数理解析研究所講究録
第2006巻 2016年 1-11

SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENT AND NON-HOMOGENEOUS TERM

KAZUNE TAKAHASHI

1. Introduction

Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary is smooth. Set $a \in L^\infty(\Omega)$ be a function so that $a \geq \kappa$ in Ω for some constant $\kappa > -\kappa_1$, where κ_1 be the first eigenvalue of $-\Delta$ under zero Dirichlet condition on Ω. Let $b \in L^\infty(\Omega)$ be a function with $b \geq 0$ in Ω and $b \not\equiv 0$. We discuss the following equation:

$$(\spadesuit)_\lambda \qquad \begin{cases} -\Delta u + au = bu^p + \lambda f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $p = (N+2)/(N-2)$ is the critical Sobolev exponent, and $f \in H^{-1}(\Omega)$ is a non-homogeneous perturbation with $f \geq 0$ and $f \not\equiv 0$. $\lambda > 0$ is a parameter.

This equation is involving the critical Sobolev exponent. It is known the existence and nonexistence of positive solution depends on the dimension N and the shape of the domain Ω.

Let us recall some facts of the Sobolev embedding. Now we assume Ω is a bounded domain, where the Sobolev space $H_0^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $1 \leq q < p+1$. The Sobolev space $H_0^1(\Omega)$ is also a subspace of $L^{p+1}(\Omega)$. However, it is not compactly embedded. This fact make it difficult to analyze critical equations.

Let us recall the definition of a weak solution. $u \in H_0^1(\Omega)$ is a weak solution of $(\spadesuit)_\lambda$ if the following holds for all $\psi \in H_0^1(\Omega)$:

$$\int_\Omega (Du \cdot D\psi + au\psi)\, dx = \int_\Omega bu^p \psi dx + \lambda \int_\Omega f\psi dx.$$

Here, we have to consider the integral of $(p+1)$-th power of H_0^1-function. If $H_0^1(\Omega)$ were compactly embedded in $L^{p+1}(\Omega)$, each bounded sequence in $H_0^1(\Omega)$ would have a convergent subsequence in $L^{p+1}(\Omega)$. We would have a weak solution easily. But it is not the case. To get solutions, we have to use other methods. We find keys for the critical equations in the next section.

2. Known Results and Main Theorem

2.1. **Known Results.** We recall known results of critical equations. Let us begin with the following equation:

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

This equation, as well as the main problem, is non-homogeneous and we are interested in positive solutions. We introduce two type of solutions. They are called the minimal solution and a second solution. $\underline{u}_\lambda$ is called the minimal solution if $u \geq \underline{u}_\lambda$ in Ω holds for any solution u. We consider another solution $\overline{u}_\lambda$ other than the minimal solution. For convenience, we call it a second solution.

The facts of the minimal solution of (1) are stated as following:

Theorem 1 ([KK74], [CR75]). $0 < \overline{\lambda} < \infty$ *exists which satisfies the following (i) – (iii) :*

(i) (1) *has the minimal solution* $\underline{u}_\lambda$ *for* $0 < \lambda < \overline{\lambda}$.
(ii) (1) *has only one weak solution* $\underline{u}_{\overline{\lambda}}$ *when* $\lambda = \overline{\lambda}$.
(iii) (1) *has no weak solution for* $\lambda > \overline{\lambda}$.

$\overline{\lambda}$ is often called the extremal value. The facts of a second solution of (1) are stated as following:

Theorem 2. *There exists a second solution* $\overline{u}_\lambda$ *for* $0 < \lambda < \overline{\lambda}$ *satisfying* $\overline{u}_\lambda \geq \underline{u}_\lambda$.

This fact was shown by Joseph and Lundgren [JL73] if Ω was a ball. They used radially ODE methods. On the other hand, Brézis and Nirenberg [BN83] showed this fact without restriction of Ω. They used two tools, the mountain pass theorem without (PS) condition and the Talenti function.

