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Free Burnside groups and their group rings

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Let F_m be a free group of rank $m>1$ and F_m^n the subgroup of F_m generated by all nth powers. The quotient group F_m/F_m^n is denoted by $B(m,n)$ and called the free m-generator Burnside group of exponent n. According to Ivanov [4] and Ol'shanskii [9] (see also [10]), for sufficiently large exponent n, $B(m,n)$ is constructed as the direct limit $B(m,n,\infty)$ of certain quotient groups $B(m,n,i)$ $(i\geq 0)$ of F_m. It is known that $B(m,n,0)$ and $B(m,n,i)$ are resudually finite (that is, each nontrivial element of those groups can be mapped to a non-identity element in some homomorphism onto a finite group) and also that group rings $KB(m,n,0)$ and $KB(m,n,1)$ over a field K are primitive (that is, it has a faithful irreducible (right) R-module). In this note, we shall show that $KB(2,n,1)$ is residually finite and also that $KB(2,n,1)$ is primitive for any K.

1 Introduction

Let F_m be a free group of rank $m>1$ and F_m^n the subgroup of F_m generated by all nth powers. The quotient group F_m/F_m^n is denoted by $B(m,n)$ and called the free m-generator Burnside group of exponent n. Due to Novikov-Adian [5] in 1968 and Ivanov [4] in 1994, $B(m,n)$ is not finite for sufficiently large exponent n, which is known as the negative solution for the famous Burnside problem on periodic groups. Moreover, in 1991, Zelmanov [12] and [13] gave the complete solution for the restricted Burnside problem; thus the orders of all finite m-generator groups of exponent n are bounded

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above by a function m and n. These two remarkable results says that $B(m,n)$ is not residually finite for sufficiently large exponent n, where a group G is residually finite provided that the intersection of all normal subgroups having finite index in G is trivial.

On the other hand, the present author has studied primitivity of group rings of non-noetherian groups ([6], [7],[8]), where a ring is (right) primitive provided it has a faithful irreducible (right) R-module. If a group G is non-noetherian with a non-abelian free subgroup, then the group algebra KG over a field K is often primitive [8]. $B(m,n)$ is also non-noetherian for sufficiently large exponent n, but it has no non-abelian free subgroups. We wish to know whether $KB(m,n)$ is primitive or not if n is sufficiently large.

Now, according to Ivanov [4] and Ol'shanskii [9] (see also [10]), for sufficiently large exponent n, $B(m,n)$ is constructed as the direct limit $B(m,n,\infty)$ of certain quotient groups $B(m,n,i)$ $(i \geq 0)$ of F_m. It can be easily verified that $B(m,n)$ is itself residually finite if $B(m,n,i)$ is residually finite for each $i \geq 0$. Therefore, if n is a sufficiently large integer, then there exists $i \geq 0$ such that $B(m,n,i)$ is not residually finite. On the other hand, if $i = 0$, $B(m,n,0)$ is a free group, and if $i = 1$, $B(m,n,1)$ is a free product of cyclic groups of order n. As is well known, these types of groups are residually finite and their group algebras are primitive. For the time being, we would like to know whether $B(m,n,2)$ is residually finite or not, and also whether $KB(m,n,2)$ is primitive or not.

In the present note, for the sake of simplicity, we cansider the case $m = 2$. If $m = 2$ and $F_2 = \langle x, y \rangle$, then

$$B(2,n,2) = \langle x, y \mid x^n, y^n, (xy)^n, (xy^{-1})^n \rangle.$$

In connection with the form of $B(2,n,2)$, the residual finiteness has been established for $\langle x, y \mid (xy)^n \rangle$ and for $\langle x, y \mid x^n, y^n, (xy)^n \rangle$ as a special case of the results given in [2] (see also [1]) and in [3] respectively. We can show the next theorem which follows both residual finiteness of $B(2,n,2)$ and primitivity of its group algebra:

