数理解析研究所講究録
第2004巻 2016年 101-107

An explicit relation between knot groups in lens spaces and those in S^3

Yuta Nozaki

Graduate School of Mathematical Sciences, the University of Tokyo

1 Introduction

We consider the symmetry of knots, precisely, free periods. The details of proofs in this article can be found in author's preprint [12]. A knot K in S^3 is said to *have free period* $p\in\mathbb{Z}_{\geq 1}$ if there exists $f\in \mathrm{Diff}(S^3,K)$ such that f^i has no fixed point for $0<i<p$ and $f^p=\mathrm{id}_{S^3}$, namely, (S^3,K) admits a free action by $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$. Whether a knot K has free period p or not is interesting problem and studied by many people.

We first review previous researches on the existence of free periods. In [7], Hartley proved that the torus knot $T_{m,n}$ has free period p if and only if $\gcd(mn,p)=1$. For example, the trefoil $T_{3,2}$ does not admit free involution. The Alexander polynomial of a torus knot was used in his proof.

Let K be a knot such that the outer automorphism group $\mathrm{Out}(G(K))$ of $G(K)=\pi_1(S^3\setminus K)$ is trivial. For instance, 9_{32}, 9_{33} and 24 more prime knots with 10 crossings (and their mirror images) satisfy this condition (see Kawauchi [9, Appendix F.2] or Kodama-Sakuma [10, Table 3.1]). Then it follows from Conner-Raymond [6, Theorem 3.2] and Burde-Zieschang [3] that K has no free period.

The purpose of this article is to deduce the above facts from a single result. Before stating our results, we review previous researches on the uniqueness of free periods. Sakuma [14], Boileau and Flapan [2] independently proved that for an oriented *prime* knot K, if $f,g\in \mathrm{Diff}(S^3,K)$ have free period p, then f is conjugate to g in the subgroup of $\mathrm{Diff}(S^3,K)$ consisting of diffeomorphisms that preserve the orientations of both S^3 and K. They also showed that the same is true for composite knots under a condition regarding "slopes". Recently, Manfredi [11] gave an interesting example regarding the uniqueness.

In order to state the main result, we describe free periods in another way. Suppose a knot K has period p. Then we obtain a knot $K'=K/\mathbb{Z}_p$ in the lens space $L(p,q)=S^3/\mathbb{Z}_p$ for some integer q coprime to p. Conversely, if K is a preimage of a knot K' under the covering map $\pi\colon S^3 \to L(p,q)$, then (a generator of) the deck transformation group realizes a free period of p. Therefore, we focus on a knot in a lens space, especially, on the fundamental group of its complement.

Theorem 1 (Theorem 2.6). Let K' be a knot in $L(p,q)$ with the connected preimage $K := \pi^{-1}(K')$. Then the image of $\pi_*\colon \pi_1(S^3 \setminus K) \rightarrowtail \pi_1(L(p,q) \setminus K')$ coincides with $\mathrm{C}^p(\pi_1(L(p,q) \setminus K'))$. In particular, the knot group $\pi_1(S^3 \setminus K)$ is a C^p-group (see Definition 2.1).

As a corollary of this result, the facts mentioned above is deduced.

Corollary 1 (Corollary 3.5). A knot K in S^3 with $\mathrm{Out}(G(K)) = 1$ cannot be represented as the preimage of any knot in any lens space.

Corollary 2 (Corollary 3.7). Let $m, n, p \in \mathbb{Z}_{\geq 2}$ with $\gcd(m,n) = 1$. There exists an integer q and a knot K' in $L(p,q)$ such that $\pi^{-1}(K')$ is ambient isotopic to the torus knot $T_{m,n}$ or its mirror image if and only if $\gcd(mn,p) = 1$.