Proposition (Mountain pass theorem without (PS) condition [AR73]). *Let* X *be a Banach space. Let* I *be a* C^1*-class functional on* X. *Suppose that there exist a neighborhood* U *of* $0 \in X$ *and a constant* ρ *so that* $\Phi(u) \geq \rho$ *for all* $u \in \partial U$. *Assume that* $I(0) < \rho$ *and* $I(v) < \rho$ *for some* $v \in X \setminus U$. *Let* $c = \inf_{P \in \mathcal{P}} \max_{w \in P} I(w) \geq \rho$ *where* $\mathcal{P}$ *is the set of paths from* 0 *to* v *in* X. *Then, there exists a sequence* $\{v_k\}_{k=0}^{\infty}$ *of* X *satisfying the following (i) and (ii):*

(i) $\lim_{k\to\infty} I_\lambda(v_k) = c$.
(ii) $\lim_{k\to\infty} I'_\lambda(v_k) = 0$ *in* X^*.

If we use the variational method, critical points are solutions of the equation. To apply this theorem for critical equations, we often need $(\mathrm{PS})_c$ condition instead of (PS) condition.

Definition. Let X be a Banach space. Let I be a functional on X. Let $c \in \mathbb{R}$. We say I satisfies **$(\mathrm{PS})_c$ condition** if any sequence $\{v_k\}_{k=0}^{\infty}$ of X satisfying (i) and (ii) stated in Proposition above has a convergent subsequence.

Broadly speaking, in our case, if the highest point c is not so high, we can rule out the possibility of noncompact sequences. Actually, it is related to the Talenti function. We discuss it in the next section.

We summarize solutions of (1) by using diagrams (Figure 1). On the horizontal axis is λ, On the vertical axis we would have $H_0^1(\Omega)$ if we could. Of course, $H_0^1(\Omega)$ is infinitely dimensional. We use the L^∞-norm instead of that. If we fix a $\lambda > 0$, we find that the vertical line and the diagram cross at two points. That means we have two positive solutions. The curve of the minimal solution starts at 0 and ends at $\lambda = \overline{\lambda}$. The curve increases. It means the bigger λ is, the bigger L^∞-norm of the minimal solution is. This fact is actually proved. The curve of the second solution starts at $\lambda = \overline{\lambda}$ and goes reversely. When $\lambda = \overline{\lambda}$, (1) has only one solution.

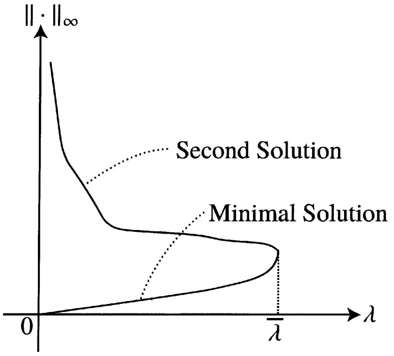

FIGURE 1. The diagram of solutions of (1). On the horizontal axis is λ. On the vertical axis is L^∞-norm. The branch of the minimal solution starts from 0 and ends at $\lambda = \overline{\lambda}$. The branch of a second solution starts from that end point and goes reversely.

The mountain pass theorem without (PS) condition and the Talenti function are efficient for our types. The following equation is the main problem $(\spadesuit)_\lambda$ with $a = 0$ and $b = 1$:

$$
\begin{cases} -\Delta u = u^p + \lambda f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}
$$

Tarantello [Tar92] proved that it has two positive solutions for sufficiently small $\lambda > 0$.

The following equation is the main problem $(\spadesuit)_\lambda$ with a be a constant κ and $b = 1$:

$$
\begin{cases} -\Delta u + \kappa u = u^p + \lambda f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}
$$

It has a linear term of u. Naito and Sato [NS12] investigated this equation. The results of the minimal solution are summarized in the same way as Theorem 1. On the other hand, the results of the second solutions are different from (1) and (2).

Theorem 3 ([NS12] Theorem 1.3)**.** *Let $0 < \lambda < \overline{\lambda}$. Assume that either (i) or (ii) holds.*

(i) $-\kappa_1 < \kappa \leq 0$ and $N \geq 3$.

