数理解析研究所講究録
第2004巻 2016年 77-88

Surface-links and marked graph diagrams

Sang Youl Lee
Department of Mathematics, Pusan National University

1 Introduction

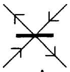
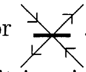
A *surface-link* is a closed surface smoothly embedded in Euclidean 4-space $\mathbb{R}^4$. A *surface-knot* is a one component surface-link. A 2-sphere-link is sometimes called a 2-*link*. A 2-link of one component is called a 2-*knot*. Two surface-links $\mathcal{L}$ and $\mathcal{L}'$ in $\mathbb{R}^4$ are *equivalent* if they are ambient isotopic, that is, there is an orientation preserving homeomorphism $h:\mathbb{R}^4\rightarrow\mathbb{R}^4$ such that $h(\mathcal{L})=\mathcal{L}'$ or, equivalently, there exists a smooth family of diffeomorphisms $f_s:\mathbb{R}^4\rightarrow\mathbb{R}^4(s\in[0,1])$ such that $f_0=\mathrm{id}_{\mathbb{R}^4}$, the identity of $\mathbb{R}^4$, and $f_1(\mathcal{L})=\mathcal{L}'$. If each component $\mathcal{K}_i$ of a surface-link $\mathcal{L}=\mathcal{K}_1\cup\cdots\cup\mathcal{K}_\mu(\mu\geq 1)$ is oriented, then $\mathcal{L}$ is called an *oriented surface-link*. Two oriented surface-links $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent* if the restriction $h|_{\mathcal{L}}:\mathcal{L}\rightarrow\mathcal{L}'$ of h is also orientation preserving.

A *marked graph diagram* is a link diagram in $\mathbb{R}^2$ possibly with some 4-valent vertices in which each 4-valent vertex has a marker indicated by a small segment "▁▁". S. J. Lomonaco, Jr. [15] and K. Yoshikawa [18] introduced a method of presenting surface-links using marked graph diagrams. Indeed, every surface-link is presented by a marked graph diagram (cf. [15, 18]) and such a presentation diagram is unique up to Yoshikawa moves (see Theorem 2.3). By using marked graph diagram presentation for surface-links, some properties and invariants of surface-links were studied in [1, 2, 4, 6, 8, 9, 12, 13, 14, 16, 18].

In this short survey paper, we give a brief introduction to marked graph diagram presentation of surface-links and a method of constructing ideal coset invariants for surface-links introduced in [4, 14] by means of a polynomial invariant $\ll\cdot\gg$ for marked graphs in $\mathbb{R}^3$ defined by using a state-sum model with classical link invariants as its state evaluation. Section 2 presents marked graph diagram presentation of surface-links. Section 3 deals with the polynomial invariant $\ll\cdot\gg$ for marked graphs in $\mathbb{R}^3$. Section 4 discusses ideal coset invariants derived from the polynomial $\ll\cdot\gg$. An extended version of this paper will be appear in elsewhere.

2 Marked graph diagrams of surface-links

A *marked graph* is a spatial graph G in $\mathbb{R}^3$ such that G is a finite regular graph possibly with 4-valent vertices, say $v_1,v_2,\ldots,v_n$; each v_i is a rigid vertex, i.e., we fix a sufficiently small rectangular neighborhood $N_i\cong\{(x,y)\in\mathbb{R}^2|-1\leq x,y\leq 1\}$, where v_i corresponds to the origin and the edges incident to v_i are represented by $x^2=y^2$; each v_i has a *marker*, which is the interval on N_i given by $\{(x,0)\in\mathbb{R}^2|-\frac{1}{2}\leq x\leq\frac{1}{2}\}$. Two marked graphs are *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping rectangular neighborhoods and markers.

An *orientation* of a marked graph G is a choice of an orientation for each edge of G in such a way that every vertex in G looks like  or . A marked graph is said to be *orientable* if it admits an orientation. Otherwise, it is said to be *nonorientable.* By an *oriented marked graph* we mean an orientable marked graph with a fixed orientation. Two oriented marked graphs are *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping rectangular neighborhoods, orientation and markers. An oriented marked graph G in $\mathbb{R}^3$ can be described as usual by a diagram D in $\mathbb{R}^2$, which is an oriented link diagram in $\mathbb{R}^2$ possibly with some marked 4-valent vertices whose incident four edges have orientations illustrated as above, and is called an *oriented marked graph diagram* of G (cf. Figure 1).

