\mathbf{K}_\alpha, \leqslant_\alpha)$-generic structure. There are A, B in $\mathbf{K}_\alpha$ with $A \leqslant_\alpha B$ and $\left|\binom{B}{A}\right| = 6$ such that $(A;B)$ is a tree-pair. Hence,* $\mathrm{Aut}(\mathbf{M}_\alpha)$ *does not have the convex $\leqslant_\alpha$-Ramsey property with respect to $(\mathbf{K}_\alpha, \leqslant_\alpha)$ and* $\mathrm{Aut}(\mathbf{M}_\alpha)$ *is not amenable.*

Moreover, it has been shown that

Theorem 15. *[Theorem 45. in* [GKP15]*] The automorphism groups of ordered Hrushovski generic structures that are obtained from pre-dimension functions with rational coefficients are not extremely amenable.*

The method in [GKP15] provides an explicit coloring function for a tree-pair that the convex Ramsey and Ramsey property fails. David M. Evans independently using a different method shows, the automorphism groups of generic structures that are obtained from pre-dimension functions with irrational coefficients and the ω-categorical generic structures are not amenable. However, it is interesting to provide an example of a generic structure that its automorphism group is amenable and it is obtained from a smooth class, which is a not Fraïssé-class.

6. Group-reducts

Suppose G is a closed subgroup of S_ω. A closed subgroup $H \lneqq S_\omega$ that $G \lneqq H$ is called a *(proper) group-reduct*. When G is the automorphism group of a countable ω-categorical structure then group-reducts and automorphism of (proper) *definable* reducts are in one-to-one correspondence. Therefore, the question about the full classification of group-reducts of ω-categorical structures are especially interesting. For a number of ω-catgorical structures the full classification of (group or definable) reducts is known (see [Cam76, Tho91, Tho96, JZ08]). However, the main question asked by Thomas remains unresolved: we do not know whether every homogeneous structure on a finite relational language has only finitely many reducts.

Generic structures do not necessarily fall in the case of homogeneous structures over a finite relational language, however, it turns out still many techniques of homogeneous structures can be adopted. It has been show in [HM13], that the rank ω structure obtained by the un-collapsed Hrushovski generic structure has a proper definable reduct. In [Gha14] it has been shown the automorphism groups of Hrushovski's ab-initio generic structures that are obtained from pre-dimensions with rational coefficients in both case of collapsed and un-collapsed, have uncountably many group-reducts.

The automorphism groups of definable reducts in [HM13] and group-reducts in [Gha14] all seem to preserve the geometry of the structure. In [KS16] Question 3.2, it has been specifically asked whether the automorphism group of the following construction is maximal. Let $\mathfrak{R}$ be a ternary relation symbol, and let $\alpha = 1$ define the pre-dimension δ_α on the class of 3-hypergraphs by $\delta_1(A) = |A| - |\mathfrak{R}(A)|$. Consider the family $\mathbf{K}_1^+$ of 3-hypergraphs A for which $\delta_1(A_0) \geq \min\{|A_0|, 2\}$ for any $A_0 \subseteq A$. The usual Fraïssé-Hrushovski generic structure associated with this class and pre-dimension gives a countable structure $\mathbf{M}_1^+$ equipped with a dimension function d. It is asked in [KS16] whether the set of all bijections that preserves the geometry is a maximal closed subgroup. It seems that is the case based on some partial results in [Gha14, HM13], however it is not fully answered yet.

References

[AZ86] Gisela Ahlbrandt and Martin Ziegler, *Quasi-finitely axiomatizable totally categorical theories*, Ann. Pure Appl. Logic **30** (1986), no. 1, 63–82, Stability in model theory (Trento, 1984). MR 831437

[Bal15] John T. Baldwin, *A field guide to hrushovski constructions*, preprint (2015).

[BHMP15] Thomas Blossier, Charlotte Hardouin, and Amador Martin-Pizarro, *Sur les automorphismes borné s de corps munis d'opérateurs*, Arxiv (2015).

[BS96] John T. Baldwin and Niandong Shi, *Stable generic structures*, Ann. Pure Appl. Logic **79** (1996), no. 1, 1–35. MR 1390325 (97c:03103)

[Cam76] Peter J. Cameron, *Transitivity of permutation groups on unordered sets*, Math. Z. **148** (1976), no. 2, 127–139. MR 0401885

[Cam90] ______, *Oligomorphic permutation groups*, London Mathematical Society Lecture Note Series, vol. 152, Cambridge University Press, Cambridge, 1990. MR 1066691 (92f:20002)

[CV06] P. J. Cameron and A. M. Vershik, *Some isometry groups of the Urysohn space*, Ann. Pure Appl. Logic **143** (2006), no. 1-3, 70–78. MR 2258622 (2008b:54045)

[EGT16] David M. Evans, Zaniar Ghadernezhad, and Katrin Tent, *Simplicity of the automorphism groups of some Hrushovski constructions*, Ann. Pure Appl. Logic **167** (2016), no. 1, 22–48. MR 3413492

[Eva02] David M. Evans, *$\aleph_0$-categorical structures with a predimension*, Annals of Pure and Applied Logic (2002), no. 116, 157–186.

[Gha13] Zaniar Ghadernezhad, *Automorphism groups of generic structures.*, Münster: Univ. Münster, Mathematisch-Naturwissenschaftliche Fakultät, Fachbereich Mathematik und Informatik (Diss.), 2013 (English).