Theorem 1.1. *Let n be a positive integer and G_n the group with two generators x, y and defining relations $x^n = 1$, $y^n = 1$, $(xy)^n = 1$ and $(xy^{-1})^n = 1$.*
(1) If $n \leq 3$, then G_n is isomorphic to the 2-generator free Burnside group $B(2,n)$.
(2) If $n \geq 4$, then there exist normal subgroups N_n and N_n^* of the derived subgroup G_n' of G_n such that
(i) N_n and N_n^* are free groups with $N_n^* \subseteq N_n$, and in particular, N_4 is finitely generated,
(ii) G_n'/N_n is isomorphic to the cyclic group of infinite order,
(iii) G_n'/N_n^* is isomorphic to the group

$$\langle a, b, c \mid aba^{-1} = c, aca^{-1} = b, [b,c] = 1 \rangle.$$

2 Preliminaries

Throughout this note, if X is a set, $\mathcal{F}(X)$ denotes the free group with the basis X. Let H be a subgroup of $\mathcal{F}(X)$. If S is a subset of H, $\mathcal{N}_H(S)$ denotes the normal closure of S in H.

Let Y be a non-empty subset of X and U a reduced word in X. Then we define the Y-image $U^{\nu_X^Y}$ of U on X as follws; if U in $\mathcal{F}(X \setminus Y)$, $U^{\nu_X^Y} = 1$ and if $U = u_1 y_2 u_2 y_3 u_3 \cdots y_m u_m$ for some y_i in $Y^{\pm 1}$ and u_i in $\mathcal{F}(X \setminus Y)$, $u^{\nu_X^Y} = y_1 \cdots y_m$. Note that $u^{\nu_X^Y}$ need not be reduced in $\mathcal{F}(Y)$ even if u is reduced in X, and also that $u^{\nu_X^Y} = u$ if u is a word in $\mathcal{F}(Y)$.

Definition 2.1. *Let X be a nonempty subset, and let $U = x_1^{\epsilon_1} \cdots x_m^{\epsilon_1}$ is a reduced word in $\mathcal{F}(X)$, where $x_i \in X$ and $\epsilon = \pm 1$. Then (U, x_i)*

is a BT-pair on X provided that $x_i \neq x_j$ *for* $i \neq j$. *Let* Λ *be a well ordered set and* $\mathfrak{U} = \{(U_\lambda, x_\lambda) \mid \lambda \in \Lambda\}$ *a set of BT-pairs on X. We say that* $\mathfrak{U}$ *is a BT-set on X if* U_λ *does not contain* $x_{\lambda'}$ *for* $\lambda < \lambda'$.

Obviously, if (U, x) is a BT-pair on X, then $\{U\} \cup X \setminus \{x\}$ is an another basis of $\mathcal{F}(X)$. More generally, we can easily have

Lemma 2.2. *Let* $\mathcal{F}(X)$ *be a free group with the basis X. If* $\mathfrak{U} = \{(U_\lambda, x_\lambda) \mid \lambda \in \Lambda\}$ *is a BT-set on X, then* $U \cup Y$ *is a basis of* $\mathcal{F}(X)$, *where* $U = \{U_\lambda \mid \lambda \in \Lambda\}$ *and* $Y = X \setminus \{u_\lambda \mid \lambda \in \Lambda\}$.

3 Outline of the proof of Theorem 1.1

In what follows, $\mathbb{Z}$ denotes the rational integers. Let $F = \mathcal{F}(\{x, y\})$ be the free group generated by $\{x, y\}$, n a positive integer, and ρ the map on $\mathbb{Z}$ to $\{0, 1, 2, \cdots, n-1\}$ such that $\rho(i) \equiv i \ (mod\ n)$. We shall first consider the subgroup $L_n = \mathcal{N}_F(x^n, y^n, [x, y])$ of F, where $[x, y] = xyx^{-1}y^{-1}$. Let $n \geq 3$ and i, j integers with $0 \leq i \leq n-1$ and $1 \leq j \leq n-1$. We set ε_{ij} as follows;

$$
(3.1)\quad \begin{aligned}
\varepsilon_{i0} &= x^i y^n x^{-i}, \\
\varepsilon_{ij} &= x^i y^j x y^{-j} x^{-(i+1)} \quad \text{for} \quad 0 \leq i \leq n-2, \\
\varepsilon_{n-1j} &= x^{n-1} y^j x y^{-j}.
\end{aligned}
$$