(ii) $\kappa > 0$ and $N = 3, 4, 5$.

Then, (3) has a second solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$ in Ω.

Naito and Sato also proved that, if $\kappa > 0$ and $N \geq 6$, there is a case the equation has only one solution for sufficiently small $\lambda > 0$. In this sense, the statement of (ii) gives the best result.

We use the diagrams to view these results (Figure. 2). Let $\kappa > 0$. If $N = 3, 4, 5$, the branch of a second solution is approaching to the extent of $\lambda = 0$. However, if $N \geq 6$, the climate changes. The branch of second solution cannot go beyond some positive λ.

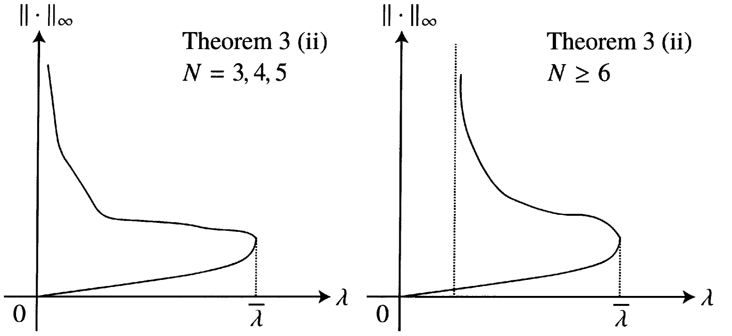

FIGURE 2. The diagram of solutions of (3) depends on the dimension N when $\kappa > 0$. If $N = 3, 4, 5$, the branch of a second solution is approaching to the extent of $\lambda = 0$ like the left figure, while if $N \geq 6$, there is a case that it cannot go beyond some λ like the right figure.

2.2. **Main Theorem.** Now we consider the main problem $(\spadesuit)_\lambda$. First we state the results of the minimal solution.

Theorem 4 ([Tak15] Theorem 1.1, 1.2)**.** $0 < \overline{\lambda} < \infty$ *exists which satisfies the following (i) – (iii) :*

(i) (1) *has the minimal solution* $\underline{u}_\lambda$ *for* $0 < \lambda < \overline{\lambda}$.

(ii) (1) *has only one weak solution* $\underline{u}_{\overline{\lambda}}$ *when* $\lambda = \overline{\lambda}$ *if* $b > 0$ *in* Ω.

(iii) (1) *has no weak solution for* $\lambda > \overline{\lambda}$.

The main result of the minimal solution is almost the same as Theorem 1. A miner difference occurs when $\lambda = \overline{\lambda}$. Since b is a function, we assume $b > 0$ in Ω to get the uniqueness of the solution.

Now we state the main result.

Theorem 5 ([Tak15] Theorem 1.4)**.** *Let* $0 < \lambda < \overline{\lambda}$. *Suppose that* b *achieves its maximum* $M = \|b\|_{L^\infty(\Omega)} > 0$ *at a point* x_0 *in* Ω. *Suppose that there exists* $r_0 > 0$ *so that* $\{|x - x_0| < 2r_0\} \subset \Omega$, b *is continuous in* $\{|x - x_0| < r_0\}$, *and*

$$a(x) = m_1 + m_2|x - x_0|^q + o(|x - x_0|^q) \quad \text{in } \{|x - x_0| < r_0\}.$$

Here, $q > 0$, $m_1 > \kappa$, *and* $m_2 \neq 0$ *are constants. Assume that either (i) – (iv) holds.*

(i) $m_1 < 0$ *and* $N \geq 3$.

(ii) $m_1 > 0$ *and* $N = 3, 4, 5$.

(iii) $m_1 = 0$, $m_2 < 0$ *and* $N \geq 3$.

(iv) $m_1 = 0$, $m_2 > 0$ *and* $3 \leq N < 6 + 2q$.

Then, $(\spadesuit)_\lambda$ *has a second solution* $\overline{u}_\lambda \in H_0^1(\Omega)$ *satisfying* $\overline{u}_\lambda > \underline{u}_\lambda$ *in* Ω.