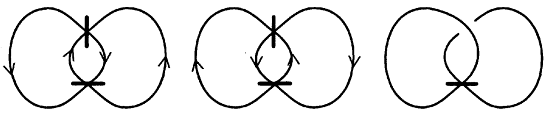


Figure 1: Oriented marked graph diagrams and a nonorientable marked graph diagram

Two oriented marked graph diagrams in $\mathbb{R}^2$ represent equivalent oriented marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of oriented mark preserving rigid vertex 4-regular spatial graph moves (simply, *oriented mark preserving RV4 moves*) $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_4'$ and Γ_5 shown in Figure 2, which consists Yoshikawa moves of type I (see Theorem 2.3).

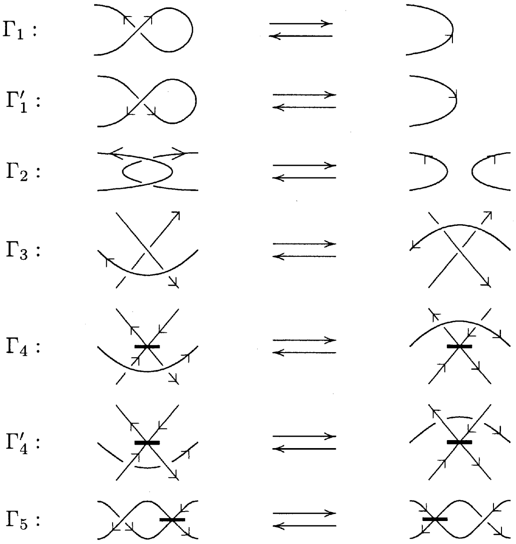
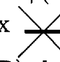
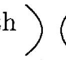


Figure 2: Oriented mark preserving RV4 moves

An *unoriented* marked graph diagram or, simply, a marked graph diagram is a nonorientable or an orientable but not oriented marked graph diagram in $\mathbb{R}^2$. Two marked graph diagrams in $\mathbb{R}^2$ represent equivalent marked graphs in $\mathbb{R}^3$ if and only if they are transformed into each other by a finite sequence of the moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_4'$ and Ω_5, where Ω_i stands for the move Γ_i without orientation.

For an (oriented) marked graph diagram D, let $L_-(D)$ and $L_+(D)$ be the (oriented) link diagrams obtained from D by replacing each marked vertex  with  and , respectively, as illustrated in Figure 3. We call $L_-(D)$ and $L_+(D)$ the *negative resolution* and the *positive resolution* of D, respectively. An (oriented) marked graph diagram D is *admissible* if both resolutions $L_-(D)$ and $L_+(D)$ are trivial link diagrams.

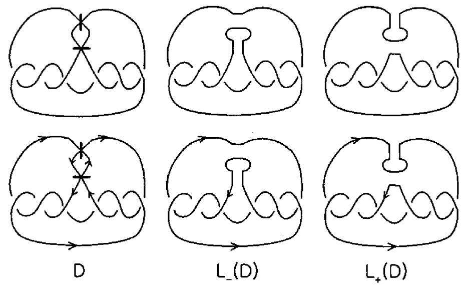


Figure 3: Marked graph diagrams and their resolutions

Let D be a given admissible marked graph diagram with marked vertices $v_1, \ldots, v_n$. Define a surface $F(D) \subset \mathbb{R}^3 \times [-1, 1]$ by

$$(\mathbb{R}^3_t, F(D) \cap \mathbb{R}^3_t) = \begin{cases} (\mathbb{R}^3, L_+(D)) & \text{for } 0 < t \leq 1, \\ \left(\mathbb{R}^3, L_-(D) \cup \left(\bigcup_{i=1}^{n} B_i\right)\right) & \text{for } t = 0, \\ (\mathbb{R}^3, L_-(D)) & \text{for } -1 \leq t < 0, \end{cases}$$

where $\mathbb{R}^3_t := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$ and $B_i (1 \leq i \leq n)$ is a band attached to $L_-(D)$ at each marked vertex v_i as illustrated in Figure 4. We call $F(D)$ the *proper surface associated with D*.