[Gha14] Zaniar Ghadernezhad, *Uncountably many group reducts of hrushovski's generic structure*, Preprint, 2014.

[Gha15] Z. Ghadernezhad, *The small index property of automorphism groups of ab-initio generic structures*, submitted (2015).

[GKP15] Z. Ghadernezhad, H. Khalilian, and M. Pourmahdian, *Automorphism of generic strutures: Extreme amenability and amenability*, submitted (2015).

[GT14] Z. Ghadernezhad and K. Tent, *New simple groups with a BN-pair*, J. Algebra **414** (2014), 72–81. MR 3223389

[Her95] Bernhard Herwig, *Extending partial isomorphisms on finite structures*, Combinatorica **15** (1995), no. 3, 365–371. MR 1357282 (97a:03044)

[HHLS93a] Wilfrid Hodges, Ian Hodkinson, Daniel Lascar, and Saharon Shelah, *The small index property for ω-stable ω-categorical structures and for the random graph*, J. London Math. Soc. (2) **48** (1993), no. 2, 204–218. MR 1231710 (94d:03063)

[HHLS93b] ______, *The small index property for ω-stable ω-categorical structures and for the random graph*, J. London Math. Soc. (2) **48** (1993), no. 2, 204–218. MR 1231710 (94d:03063)

[HM13] Assaf Hasson and Omer Mermelstein, *On reducts of Hrushovski's construction - the non-collapsed case*, ArXiv (2013).

[Hod93] Wilfrid Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993. MR 1221741

[Hod97] ______, *A shorter model theory*, Cambridge University Press, Cambridge, 1997. MR 1462612 (98i:03041)

[Hru92] Ehud Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), no. 4, 411–416. MR 1194731 (93m:05089)

[Hru93] ______, *A new strongly minimal set*, Ann. Pure Appl. Logic **62** (1993), no. 2, 147–166, Stability in model theory, III (Trento, 1991). MR 1226304 (94d:03064)

[JZ08] Markus Junker and Martin Ziegler, *The 116 reducts of* $(\mathbb{Q}, <, a)$, J. Symbolic Logic **73** (2008), no. 3, 861–884. MR 2444273

[KL92] D. W. Kueker and M. C. Laskowski, *On generic structures*, Notre Dame J. Formal Logic **33** (1992), no. 2, 175–183. MR 1167973 (93k:03032)

[KPT05] A. Kechris, V. Pestov, and S. Todorcevic, *Fraïssé limites, ramsey theory, and topological dynamics of automorphism groups*, GAFA (2005), no. 15, 106 –189.

[KS16] Itay Kaplan and Pierre Simon, *The affine and projective groups are maximal*, Trans. Amer. Math. Soc. **368** (2016), no. 7, 5229–5245. MR 3456178

[Las92] Daniel Lascar, *Les automorphismes d'un ensemble fortement minimal*, J. Symbolic Logic **57** (1992), no. 1, 238–251. MR 1150937 (93c:03042)

[Las02] D. Lascar, *Automorphism groups of saturated structures; a review*, Proceedings of the ICM **vol. 2** (Beijing 2002), 25–36.

[Mac11] Dugald Macpherson, *A survey of homogeneous structures*, Discrete Math. **311** (2011), no. 15, 1599–1634. MR 2800979 (2012e:03063)

[Moo13] Justin Tatch Moore, *Amenability and Ramsey theory*, Fund. Math. **220** (2013), no. 3, 263–280. MR 3040674

[MT11] Dugald Macpherson and Katrin Tent, *Simplicity of some automorphism groups*, J. Algebra **342** (2011), 40–52. MR 2824528

[Pou02] Massoud Pourmahdian, *Smooth classes without AC and Robinson theories*, J. Symbolic Logic **67** (2002), no. 4, 1274–1294. MR 1955238 (2004f:03066)

[Ten00] K. Tent, *Very homogeneous generalized n-gons of finite Morley rank*, J. London Math. Soc. (2) **62** (2000), no. 1, 1–15. MR 1771846 (2001f:51011)

[Tho91] Simon Thomas, *Reducts of the random graph*, J. Symbolic Logic **56** (1991), no. 1, 176–181. MR 1131738

[Tho96] ______, *Reducts of random hypergraphs*, Ann. Pure Appl. Logic **80** (1996), no. 2, 165–193. MR 1402977

[TZ13] Katrin Tent and Martin Ziegler, *On the isometry group of the Urysohn space*, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 289–303. MR 3022717

[Wag94] Frank O. Wagner, *Relational structures and dimensions*, Automorphisms of first-order structures, Oxford Sci. Publ., Oxford Univ. Press, New York, 1994, pp. 153–180. MR 1325473

[Wag09] ———, *Hrushovski's amalgamation construction*, Logic Colloquium 2006, Lect. Notes Log., vol. 32, Assoc. Symbol. Logic, Chicago, IL, 2009, pp. 361–373. MR 2562559 (2011a:03036)

[Wag15] Frank Olaf Wagner, *Bounded automorphisms in simple structures*, Arxiv (2015).

[Zie13] Martin Ziegler, *An exposition of Hrushovski's new strongly minimal set*, Ann. Pure Appl. Logic **164** (2013), no. 12, 1507–1519. MR 3093399

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.

E-mail address: zaniar.gh@gmail.com