The main theorem gives sufficient conditions where we have two solutions for $0 < \lambda < \overline{\lambda}$. Though (i)–(iii) are like Naito and Sato's results, (iv) seems to be new. In (iv), we have two solutions if $3 \leq N < 6 + 2q$. This occurs since coefficients a and b are functions on Ω.

We compare (3) and the main problem $(\spadesuit)_\lambda$. We focus on the dimension where there exists a second solution for any $0 < \lambda < \overline{\lambda}$. In Theorem 3 (i) and (ii), when κ stride across zero, the upper bound of the dimension jumps from ∞ to 6. We consider this result from the main problem $(\spadesuit)_\lambda$. Theorem 3 is regarded as the case that a is a constant κ. On the other hand, in the main problem $(\spadesuit)_\lambda$, a is a function. We can think of a function whose order is q at x_0, which appears in Theorem 5 (iv). In this case, the upper bound of the dimension is $6 + 2q$. Therefore, this can be regarded as a intermediate case of Naito and Sato's results (Figure 3).

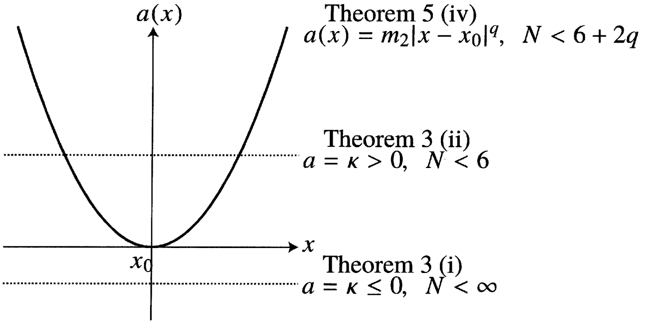

FIGURE 3. The dimension where $(\spadesuit)_\lambda$ has a second solution varies depending on the function $a(x)$. Theorem 5 (iv), in which a increases in order q around zero point x_0, can be interpreted as a intermediate case between Theorem 3 (i) and (ii), in which a is a constant κ.

3. Proof of Main Theorem

We move on to the summary of the proof of Theorem 5. We start at the point that we already have investigated the minimal solution $\underline{u}_\lambda$ of $(\spadesuit)_\lambda$. We assume $0 < \lambda < \overline{\lambda}$. We can also assume that $x_0 = 0$ by translation of axes if we need. We are now interested in the second solutions. Instead of chasing a second solution directly, we consider the difference between a second solution $\overline{u}_\lambda$ and the minimal solution $\underline{u}_\lambda$. We name it $v = \overline{u}_\lambda - \underline{u}_\lambda$, which satisfies the following:

$$(\heartsuit)_\lambda \qquad \begin{cases} -\Delta v + av = b\left((v + \underline{u}_\lambda)^p - \underline{u}_\lambda^p\right) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The main problem $(\spadesuit)_\lambda$ has a second solution if and only if $(\heartsuit)_\lambda$ has a solution. It follows that $v > 0$ in Ω because of the definition of the minimal solution and the strong maximum principle.

We use the variational method to prove the existence of v. We define a functional I_λ as following:

$$I_\lambda(v) = \frac{1}{2}\int_\Omega |Dv|^2 dx - \int_\Omega G(v, \underline{u}_\lambda) dx, \tag{4}$$

where

$$g(t, s, x) = b(x)\left((t_+ + s)^p - s^p\right) - a(x)t_+, \tag{5}$$

$$\begin{aligned} G(t, s, x) &= \int_0^{t_+} g(t, s, x)dt \\ &= b(x)\left(\frac{1}{p+1}(t_+ + s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t_+\right) - \frac{1}{2}a(x)t_+^2, \end{aligned} \tag{6}$$

and we write $g(v, \underline{u}_\lambda)$ as $g(v, \underline{u}_\lambda, x)$ and $G(v, \underline{u}_\lambda)$ as $G(v, \underline{u}_\lambda, x)$. If we find a critical point v of I_λ, we get a solution of $(\heartsuit)_\lambda$. It means that we get a second solution $\overline{u}_\lambda$ as the sum of the minimal solution $\underline{u}_\lambda$ and the critical point v.