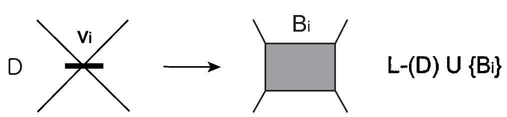


Figure 4: A band attached to $L_-(D)$ at v_i

When D is oriented, $L_-(D)$ and $L_+(D)$ have the orientations induced from the orientation of D (cf. Figure 3). We assume that the proper surface $F(D)$ is oriented so that the induced orientation on $L_+(D) = \partial F(D) \cap \mathbb{R}^3_1$ matches the orientation of $L_+(D)$.

Since D is admissible, we can obtain a surface-link from $F(D)$ by attaching trivial disks in $\mathbb{R}^3 \times [1, \infty)$ and another trivial disks in $\mathbb{R}^3 \times (-\infty, 1]$. We denote the resulting (oriented) surface-link by $\mathcal{L}(D)$, and call it the *(oriented) surface-link associated with D*. It is well known that the isotopy type of $\mathcal{L}(D)$ does not depend on the choices of trivial disks (cf. [5, 7]). Figure 5 shows a schematic picture of the surface-link $\mathcal{L}(D)$ associated with a marked graph diagram D.

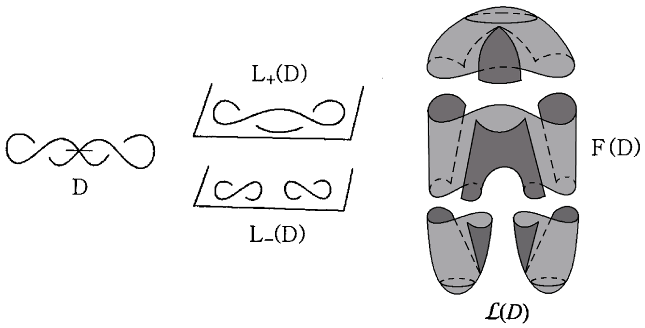


Figure 5: A surface-link $\mathcal{L}(D)$ associated with a marked graph diagram D

Definition 2.1. Let $\mathcal{L}$ be an (oriented) surface-link in $\mathbb{R}^4$. We say that $\mathcal{L}$ is *presented* by an (oriented) marked graph diagram D if $\mathcal{L}$ is ambient isotopic to the (oriented) surface-link $\mathcal{L}(D)$ in $\mathbb{R}^4$.

Let D be an admissible (oriented) marked graph diagram. By definition, $\mathcal{L}(D)$ is presented by D.

From now on, we show that any (oriented) surface-link is presented by an admissible (oriented) marked graph diagram. It is well known [7] that any surface-link $\mathcal{L}$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be deformed into a surface-link $\mathcal{L}'$, called a *hyperbolic splitting* of $\mathcal{L}$, by an ambient isotopy of $\mathbb{R}^4$ in such a way that the projection $p : \mathcal{L}' \to \mathbb{R}$ satisfies the followings:

- all critical points are non-degenerate,
- all the index 0 critical points (minimal points) are in $\mathbb{R}^3_{-1}$,
- all the index 1 critical points (saddle points) are in $\mathbb{R}^3_0$,
- all the index 2 critical points (maximal points) are in $\mathbb{R}^3_1$.

Let $\mathcal{L}$ be a surface-link and let $\mathcal{L}'$ be a hyperbolic splitting of $\mathcal{L}$. Then the cross-section

$$\mathcal{L}'_0 = \mathcal{L}' \cap \mathbb{R}^3_0 \text{ at } t = 0$$

is a spatial 4-valent regular graph in $\mathbb{R}^3_0$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Figure 6.