Furthermore, if $n = 2m + 1$ with $m > 0$, then we set $f_{i0}^n, f_{i1}^n, f_{i2}^n$ as follows;

$$
(3.2)\quad \begin{aligned}
f_{i0}^n &= y^{\rho(2i)} (xy^{-1})^n y^{-\rho(2i)}, \\
f_{01}^n &= (xy)^n, \\
f_{i1}^n &= x^{n-i-1} y^{i-1} (xy)^{n-1} x y^{-(i-2)} x^{-(n-i-1)} \text{ for } \quad 1 \leq i \leq n-1, \\
f_{02}^n &= x^n, \\
f_{i2}^n &= x^{\rho(n-i-2)} y^i x^n y^{-i} x^{-\rho(n-i-2)} \text{ for } \quad 1 \leq i \leq n-1,
\end{aligned}
$$

and if $n = 2m$ with $m > 1$, then we set $f_{i0}^n, f_{im-1}^n, f_{im}^n$ as follows;

$$
(3.3)\quad
\begin{array}{lcl}
f^n_{m-10} & = & x^{m+1}y^{m-1}x^n y^{-(m-1)}x^{-(m+1)}, \\
f^n_{m0} & = & x^{m+2}y^m x^n y^{-m}x^{-(m+2)}, \\
f^n_{i0} & = & y^i x^n y^{-i} \quad \text{for} \quad i \in \{0,1,\cdots,n-1\} \setminus \{m-1,m\}, \\
f^n_{im-1} & = & x^i y^{m-1}(xy^{-1})^n y^{-(m-1)}x^{-i}, \\
f^n_{im} & = & x^i y^m (xy)^{n-1} xy^{-(m-1)}x^{-i}.
\end{array}
$$

In addition, we set $X_n = \{\varepsilon_{ij}, x^n \mid 0 \le i,j \le n-1\}$. Then we can get the following lemma:

Lemma 3.1. (1) *X_n is a basis of L_n for each $n \ge 3$.*
(2) *Let $n = 2m+1$ with $m > 0$ (resp. $n = 2m$ with $m > 1$) . If $0 \le i \le n-1$ and $1 \le j \le n-1$, then each of $f^n_{i0}, f^n_{i1}, f^n_{j2}$ (resp. $f^n_{j0}, f^n_{im-1}, f^n_{im}$) is expressed as a reduced word in X_n as follows;*

$$
\begin{array}{ll}
f^n_{00} = \prod_{t=0}^{n-1} f^{n^t}_{00}, & f^{n^t}_{00} = \begin{cases} \varepsilon_{10}^{-1} \text{ for } & t = 0, \\ \varepsilon_{t(n-t)} \text{ for } & t > 0, \end{cases} \\
f^n_{j0} = \prod_{t=0}^{n-1} f^{n^t}_{j0}, & f^{n^t}_{j0} = \begin{cases} x^n \varepsilon_{00}^{-1} \text{ for } & \rho(2j-t) = 0, j = m, \\ \varepsilon^{-1}_{\rho(2j+1)0} \text{ for } & \rho(2j-t) = 0, j \neq m, \\ \varepsilon_{t\rho(2j-t)} \text{ for } & \rho(2j-t) \neq 0, \end{cases} \\
f^n_{01} = \prod_{t=0}^{n-1} f^{n^t}_{01}, & f^{n^t}_{01} = \varepsilon_{\rho(t+1)\rho(t+1)} \\
f^n_{j1} = \prod_{t=0}^{n-1} f^{n^t}_{j1}, & f^{n^t}_{j1} = \begin{cases} \varepsilon_{\rho(n-1+t)\rho(t+1)} & (j = 1), \\ \varepsilon_{(n-1)0}x^n & (j = m+1,\ t = m+1), \\ \varepsilon_{\rho(n-j-1+t)\rho(j-1+t)} & \text{for the others}, \end{cases} \\
f^n_{j2} = \prod_{t=0}^{n-1} f^{n^t}_{j2}, & f^{n^t}_{j2} = \varepsilon_{\rho(n-j-2+t)j},
\end{array}
$$