We combine the mountain pass theorem and the Talenti function. We need $(PS)_c$ condition. Through some argument, we can find a sufficient condition of $(PS)_c$ condition, that is, there exists a function $v_0 \in H_0^1(\Omega)$ so that $v_0 \geq 0$ in Ω and

$$\int_\Omega b v_0^{p+1} dx > 0, \tag{7}$$

and

$$\sup_{t>0} I_\lambda(tv_0) < \frac{1}{NM^{(N-2)/2}} S^{N/2}. \tag{8}$$

Here, M is the maximum value of b. S is called the best Sobolev constant. The best Sobolev constant S is defined by following:

$$S = \inf_{u \in H_0^1(V), u \not\equiv 0} \frac{\|Du\|^2_{L^2(V)}}{\|u\|^2_{L^{p+1}(V)}}, \tag{9}$$

where $V \subset \mathbb{R}^N$ is a domain. It is known that S does not depend on V. This infimum is actually achieved by the Tarenti function. This fact is important when we compute the condition (8). To get this v_0, we use the Talenti function. The Talenti function U is given as following:

$$U(x) = \frac{1}{(1 + |x|^2)^{(N-2)/2}}.$$

Our aim is to get v_0 of the condition (8). We define u_ϵ and v_ϵ as following:

$$u_\epsilon(x) = \frac{\eta(x)}{(\epsilon + |x|^2)^{(N-2)/2}}, \tag{10}$$

$$v_\epsilon(x) = \frac{u_\epsilon(x)}{\|b^{1/(p+1)} u_\epsilon\|_{L^{p+1}(\Omega)}}. \tag{11}$$

Here, $\epsilon > 0$ and η is a cut-off function around 0. v_ϵ is some kind of normalization of u_ϵ. Note that, in case (i) or (iii), we can change $r_0 > 0$ for smaller one if we need so that

$$\int_\Omega a v_\epsilon^2 dx \le 0. \tag{12}$$

Note also that, in case (ii) or (iv), we can change $r_0 > 0$ for smaller one if we need so that

$$\int_\Omega a v_\epsilon^2 dx \ge 0. \tag{13}$$

Now we compute the condition (8). Through some argument, We can see the supremum in (8) is actually achieved at some positive $t = t_\epsilon$. We are going to estimate $I_\lambda(t_\epsilon v_\epsilon)$. We define

$$H'(v, \underline{u}_\lambda) = \frac{1}{p+1}(v + \underline{u}_\lambda)^{p+1} - \frac{1}{p+1} v^{p+1} - \frac{1}{p+1}\underline{u}_\lambda^{p+1} - \underline{u}_\lambda^p v.$$

Then, we have

$$\begin{aligned}\sup_{t>0} I_\lambda(t v_\epsilon) &= I_\lambda(t_\epsilon v_\epsilon) \\ &= \frac{1}{2} t_\epsilon^2 \|v_\epsilon\|^2 - \frac{1}{p+1} t_\epsilon^{p+1} - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda) dx + t_\epsilon^2 \int_\Omega a v_\epsilon^2 dx, \end{aligned}\tag{14}$$

where

$$\|v_\epsilon\| = \left(\int_\Omega |Dv_\epsilon|^2\, dx\right)^{1/2}.$$

The term containing a is important for this argument. The ϵ-order of that term depends on q. We divide the proof into two cases.