When $\mathcal{L}$ is an oriented surface-link, we choose an orientation for each edge of $\mathcal{L}'_0$ so that it coincides with the induced orientation on the boundary of $\mathcal{L}' \cap \mathbb{R}^3 \times (-\infty, 0]$ by

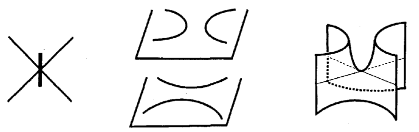


Figure 6: A marker at a 4-valent vertex

the orientation of $\mathcal{L}'$ inherited from the orientation of $\mathcal{L}$. The resulting (oriented) marked graph $G := \mathcal{L}'_0$ is called an *(oriented) marked graph presenting* $\mathcal{L}$. A diagram D of the (oriented) marked graph G is clearly admissible, and is called an *(oriented) marked graph diagram* or *(oriented) ch-diagram presenting* $\mathcal{L}$. In conclusion, we state the followings.

Theorem 2.2 ([7]). (1) Let D be an admissible (oriented) marked graph diagram. Then there is an (oriented) surface-link $\mathcal{L}$ presented by D.

(2) Let $\mathcal{L}$ be an (oriented) surface-link. Then there is an admissible (oriented) marked graph diagram D presenting $\mathcal{L}$.

Theorem 2.3 ([9, 10, 17]). (1) Two oriented marked graph diagrams present the same oriented surface-link if and only if they are transformed into each other by a finite sequence of oriented mark preserving RV4 moves in Figure 2, called *oriented Yoshikawa moves of type I*, and *oriented Yoshikawa moves of type II* in Figure 7.

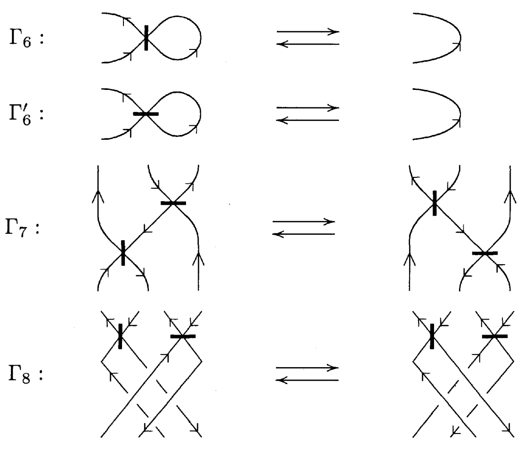


Figure 7: Oriented Yoshikawa moves of type II

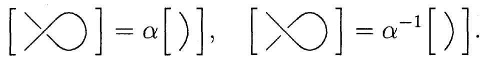
(2) Two unoriented marked graph diagrams present the same unoriented surface-link if and only if they are transformed into each other by a finite sequence of unoriented mark preserving RV4 moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega'_4, \Omega_5$, called *unoriented Yoshikawa moves of type I*, and *unoriented Yoshikawa moves of type II* $\Omega_6, \Omega'_6, \Omega_7$ and Ω_8, where Ω_i stands for the move Γ_i without orientation.

3 Polynomial invariants for marked graphs in $\mathbb{R}^3$ via classical link invariants

Let R be a commutative ring with the additive identity 0 and the multiplicative identity 1 and let

$$[\quad]: \{\text{classical knots and links in } \mathbb{R}^3\} \longrightarrow R$$

be a regular or an ambient isotopy invariant such that for a unit $\alpha \in R$ and an element $\delta \in R$,



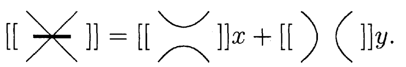
(3.1)

$$\left[K \bigcirc\right] = \delta\left[K\right], \tag{3.2}$$

where $K \bigcirc$ denotes any addition of a disjoint circle $\bigcirc$ to a classical knot or link diagram K.