(*resp.*

$$
\begin{array}{ll}
f^n_{j0} = \prod_{t=0}^{n-1} f^{n^t}_{j0}, & f^{n^t}_{j0} = \begin{cases} \varepsilon_{\rho(m+1+t)(m-1)}, & (j = m-1), \\ \varepsilon_{\rho(m+2+t)m}, & (j = m), \\ \varepsilon_{tj}, & (j \neq m-1, m), \end{cases} \\
f^n_{i(m-1)} = \prod_{t=0}^{n-1} f^{n^t}_{i(m-1)}, & f^{n^t}_{i(m-1)} = \begin{cases} x^n \varepsilon_{00}^{-1} & (t = m-1, i = m,) \\ \varepsilon^{-1}_{\rho(i+m)0} & (t = m-1, i \neq m), \\ \varepsilon_{\rho(i+t)\rho(m-1-t)} & (t \neq m-1), \end{cases} \\
f^n_{im} = \prod_{t=0}^{n-1} f^{n^t}_{im}, & f^{n^t}_{im} = \begin{cases} \varepsilon_{(n-1)0}x^n & (i = m-1, t = m), \\ \varepsilon_{\rho(i+t)\rho(m+t)} & \text{for the others. }). \end{cases}
\end{array}
$$

(3) *Let* $H_n = \mathcal{N}_F(x^n, y^n, (xy)^n, (xy^{-1})^n)$. *If* $n = 2m+1$ *with* $m > 0$ (*resp.* $n = 2m$ *with* $m > 1$), *then*

$$\mathcal{N}_{L_n}(f_{i0}^n, f_{i1}^n, f_{i2}^n, \varepsilon_{i0} \mid 0 \le i \le n-1) = H_n$$
$$(\textit{resp.}\ \ \mathcal{N}_{L_n}(f_{i0}^n, f_{i(m-1)}^n, f_{im}^n, \varepsilon_{i0} \mid 0 \le i \le n-1) = H_n\ \ ).$$

We express X_n as a union of pairwise disjoint subsets:

$$X_n = X_n^{(1)} \cup X_n^{(2)} \cup X_n^{(3)},$$

where if $n = 2m+1$ with $m \ge 2$,

$$\begin{aligned} X_n^{(1)} &= \{\varepsilon_{01}, \varepsilon_{(m-1)(m+1)}, \varepsilon_{(m-2)(m+2)}, \varepsilon_{(m+1)(m-1)}\}, \\ X_n^{(2)} &= \{\varepsilon_{ii}, \varepsilon_{j(n-j)}, \varepsilon_{\rho(n-2-j)j} \mid 1 \le i \le n-2,\ 1 \le j \le n-1\}, \\ X_n^{(3)} &= X_n \setminus (X_n^{(1)} \cup X_n^{(2)}), \end{aligned}$$

if $n = 2m$ with $m \ge 2$,

$$\begin{aligned} X_n^{(1)} &= \{\varepsilon_{(n-3)(n-1)}, \varepsilon_{(n-1)(n-2)}, \varepsilon_{(n-2)(n-1)}, \varepsilon_{(n-2)(n-2)}, \varepsilon_{(n-1)(n-1)}\}, \\ X_n^{(2)} &= \{\varepsilon_{(n-3)(n-2)}, \varepsilon_{0i}, \varepsilon_{j(m-1)}, \varepsilon_{tm} \mid i \in I_n^{m-1},\ j \in I_n,\ t \in, I_n^m\}, \\ X_n^{(3)} &= X_n \setminus (X_n^{(1)} \cup X_n^{(2)}), \end{aligned}$$

where $I_n = \{0, 1, 2, \cdots, n-1\}$, $I_n^{m-1} = I_n \setminus \{m-1, m, 0, n-2\}$ and $I_n^m = I_n \setminus \{m, m+1\}$.

If we set

$$X_n^{(2)*} = \begin{cases} \{f_{i0}^n, f_{jt}^n \mid 1 \le i \le n-2,\ 1 \le j \le n-1,\ t = 1, 2\} \\ \quad \text{for } \ n = 2m+1 \\ \{f_{i0}^n, f_{j(m-1)}^n, f_{tm}^n, \mid i \in I_n^{m-1} \setminus \{0\},\ j \in I_n,\ t \in I_n^m\} \\ \quad \text{for } \ n = 2m, \end{cases}$$

then $X_n^* = X_n^{(1)} \cup X_n^{(2)*} \cup X_n^{(3)'}$ is a basis of L_n.