2 Definitions and main theorem

Definition 2.1. For a group G and $p \in \mathbb{Z}_{\geq 1}$, let $\mathrm{C}^p(G)$ denote the subgroup of G generated by the set $\{g^p \mid g \in G\} \cup \{[g,h] \mid g,h \in G\}$, where $[g,h] := ghg^{-1}h^{-1}$. A group G is called a C^p-*group* if there exists a group G' such that $G \cong \mathrm{C}^p(G')$.

Remark 2.2. The subgroup $\mathrm{C}^p(G)$ coincides with the kernel of the composite map $G \twoheadrightarrow G_{\mathrm{ab}} \twoheadrightarrow G_{\mathrm{ab}}/pG_{\mathrm{ab}}$.

Remark 2.3. $\mathrm{C}^2(G)$ is denoted by G^2 in [17] and by $S(G)$ in [8]. For a prime p, $\mathrm{C}^p(G)$ coincides with the first term of the p-lower central series in [16] and with the first term of the derived p-series in [5].

Let $\pi\colon \Sigma \to \Sigma'$ be a p-fold cyclic covering, where Σ is an integral homology 3-sphere, and K' be a knot in Σ' with the connected preimage $K := \pi^{-1}(K')$.

Remark 2.4. K is connected if and only if $[K']$ generates $H_1(\Sigma') \cong \mathbb{Z}_p$. The last isomorphism is confirmed by using the five-term exact sequence for the short exact sequence $1 \to \pi_1(\Sigma) \to \pi_1(\Sigma') \to \mathbb{Z}_p \to 1$.

Lemma 2.5. *For Σ' and K' as above,* $H_*(\Sigma' \setminus K') \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 1, \\ 0 & \text{otherwise} \end{cases}$. *The homology class represented by a meridian of K' corresponds to $\pm p \in \mathbb{Z}$.*

Theorem 2.6. *The image of* $\pi_*\colon \pi_1(\Sigma \setminus K) \rightarrowtail \pi_1(\Sigma' \setminus K')$ *coincides with* $\mathrm{C}^p(\pi_1(\Sigma' \setminus K'))$.

Proof. Set $G := \pi_1(\Sigma \setminus K)$ and $G' := \pi_1(\Sigma' \setminus K')$. The covering map π induces the exact sequence

$$1 \to G \xrightarrow{\pi_*} G' \xrightarrow{\psi} \mathbb{Z}_p \to 1.$$

Here, ψ factors through $G'^{\mathrm{ab}}/pG'^{\mathrm{ab}} \cong \mathbb{Z}_p$ (Lemma 2.5), and thus $\mathrm{Im}\,\pi_* = \mathrm{Ker}\,\psi = \mathrm{C}^p(G')$ (Remark 2.2). □

3 Corollaries

We start this section with a remark in group theory.

Remark 3.1. For a normal subgroup H of a group G, the restriction map $\mathrm{Inn}(G) \to \mathrm{Aut}(H)$, $\mathrm{Ad}_g \mapsto \mathrm{Ad}_g|_H$, is induced by definition. Furthermore, $\mathrm{Aut}(G) \to \mathrm{Aut}(H)$ is defined if H is *characteristic*, that is, $f(H) = H$ for all $f \in \mathrm{Aut}(G)$. However, the restriction map $\mathrm{Inn}(G) \to \mathrm{Inn}(H)$ is not induced in general.

The following lemma is a refinement of the well-known fact [13, 13.5.8] for a complete group H. (A group G is said to be *complete* if the center $Z(G)$ and the outer automorphism group $\mathrm{Out}(G)$ are trivial.)

Lemma 3.2. *Let* G, H *be groups such that* $H \lhd G$ *and* $\mathrm{Ad}_g|_H \in \mathrm{Inn}(H)$ *for any* $g \in G$. *Then the sequence of groups*

$$1 \to Z(H) \xrightarrow{\phi} H \times C_G(H) \xrightarrow{\psi} G \to 1$$

is exact, where $C_G(H)$ *is the centralizer of* H *in* G, *and* $\phi(h) := (h, h^{-1})$, $\psi(h, g) := hg$.