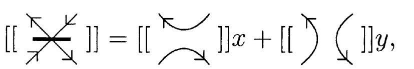
For a given marked graph diagram D, let $[[D]](x, y)$ ([[D]] for short) be a polynomial in $R[x, y]$ defined by the following two rules:

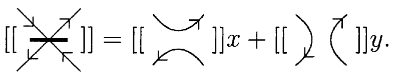
(L1) $[[D]] = [D]$ if D is a link diagram,

(L2) 

When D is an oriented marked graph diagram and $[\quad]$ is an invariant for oriented links, then $[[D]]$ is defined by the rules:

(L1) $[[D]] = [D]$ if D is an oriented link diagram,

(L2) 

(L3) 

Let $D = D_1 \cup \cdots \cup D_m$ be an oriented link diagram and let $w(D_i)$ be the usual writhe of the component D_i. The *self-writhe* $sw(D)$ of D is defined to be the sum

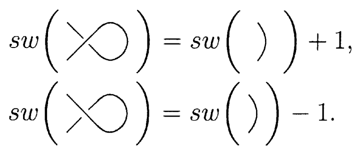
$$sw(D) = \sum_{i=1}^{m} w(D_i).$$

Now let D be a marked graph diagram. We choose an arbitrary orientation for each component of $L_+(D)$ and $L_-(D)$. When D is oriented, we choose orientations for $L_+(D)$ and $L_-(D)$ induced from the orientation of D. We define the *self-writhe* $sw(D)$ of D by

$$sw(D) = \frac{sw(L_+(D)) + sw(L_-(D))}{2},$$

where $sw(L_+(D))$ and $sw(L_-(D))$ are independent of the choice of orientations because the writhe of each component of $L_+(D)$ and $L_-(D)$ is independent of the choice of orientation for the component.

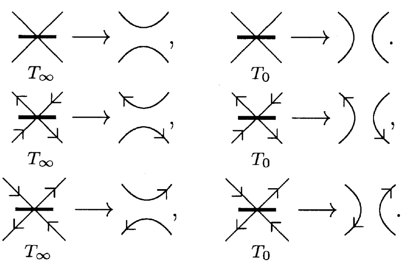
It is noted that the self-writhe $sw(D)$ is invariant under Yoshikawa moves except the move Ω_1. For Ω_1 and its mirror move, we have



Definition 3.1. Let D be an (oriented) marked graph diagram. We define $\ll D \gg (x, y)$ ($\ll D \gg$ for short) to be the polynomial in variables x and y with coefficients in R given by

$$\ll D \gg = \alpha^{-sw(D)}[[D]](x, y) \in R[x, y].$$

Let D be an (oriented) marked diagram. A *state* of D is an assignment of T_∞ or T_0 to each marked vertex in D. Let $\mathcal{S}(D)$ be the set of all states of D. For each state $\sigma \in \mathcal{S}(D)$, let D_σ denote the (oriented) link diagram obtained from D by replacing marked vertices of D with two trivial 2-tangles according to the assignment T_∞ or T_0 by the state σ:



Then $\ll D \gg$ has the following *state-sum formula*:

$$\ll D \gg = \alpha^{-sw(D)} \sum_{\sigma \in \mathcal{S}(D)} [D_\sigma] \, x^{\sigma(\infty)} y^{\sigma(0)},$$

where $\sigma(\infty)$ and $\sigma(0)$ denote the numbers of the assignment T_∞ and T_0 of the state σ, respectively.

Theorem 3.2 ([14]). Let G be an (oriented) marked graph in $\mathbb{R}^3$ and let D be an (oriented) marked graph diagram of G. For any given regular or ambient isotopy invariant

$$[\quad] : \{\text{classical (oriented) links in } \mathbb{R}^3\} \longrightarrow R$$

satisfying the properties (3.1) and (3.2), the polynomial $\ll D \gg$ is an invariant for (oriented) Yoshikawa moves of type I, and therefore it is an invariant of the (oriented) marked graph G in $\mathbb{R}^3$.

4 Ideal coset invariants for surface-links

An *oriented n-tangle diagram* ($n \geq 1$) is an oriented link diagram $\mathcal{T}$ in the rectangle $I^2 = [0, 1] \times [0, 1]$ in $\mathbb{R}^2$ such that $\mathcal{T}$ transversely intersect with $(0, 1) \times \{0\}$ and $(0, 1) \times \{1\}$ in n distinct points, respectively, called the *endpoints* of $\mathcal{T}$.

Let $\mathcal{T}_3^{\text{ori}}$ and $\mathcal{T}_4^{\text{ori}}$ denote the set of all oriented 3- and 4-tangle diagrams such that the orientations of the arcs of the tangles intersecting the boundary of I^2 coincide with the orientations as shown in (a) and (b) of Figure 8, respectively.