We set $Y_n = \{\varepsilon^{(i)} \mid \varepsilon \in X_n^*,\ i \in \mathbb{Z}\} \setminus \{1\}$. Since $\mathfrak{T}_2 = \{\alpha_n^i \mid i \in \mathbb{Z}\}$ is a Schreier transversal to M_n in L_n, Y_n is a basis of M_n. We express Y_n as a union of disjoint subsets:

$$Y_n = Y_n^{(1)} \cup Y_n^{(2)} \cup Y_n^{(3)},$$

where $Y_n^{(1)} = \{\varepsilon^{(i)} \mid \varepsilon \in X_n^{(1)}, i \in \mathbb{Z}\} \setminus \{1\}$, $Y_n^{(2)} = \{f^{n^{(i)}} \mid f^n \in X_n^{(2)*}, i \in \mathbb{Z}\} \cup \{\varepsilon_{j0}^{(i)} \mid 0 \leq j \leq n-1, i \in \mathbb{Z}\}$, and $Y_n^{(3)} = Y_n \setminus (Y_n^{(1)} \cup Y_n^{(2)})$. Then note that $Y_4^{(3)} = \emptyset$ and $Y_n^{(3)} \neq \emptyset$ for $n \geq 5$.

Outline of the proof of Theorem 1.1 (1): If $n = 1$, then we have nothing to prove.

Since relations $x^2 = 1$, $y^2 = 1$ and $(xy)^2 = 1$ implies the relation $[x, y] = xyx^{-1}y^{-1} = 1$, it is trivial that G_2 is isomorphic to $B(2, 2)$.

Now, as is well known, $B(2, 3)$ is finite and has order 3^3. In addition, $B(2, 3)$ is isomorphic to a homomorphic image of G_3. Hence to get the conclusion, it suffices to show that G_3 is finite and has order 3^3. Let F be free group generated by $\{x, y\}$, and $L_3 = \mathcal{N}_F(x^3, y^3, [x, y])$. By Lemma 3.1 (1), $X_3 = \{\varepsilon_{ij}, x^3 \mid 0 \leq i, j \leq 2\}$ is a basis of L_3, and

$$\begin{array}{lll}
f_{00}^3 = \varepsilon_{10}^{-1}\varepsilon_{12}\varepsilon_{21}, & f_{01}^3 = \varepsilon_{11}\varepsilon_{22}\varepsilon_{00}, & f_{12}^3 = \varepsilon_{01}\varepsilon_{11}\varepsilon_{21}, \\
f_{10}^3 = \varepsilon_{02}\varepsilon_{11}x^3\varepsilon_{00}^{-1}, & f_{11}^3 = \varepsilon_{21}\varepsilon_{02}\varepsilon_{10}, & f_{22}^3 = \varepsilon_{22}\varepsilon_{02}\varepsilon_{12}, \\
f_{20}^3 = \varepsilon_{01}\varepsilon_{20}^{-1}\varepsilon_{22}, & f_{21}^3 = \varepsilon_{01}\varepsilon_{12}\varepsilon_{20}x^3, &
\end{array}$$

where f_{ij}^3 is as described in (3.2). We set

$$\begin{array}{l}
(V_1, v_1) = (f_{10}^3, \varepsilon_{11}), \ (V_2, v_2) = (f_{11}^3, \varepsilon_{21}), \ (V_3, v_3) = (f_{12}^3, \varepsilon_{01}), \\
(V_4, v_4) = (f_{21}^3, \varepsilon_{12}), \ (V_5, v_5) = (f_{22}^3, \varepsilon_{22}).
\end{array}$$