Proof. We only confirm the surjectivity of ψ. Let $g \in G$. Then $\mathrm{Ad}_g|_H = \mathrm{Ad}_h$ for some $h \in H$. For any $h' \in H$, we have

$$[h^{-1}g, h'] = h^{-1}gh'g^{-1}hh'^{-1} = \mathrm{Ad}_{h^{-1}}(\mathrm{Ad}_g(h'))h'^{-1} = 1.$$

Hence, $h^{-1}g \in C_G(H)$ and $\psi(h, h^{-1}g) = g$. □

The next lemma is a generalization of [8, Theorem 1].

Lemma 3.3. *Let* G, H *be as in Lemma 3.2 and suppose* $\mathrm{C}^p(G) = H$, $Z(H) = 1$. *Then* $\mathrm{C}^p(H) = H$.

Proof. By Lemma 3.2, $\psi\colon H \times C_G(H) \to G$ and its restriction

$$\psi|\colon \mathrm{C}^p(H) \times \mathrm{C}^p(K) \to \mathrm{C}^p(G) = H, \tag{1}$$

are isomorphisms, where $K := C_G(H)$. Since $\psi(\mathrm{C}^p(H) \times \{1\}) = \mathrm{C}^p(H)$, we have $\mathrm{C}^p(K) \cong H/\mathrm{C}^p(H)$, and thus $Z(\mathrm{C}^p(K)) = \mathrm{C}^p(K)$. On the other hand, taking the center of (1), we have $Z(\mathrm{C}^p(H)) \times Z(\mathrm{C}^p(K)) \cong Z(H) = 1$ and $Z(\mathrm{C}^p(K)) = 1$. Hence, we conclude $\mathrm{C}^p(K) = 1$, and thus $\psi|$ is the identity map. □

The quotient $G/\mathrm{C}^p(G)$ plays a key role in our argument. The next lemma follows from Remark 2.2 and the homomorphism theorem for the abelianization $G \twoheadrightarrow G_{\mathrm{ab}}$.

Lemma 3.4. *For a group G whose abelianization is isomorphic to $\mathbb{Z}$, the quotient group $G/\mathrm{C}^p(G)$ is isomorphic to $\mathbb{Z}_p$.*

Corollary 3.5 ([6, Theorem 3.2], [3]). *A knot K in S^3 with* $\mathrm{Out}(G(K)) = 1$ *is not represented as the preimage of any knot in any lens space.*

Proof. Since $\mathrm{Out}(G(T_{m,n})) \cong \mathbb{Z}_2$ ([15]), K is not a torus knot, and thus $Z(G(K)) = 1$ ([3]). Hence, $G(K)$ is complete.

Assume that there exists a knot K' in $L(p,q)$ whose preimage is isotopic to K. Then we have $G(K) = \mathrm{C}^p(\pi_1(L(p,q) \setminus K'))$ by Theorem 2.6. Since $G(K)$ is complete, by Lemme 3.3, we conclude $\mathrm{C}^p G(K) = G(K)$. However, this contradicts Lemma 3.4. □

In order to prove Corollary 3.7, we quote the next lemma without proof.

Lemma 3.6 (see [12, Lemma 3.5]). *Let $m, n, p \in \mathbb{Z}_{\geq 1}$. If there exists a group G satisfying $\mathrm{C}^p(G) \cong \mathbb{Z}_m * \mathbb{Z}_n$, $G/\mathrm{C}^p(G) \cong \mathbb{Z}_p$ and $|G_{\mathrm{ab}}| = mnp$, then* $\gcd(mn,p) = 1$. *(Moreover, $H_*(G)$ is isomorphic to $H_*(\mathbb{Z}_m * \mathbb{Z}_n * \mathbb{Z}_p)$.)*

Corollary 3.7 ([7, Theorem 3.1]). *Let $m, n, p \in \mathbb{Z}_{\geq 2}$ with* $\gcd(m,n) = 1$. *There exist an integer q and a knot K' in $L(p,q)$ such that $\pi^{-1}(K')$ is isotopic to the torus knot $T_{m,n}$ or its mirror image if and only if* $\gcd(mn,p) = 1$.