For case (i) and (iii), by (12), we have

$$\begin{aligned}\sup_{t>0} I_\lambda(t v_\epsilon) &\le \frac{1}{2} t_\epsilon^2 \|v_\epsilon\|^2 - \frac{1}{p+1} t_\epsilon^{p+1} - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda) dx \\ &\le \sup_{t>0}\left(\frac{1}{2} t^2 \|v_\epsilon\|^2 - \frac{1}{p+1} t^{p+1}\right) - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda) dx \\ &= \frac{1}{N}\left(\|v_\epsilon\|^2\right)^{N/2} - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda) dx.\end{aligned}$$

By calculation, we have

$$\|v_\epsilon\|^2 = \|Dv_\epsilon\|^2_{L^2(\Omega)} = \frac{S}{M^{2/(p+1)}} + O(\epsilon^{(N-2)/2}) \tag{15}$$

as $\epsilon \searrow 0$, that is,

$$\frac{1}{N}\left(\|v_\epsilon\|^2\right)^{N/2} = \frac{1}{NM^{(N-2)/2}} S^{N/2} + O(\epsilon^{(N-2)/2})$$

as $\epsilon \searrow 0$. In H', $(p+1)$-th powered terms are canceled. After some argument, we can show that there exist $\epsilon_0 > 0$ and $C > 0$ so that for all $0 < \epsilon < \epsilon_0$, the following holds:

$$\int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda) dx \ge C\epsilon^{(N-2)/4}. \tag{16}$$

Summing up these results, there exist $\epsilon_0 > 0$ and $C, C' > 0$ so that for all $0 < \epsilon < \epsilon_0$, we have

$$\sup_{t>0} I_\lambda(tv_\epsilon) \le \frac{1}{NM^{(N-2)/2}} S^{N/2} + \left(C\epsilon^{(N-2)/2} - C'\epsilon^{(N-2)/4}\right). \tag{17}$$

For any $N \ge 3$, it holds that $(N-2)/2 > (N-2)/4$. Thus there exists $\epsilon > 0$ so that the terms in parentheses of the right side of (17) is negative. If we use this ϵ for $v_0 = v_\epsilon$, $v_0 \ge 0$ in Ω, (7), and (8) are all satisfied. Thus we have a second solution of $(\spadesuit)_\lambda$.

For case (ii) and (iv), by (12), we have

$$\begin{aligned}
\sup_{t>0} I_\lambda(tv_\epsilon) &= \frac{1}{2}t_\epsilon^2\left(\|v_\epsilon\|^2 + 2\int_\Omega av_\epsilon^2 dx\right) - \frac{1}{p+1}t_\epsilon^{p+1} - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda)dx \\
&\le \sup_{t>0}\left(\frac{1}{2}t^2\left(\|v_\epsilon\|^2 + 2\int_\Omega av_\epsilon^2 dx\right) - \frac{1}{p+1}t^{p+1}\right) - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda)dx \\
&= \frac{1}{N}\left(\|v_\epsilon\|^2 + 2\int_\Omega av_\epsilon^2 dx\right)^{N/2} - \int_\Omega H'(t_\epsilon v_\epsilon, \underline{u}_\lambda)dx.
\end{aligned}$$

Here we define $A(\epsilon)$ as following:

$$A(\epsilon) = \frac{1}{N}\left(\|v_\epsilon\|^2 + 2\int_\Omega av_\epsilon^2 dx\right)^{N/2} - \frac{1}{NM^{(N-2)/2}} S^{N/2}.$$

We are now at the point to evaluate the integral of av_ϵ^2. We define

$$\begin{aligned}
I_1 &= \int_{\{|x|<r_0\}} \frac{1}{(\epsilon + |x|^2)^{N-2}} dx, \\
I_2 &= \int_{\{|x|<r_0\}} \frac{|x|^q}{(\epsilon + |x|^2)^{N-2}} dx.
\end{aligned}$$

The estimate of I_1 is famous. Many paper refers to Brézis–Nirenberg [BN83]. On the other hand, I_2 is not on Brézis–Nirenberg. We use almost the same method of Brézis–Nirenberg to evaluate I_2. To describe only the results, we have the following:

$$\begin{cases}
I_1 = \begin{cases} O(\epsilon^{-(N-4)/2}) & (N \ge 5), \\ O(|\log \epsilon|) & (N = 4), \\ O(1) & (N = 3), \end{cases} \\
I_2 = \begin{cases} O(\epsilon^{-(N-q-4)/2}) & (N > q+4), \\ O(|\log \epsilon|) & (N = q+4), \\ O(1) & (N < q+4) \end{cases}
\end{cases} \tag{18}$$