Then it is easily verified that (V_i, v_i) is a BT-pair on X_3 for each $i \in \{1, 2, 3, 4, 5\}$, and also that the expression of V_i on X_3 does not contain v_j for each $i, j \in \{1, 2, 3, 4, 5\}$ with $i < j$. Hence $\{(V_i, v_i) \mid 1 \leq i \leq 5\}$ is a BT-set on X_3. By virtue of Lemma 2.2, $X_3^* = \{f_{10}^3, f_{11}^3, f_{12}^3, f_{21}^3, f_{22}^3\} \cup \{\varepsilon_{02}, x^3, \varepsilon_{i0} \mid 0 \leq i \leq 2\}$ is a basis of L_3. Let $X_3^{*\varepsilon_{02}} = X_3^* \setminus \{\varepsilon_{02}\}$, and $\widehat{\ }: L_3 \longrightarrow L_3/\mathcal{N}_{L_3}(X_3^{*\varepsilon_{02}})$ the natural epimorphism. Clearly, $\widehat{L_3} = \langle\widehat{\varepsilon_{02}}\rangle$ is cyclic of infinite order. Moreover, it is easily verified that $\widehat{f_{01}^3} = \widehat{1}$, $\widehat{f_{00}^3} = \widehat{\varepsilon_{02}}^{-3}$ and $\widehat{f_{20}^3} = \widehat{\varepsilon_{02}}^3$, and so $\mathcal{N}_{L_3}(X_3^{*\varepsilon_{02}}, \{f_{01}^3, f_{00}^3, f_{20}^3\}) = \mathcal{N}_{L_3}(X_3^{*\varepsilon_{02}}, \{\varepsilon_{02}^3\})$.

Hence $L_3/\mathcal{N}_{L_3}(X_3^{*\varepsilon_{02}}, \{f_{01}^3, f_{00}^3, f_{20}^3\})$ is isomorphic to the cyclic group of order 3. On the other hand, by Lemma 3.1 (3),

$$\mathcal{N}_{L_3}(X_3^{*\varepsilon_{02}}, \{f_{01}^3, f_{00}^3, f_{20}^3\}) = H_3 = \mathcal{N}_F(x^3, y^3, (xy)^3, (xy^{-1})^3),$$

and so L_3/H_3 is cyclic of order 3. Since the derived subgroup G_3' of G_3 is isomorphic to L_3/H_3 and G_3/G_3' is abelian of order 3^2, it follows that G_3 is finite and has order 3^3.

(2): For $n = 2m+1$ (resp. $n = 2m$) with $m \geq 2$, then we set

$$\alpha_n = \varepsilon_{01}, \quad \beta_{n1} = \varepsilon_{(m-1)(m+1)}, \quad \beta_{n2} = \varepsilon_{(m-2)(m+2)}, \quad \beta_{n3} = \varepsilon_{(m+1)(m-1)}$$

$$\left(\begin{array}{l} \text{resp.} \quad \alpha_n = \varepsilon_{(n-3)(n-1)}, \quad \beta_{n1} = \varepsilon_{(n-2)(n-1)}, \quad \beta_{n2} = \varepsilon_{(n-2)(n-2)}, \\ \qquad\quad \beta_{n0} = \varepsilon_{(n-1)(n-2)}, \quad \beta_{n3} = \varepsilon_{(n-1)(n-1)} \end{array} \right).$$

Let $Z_{n1} = X_n^* \setminus \{\alpha_n\}$, $Z_{n2} = X_n^* \setminus \{\alpha_n, \beta_{n0}, \beta_{n1}, \beta_{n2}, \beta_{n3}\}$ and

$$M_n = \begin{cases} \mathcal{N}_{L_n}(\varepsilon \mid \varepsilon \in Z_{n1}) & \text{for } n = 2m+1, \\ \mathcal{N}_{L_n}(\varepsilon, \alpha_n\beta_{n0}, \alpha_n\beta_{n1}^{-1}, \alpha_n\beta_{n2}^{-1}, \alpha_n\beta_{n3}^{-1} \mid \varepsilon \in Z_{n2}) & \text{for } n = 2m \end{cases}$$

of L_n, where $m \geq 2$. and $H_n = \mathcal{N}_F(x^n, y^n, (xy)^n, (xy^{-1})^n)$. We can see that H_n is a normal subgroup of M_n.