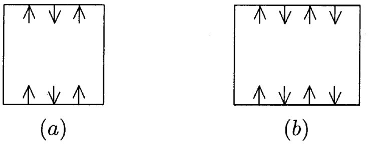


Figure 8: Boundaries of 3, 4-tangle diagrams

For $U \in \mathcal{T}_3^{\text{ori}}$ and $V \in \mathcal{T}_4^{\text{ori}}$, let $R(U), R^*(U), S(V)$ *and* $S^*(V)$ denote the oriented link diagrams obtained from the tangles U and V by closing as shown in Figures 9 and 10.

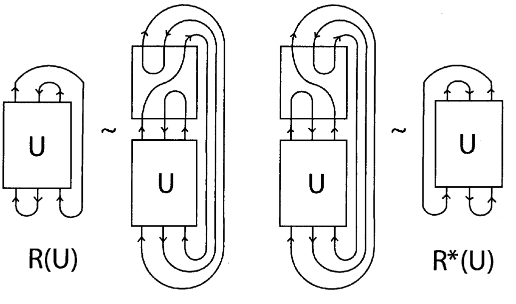


Figure 9: Closing operations R and R^* of a 3-tangle U

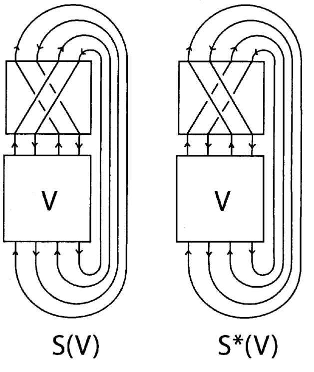


Figure 10: Closing operations S and S^* of a 4-tangle V

Let $\mathcal{T}_3$ and $\mathcal{T}_4$ denote the set of all 3- and 4-tangle diagrams without orientations, respectively. For $U \in \mathcal{T}_3$ and $V \in \mathcal{T}_4$, let $R(U), R^*(U), S(V)$ *and* $S^*(V)$ be the link diagrams obtained by the same way as above forgetting orientations.

Definition 4.1 ([4]). For any given regular or ambient isotopy invariant

$$[\ \ ] : \{\text{classical (oriented) links in } \mathbb{R}^3\} \longrightarrow R$$

satisfying the properties (3.1) and (3.2), the [ ]-*obstruction ideal* (or simply, [ ] *ideal*) I is defined to be the ideal of $R[x, y]$ generated by the polynomials in $R[x, y]$:

$$\begin{aligned} P_1 &= \delta x + y - 1, \\ P_2 &= x + \delta y - 1, \\ P_U &= ([R(U)] - [R^*(U)])xy, U \in \mathcal{T}_3\ (\mathcal{T}_3^{\text{ori}}), \\ P_V &= ([S(V)] - [S^*(V)])xy, V \in \mathcal{T}_3\ (\mathcal{T}_4^{\text{ori}}). \end{aligned}$$

Theorem 4.2 ([4]). The map

$$\overline{[\ \ ]} : \{\text{(oriented) marked graph diagrams}\} \longrightarrow R[x, y]/I$$

defined by

$$\overline{[\ \ ]}(D) = \overline{[D]} := \ll D \gg + I$$

for any (oriented) marked graph diagram D is an invariant for (oriented) surface-links.

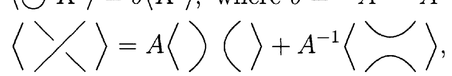
Remark 4.3. Let F be an extension field of R. By Hilbert Basis Theorem, the [ ] ideal I is completely determined by a finite number of polynomials in $F[x, y]$, say $p_1, p_2, \ldots, p_r$, i.e., $I = < p_1, p_2, \ldots, p_r >$.

In the rest of the paper, we give the ideals of Kauffman bracket for unoriented links and Kuperberg's quantum A_2 bracket for tangled trivalent graphs [11] and corresponding ideal coset invariants for unoriented surface-links and oriented surface-links, respectively. For more details, we refer to [3, 4, 14].