We can also evaluate

$$\left\|b^{1/(p+1)}u_\epsilon\right\|_{L^{p+1}(\Omega)}^2 = O(\epsilon^{-(N-2)/2}) \tag{19}$$

as $\epsilon \searrow 0$. Therefore, we admit the following:

$$(20)\qquad \begin{cases} \displaystyle\int_\Omega a v_\epsilon^2 dx = O(\epsilon^{(N-2)/2}) + m_1 I_1' + m_2 I_2', \\ \qquad I_1' = \begin{cases} O(\epsilon) & (N \geq 5), \\ O(\epsilon|\log \epsilon|) & (N = 4), \\ O(\epsilon^{1/2}) & (N = 3), \end{cases} \\ \qquad I_2' = \begin{cases} O(\epsilon^{1+q/2}) & (N > q+4), \\ O(\epsilon^{(N-2)/2}|\log \epsilon|) & (N = q+4), \\ O(\epsilon^{(N-2)/2}) & (N < q+4) \end{cases} \end{cases}$$

as $\epsilon \searrow 0$. We can see I_2' is affected with q. Note that, for all case, $I_i' \gg \epsilon^{(N-2)/2}$ or $I_i' = O(\epsilon^{(N-2)/2})$ as $\epsilon \searrow 0$ for $i = 1, 2$. Note also that $I_1 \gg I_2$ as $\epsilon \searrow 0$ for any $N \geq 3$, $q > 0$. Considering (15) and (20), we have the evaluation of $A(\epsilon)$ as following:

$$(21)\qquad A(\epsilon) = \begin{cases} O(\epsilon) & (m_1 > 0, N \geq 5), \\ O(\epsilon|\log \epsilon|) & (m_1 > 0, N = 4), \\ O(\epsilon^{1/2}) & (m_1 > 0, N = 3), \\ O(\epsilon^{1+q/2}) & (m_1 = 0, m_2 > 0, N > q+4), \\ O(\epsilon^{(N-2)/2}|\log \epsilon|) & (m_1 = 0, m_2 > 0, N = q+4), \\ O(\epsilon^{(N-2)/2}) & (m_1 = 0, m_2 > 0, N < q+4) \end{cases}$$

as $\epsilon \searrow 0$. By (16), there exist $\epsilon_0 > 0$ and $C' > 0$ so that for all $0 < \epsilon < \epsilon_0$,

$$(22)\qquad \sup_{t>0} I_\lambda(tv_\epsilon) \leq \frac{1}{NM^{(N-2)/2}} S^{N/2} + \left(A(\epsilon) - C'\epsilon^{(N-2)/4}\right).$$

As we discussed before, if there exists $\epsilon > 0$ so that the terms in parentheses of the right side of (22) is negative, we have a second solution of $(\spadesuit)_\lambda$. If $m_1 > 0$ and $N = 3, 4, 5$, it holds that $A(\epsilon) \ll \epsilon^{(N-2)/4}$ as $\epsilon \searrow 0$ by (21). Thus we have the desired ϵ. If $m_1 = 0$, $m_2 > 0$ and $N \leq q + 4$, it holds that $A(\epsilon) \ll \epsilon^{(N-2)/4}$ as $\epsilon \searrow 0$ by (21). If $m_1 = 0$, $m_2 > 0$ and $N > q + 4$, the condition where $A(\epsilon) \ll \epsilon^{(N-2)/4}$ as $\epsilon \searrow 0$ is that $1 + q/2 > (N - 2)/4$. Then we have $N < 2q + 6$. That is, if $m_1 = 0$, $m_2 > 0$ and $3 \leq N < 2q + 6$, we have the desired ϵ.

In [Tak15] and the talk of this workshop, the author mistakenly dropped t_ϵ^2 of the last term in (14). The proof should be replaced as above.