Proof. If $\gcd(mn,p) = 1$, then a construction of a desired knot K' was given in [7, Theorem 3.1].

Suppose there exists K' as in the statement. We set $\pi_1' := \pi_1(L(p,q) \setminus K')$. The covering map $\pi\colon S^3 \setminus T_{m,n} \to L(p,q) \setminus K'$ induces the exact sequence

$$1 \to G(T_{m,n}) = \langle a, b \mid a^m = b^n \rangle \xrightarrow{\pi_*} \pi_1' \to \mathbb{Z}_p \to 1.$$

Since the center $Z(G(T_{m,n})) = \langle a^m \rangle = \mathbb{Z}$ is characteristic in $G(T_{m,n})$, the subgroup $N := \pi_*(\langle a^m \rangle)$ of π_1' is normal. We deduce the exact sequence

$$1 \to \langle a, b \mid a^m = 1 = b^n \rangle \xrightarrow{\pi_*} \pi_1'/N \to \mathbb{Z}_p \to 1$$

from the third isomorphism theorem. By Theorem 2.6, the group $G := \pi_1'/N$ satisfies

$$\mathrm{C}^p(G) = \mathrm{C}^p(\pi_1')/N = G(T_{m,n})/\langle a^m \rangle = \mathbb{Z}_m * \mathbb{Z}_n$$

and $G/C^p(G) \cong \mathbb{Z}_p$. Hence, by Lemma 3.6, it suffices to prove $|G_{\mathrm{ab}}| = mnp$.

The five-term exact sequence for

$$1 \to N \to \pi_1' \to G \to 1 \tag{2}$$

is as follows:

$$0 \to H_2(G) \to \mathbb{Z}_G \to \mathbb{Z} \to H_1(G) \to 0.$$

An observation on a meridian of K' proves $H_1(G) = \mathbb{Z}_{mnp}$ (see [12, Corollary 3.4] for details). □

Remark 3.8. The above Hartley's result (Corollary 3.7) was extended by Chbili [4] to torus links. In fact, the torus link $T_{m,n}$ has free period p if and only if there exists an integer q such that $\gcd(p,q) = 1$ and $p \mid m - nq$. Note that $\gcd(mn,p) = 1$ implies that the existence of such a q, however, the converse is not true without the assumption $\gcd(m,n) = 1$.

4 Symmetric groups and braid groups

In this section, we suppose $n \geq 3$ and $p \geq 2$ for simplicity. The next lemma follows from Lemma 3.3 and the fact that $\mathfrak{S}_n$ is complete for $n \neq 2, 6$. Note that the case $n = 6$ requires an additional argument (see [12, Appendix A.2]).

Lemma 4.1. *The nth symmetric group $\mathfrak{S}_n$ is a C^p-group if and only if p is odd.*

Even if a group G is a C^p-group, G/H is not necessarily a C^p-group. However, the following lemma assert that G/H is a C^p-group for a characteristic subgroup H.

Lemma 4.2 (see [17, Theorem 1]). *Let G be a C^p-group and $f\colon G \twoheadrightarrow H$ be a surjective homomorphism whose kernel is a characteristic subgroup of G. Then H is also a C^p-group.*

Proof. Suppose $C^p(G') = G$. Then we have

$$C^p(G'/\operatorname{Ker} f) = C^p(G')/(\operatorname{Ker} f \cap C^p(G')) = G/\operatorname{Ker} f \cong H.$$