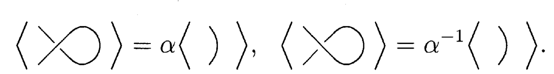
Let K be a knot or link diagram. The *Kauffman bracket* of K is a Laurent polynomial $\langle K \rangle = \langle K \rangle(A) \in R = \mathbb{Z}[A, A^{-1}]$ defined by the following rules:

(B1) $\langle \bigcirc \rangle = 1,$

(B2) $\langle \bigcirc\ K' \rangle = \delta \langle K' \rangle,$ where $\delta = -A^2 - A^{-2},$

(B3) 

where $\bigcirc\ K'$ denotes any addition of a disjoint circle $\bigcirc$ to a knot or link diagram K'. Note that the Kauffman bracket polynomial is invariant under Reidemeister moves except the move Ω_1 and for $\alpha = -A^3$, we have



Then the polynomial $\ll D \gg = \ll D \gg (A, x, y)$ in Definition 3.1 is given by

$$\begin{aligned} \ll D \gg &= (-A^3)^{-sw(D)}[[D]](A, x, y) \\ &= (-A^3)^{-sw(D)} \sum_{\sigma \in \mathcal{S}(D)} x^{\sigma(\infty)} y^{\sigma(0)} \langle D_\sigma \rangle. \end{aligned}$$

Theorem 4.4. The Kauffman bracket ideal I is the ideal of $\mathbb{Z}[A, A^{-1}, x, y]$ generated by

$$(-A^2 - A^{-2})x + y - 1,$$
$$x + (-A^2 - A^{-2})y - 1,$$
$$(A^8 + A^4 + 1)xy.$$

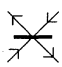
Moreover, the map $\overline{\langle\ \rangle}$: {marked graph diagrams} $\longrightarrow \mathbb{Z}[A, A^{-1}, x, y]/I$ defined by $\overline{\langle D \rangle} = \ll D \gg + I$ for any marked graph diagram D is an invariant for unoriented surface-links.

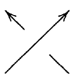
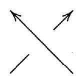
For any given oriented marked graph diagram D, let $\ll D \gg$ denote the polynomial in $\mathbb{Z}[a, a^{-1}, x, y]$ defined by the following recursive rules:

(1) $\ll$  $\gg = 1$.

(2) If D and D' are two oriented marked graph diagrams related by oriented Yoshikawa moves $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_4'$, and Γ_5, then $\ll D \gg = \ll D' \gg$.

(3) $\ll D \sqcup$  $\gg = (a^{-6} + 1 + a^6) \ll D \gg$.

(4) $\ll$  $\gg = x \ll$  $\gg + y \ll$  $\gg$.

(5) $a^{-9} \ll$  $\gg - a^9 \ll$  $\gg = (a^{-3} - a^3) \ll$  $\gg$.

Theorem 4.5. Let I be the ideal of $\mathbb{Z}[a, a^{-1}, x, y]$ generated by

$$(a^{-6} + 1 + a^6)x + y - 1,$$
$$x + (a^{-6} + 1 + a^6)y - 1,$$
$$(a^{12} + 1)(a^6 + 1)^2 xy.$$

Then the map $\overline{\langle\ \rangle}_{A_2}$: {oriented marked graph diagrams} $\longrightarrow \mathbb{Z}[a, a^{-1}, x, y]/I$ defined by $\overline{\langle D \rangle}_{A_2} = \ll D \gg + I$ for any oriented marked graph diagram D is an invariant for oriented surface-links.

We remark that the ideal I of $\mathbb{Z}[a, a^{-1}, x, y]$ in Theorem 4.5 is actually the ideal of Kuperberg's quantum A_2 bracket for oriented links and the map $\overline{\langle\ \rangle}_{A_2}$ is the corresponding ideal coset invariant for oriented surface-links (cf. [3, 11]). We close this section with the following:

Question 4.6. Is there a classical link invariant [ ] such that the [ ] ideal is trivial?

Acknowlegement

This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2012446).

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Department of Mathematics
Pusan National University
Busan 46241
KOREA
E-mail address: `sangyoul@pusan.ac.kr`