Appendix A. Proof of Boundedness of λ

In the workshop, several people asked the author about the proof of Theorem 4 (iii). In this appendix, we see the summary of that proof. To prove Theorem 4 (iii), we define

$$\overline{\lambda} = \{\lambda \geq 0 \mid \text{The main problem } (\spadesuit)_\lambda \text{ has a solution.}\}$$

and we show $\overline{\lambda} < \infty$. The following argument is almost the same as the proof appeared in [NS12].

We define $g_0 \in H_0^1(\Omega)$ as the unique solution of the following equation:

$$\begin{cases} -\Delta g_0 + a g_0 = f & \text{in } \Omega, \\ g_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

We have $g_0 > 0$ in Ω by the strong maximum principle. Next we consider the following linearized eigenvalue problem:

$$-\Delta\phi + a\phi = \mu b(g_0)^{p-1}\phi \text{ in } \Omega, \quad \phi \in H_0^1(\Omega). \tag{24}$$

It is known that the first eigenvalue μ_1 is characterized by the Rayleigh quotient:

$$\mu_1 = \inf_{\psi \in H_0^1(\Omega), \psi \not\equiv 0} \frac{\int_\Omega \left(|D\psi|^2 + a\psi^2\right) dx}{\int_\Omega b(g_0)^{p-1}\psi^2 dx}. \tag{25}$$

It is also known that if ϕ achieves the infimum of the right side of (25), ϕ is a first eigenfunction of (24). Through some argument we have $\mu_1 > 0$ and we find the first eigenfunction ϕ_1 which satisfies $\phi_1 > 0$ in Ω.

We fix $\lambda > 0$ so that there exists a solution of the main problem $(\spadesuit)_\lambda$. We write u as a solution of the main problem $(\spadesuit)_\lambda$ and we set $v = u - \lambda g_0$, which satisfies

$$-\Delta v + av = bu^p \geq 0.$$

Then we have $v > 0$ in Ω by the strong maximum principle. It means that $u > \lambda g_0$ in Ω. Therefore we get the following:

$$-\Delta u + au \geq bu^p > b\lambda^{p-1}(g_0)^{p-1}u \text{ in } \Omega. \tag{26}$$

On the other hand, we have the following:

$$-\Delta\phi_1 + a\phi_1 = \mu_1 b(g_0)^{p-1}\phi_1 \text{ in } \Omega. \tag{27}$$

Integrating (26) $\times\, \phi_1$ − (27) $\times\, u$ on Ω, we admit the following:

$$0 > (\lambda^{p-1} - \mu_1) \int_\Omega b(g_0)^{p-1}u\phi_1 dx. \tag{28}$$

Since $b \geq 0$ in Ω, $b \not\equiv 0$, and $g_0, u, \phi_1 > 0$ in Ω, the integral of the right side of (28) is positive, which implies $\lambda^{p-1} - \mu_1 < 0$. Thus we conclude that $\overline{\lambda} \leq \mu_1^{1/(p-1)} < \infty$.

References

[AR73] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[BN83] Haïm Brézis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[CR75] Michael G. Crandall and Paul H. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal.*, 58(3):207–218, 1975.

[JL73] Daniel D. Joseph and Thomas S. Lundgren. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.*, 49:241–269, 1972/73.

[KK74] James P. Keener and Herbert B. Keller. Positive solutions of convex nonlinear eigenvalue problems. *J. Differential Equations*, 16:103–125, 1974.

[NS12] Yūki Naito and Tokushi Sato. Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent. *Ann. Mat. Pura Appl. (4)*, 191(1):25–51, 2012.

[Tak15] Kazune Takahashi. Semilinear elliptic equations with critical sobolev exponent and non-homogeneous term. *Master thesis, The University of Tokyo.*, 2015.

[Tar92] Gabriella Tarantello. On nonhomogeneous elliptic equations involving critical Sobolev exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9(3):281–304, 1992.

(K. Takahashi) GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO. 3–8–1 KOMABA MEGUROKU TOKYO 153–8914, JAPAN

E-mail address: kazune@ms.u-tokyo.ac.jp