Hence, H is a C^p-group. □

Corollary 4.3. *The nth braid group B_n is not a C^p-group for even p.*

Proof. Since the nth pure braid group $P_n := \operatorname{Ker}(B_n \to \mathfrak{S}_n)$ is characteristic ([1, Theorem 3]), by Lemma 4.2, it suffices to prove that $\mathfrak{S}_n$ is not a C^p-group for even p. Lemma 4.1 completes the proof. □

Remark 4.4. One of the definition of B_n is the fundamental group of $X_n/\mathfrak{S}_n$, where X_n is the configuration space $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \ (i \neq j)\}$ of distinct n points in $\mathbb{C}$, and $\mathfrak{S}_n$ acts on X_n by permutation of coordinates. Therefore, B_n is a C^p-group if there exists a topological space Y admitting a p-fold cyclic covering $X_n/\mathfrak{S}_n \to Y$ and satisfying $H_1(X)/pH_1(X) = \mathbb{Z}_p$.

Acknowledgments

The author would like to express his gratitude to Takuya Sakasai for his various suggestions. Also, he would like to thank Makoto Sakuma for useful discussions in this topic. Finally, this work was supported by the Program for Leading Graduate Schools, MEXT, Japan and JSPS KAKENHI Grant Number 16J07859.

References

[1] E. Artin. Braids and permutations. *Ann. of Math. (2)*, 48:643–649, 1947.

[2] M. Boileau and E. Flapan. Uniqueness of free actions on S^3 respecting a knot. *Canad. J. Math.*, 39(4):969–982, 1987.

[3] G. Burde and H. Zieschang. Eine Kennzeichnung der Torusknoten. *Math. Ann.*, 167:169–176, 1966.

[4] N. Chbili. A new criterion for knots with free periods. *Ann. Fac. Sci. Toulouse Math. (6)*, 12(4):465–477, 2003.

[5] T. Cochran and S. Harvey. Homology and derived p-series of groups. *J. Lond. Math. Soc. (2)*, 78(3):677–692, 2008.

[6] P. E. Conner and F. Raymond. Manifolds with few periodic homeomorphisms. In *Proceedings of the Second Conference on Compact Transformation Groups (Univ. Massachusetts, Amherst, Mass., 1971), Part II*, pages 1–75. Lecture Notes in Math., Vol. 299. Springer, Berlin, 1972.

[7] R. Hartley. Knots with free period. *Canad. J. Math.*, 33(1):91–102, 1981.

[8] M. Haugh and D. MacHale. The subgroup generated by the squares. *Proc. Roy. Irish Acad. Sect. A*, 97(2):123–129, 1997.

[9] A. Kawauchi, (ed.). *A survey of knot theory.* Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.

[10] K. Kodama and M. Sakuma. Symmetry groups of prime knots up to 10 crossings. In *Knots 90 (Osaka, 1990)*, pages 323–340. de Gruyter, Berlin, 1992.

[11] E. Manfredi. Lift in the 3-sphere of knots and links in lens spaces. *J. Knot Theory Ramifications*, 23(5):1450022, 21, 2014.

[12] Y. Nozaki. An explicit relation between knot groups in lens spaces and those in S^3, 2016, arXiv:1602.05884.

[13] D. J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[14] M. Sakuma. Uniqueness of symmetries of knots. *Math. Z.*, 192(2):225–242, 1986.

[15] O. Schreier. Über die gruppen $A^a B^b = 1$. *Abh. Math. Sem. Univ. Hamburg*, 3(1):167–169, 1924.

[16] J. Stallings. Homology and central series of groups. *J. Algebra*, 2:170–181, 1965.

[17] H. S. Sun. On groups generated by the squares. *Fibonacci Quart.*, 17(3):241–246, 1979.

Graduate School of Mathematical Sciences
The University of Tokyo
Tokyo 153-8914
JAPAN
E-mail address: `nozaki@ms.u-tokyo.ac.jp`

東京大学大学院数理科学研究科　野崎　雄太