Let n be a positive integer with $n \geq 4$ and M_n as above. If we set

$$Y_n^{(1)**} = \begin{cases} \{f_{00}^{n^{(i)}}, f_{01}^{n^{(i)}}, f_{(n-1)0}^{n^{(i)}} \mid i \in \mathbb{Z}\} & \text{for } n = 2m+1 \\ \{f_{mm}^{n^{(i)}}, f_{(m+1)m}^{n^{(i)}}, f_{(m-1)0}^{n^{(i)}}, f_{m0}^{n^{(i)}} \mid i \in \mathbb{Z}\} & \text{for } n = 2m, \end{cases}$$

$$Y_n^{(0)*} = \begin{cases} \{\beta_{n2}^{(i)}, \beta_{n3}^{(-1)}, \delta_n^{(i)} \mid 0 \leq i \leq m-1\} \ \text{for } n = 2m+1 \\ \{\beta_{n0}^{(0)}, \beta_{n0}^{(1)}, \beta_{n0}^{(2)}, \beta_{n1}^{(0)}, \beta_{n2}^{(-1)}, \beta_{n2}^{(0)}, \beta_{n3}^{(-2)}, \beta_{n3}^{(-1)}, \beta_{n3}^{(0)}, \delta_n^{(-2)}\} \\ \qquad\qquad \text{for } n = 2m \text{ and } m > 2 \\ \{\beta_{41}^{(-1)}, \beta_{41}^{(-2)}, \beta_{42}^{(-1)}, \beta_{42}^{(0)}, \beta_{43}^{(-2)}, \beta_{43}^{(-1)}, \beta_{43}^{(0)}, \delta_4^{(-1)}, \delta_4^{(0)}\} \\ \qquad\qquad \text{for } n = 4, \end{cases}$$

and $Y_n^{**} = Y_n^{(0)*} \cup Y_n^{(1)**} \cup Y_n^{(2)} \cup Y_n^{(3)}$, then we can see that Y_n^{**} is a basis of M_n.

We set $N_n = M_n/H_n$. It follows from the above that $H_n = \mathcal{N}_{M_n}(Y_n^{(1)**} \cup Y_n^{(2)})$. Since $Y_n^{**} = Y_n^{(0)*} \cup Y_n^{(1)**} \cup Y_n^{(2)} \cup Y_n^{(3)}$ is a basis of M_n, we have that N_n is isomorphic to the free group generated by $Y_n^{(0)*} \cup Y_n^{(3)}$. Let G_n' be the derived subgroup of G_n and $L_n = \mathcal{N}_F(x^n, y^n, [x,y])$. It obvious that G_n' coincides with L_n/H_n. Hence, by definition of M_n, $N_n = M_n/H_n$ is a normal subgroup of G_n', and G_n'/N_n is isomorphic to L_n/M_n which is isomorphic to $\langle \alpha_n \rangle$, the cyclic group of infinite order.

Now, if $n = 4$ then $Y_4^{(3)} = \emptyset$, and so N_4 is isomorphic to the free group generated by the finite basis

$$Y_n^{(0)*} = \{\beta_{41}^{(-2)}, \beta_{41}^{(-1)}, \beta_{42}^{(-1)}, \beta_{42}^{(0)}, \beta_{43}^{(-2)}, \beta_{43}^{(-1)}, \beta_{43}^{(0)}, \delta_4^{(-1)}, \delta_4^{(0)}\}$$

. We set $N_4' = [N_4, N_4]$ and

$$N_4^* = \langle \beta_{41}^{(-2)}, \beta_{41}^{(-1)}, \beta_{43}^{(-2)}, \beta_{43}^{(-1)}, \beta_{43}^{(0)}, \delta_4^{(-1)}, \delta_4^{(0)} \rangle N_4'.$$

We have then that

$$f_{22}^{4(i-1)\sigma_4} = \beta_{42}^{(i-1)} \beta_{43}^{(i)} \beta_{43}^{(i+1)^{-1}} \beta_{43}^{(i-1)^{-1}}.$$

Since $\{(f_{22}^{4(i-1)}, \beta_{42}^{(i+1)}), (f_{22}^{4(j-1)}, \beta_{42}^{(j-1)}) \mid i \geq 0,\ j < 0\}$ is a subset of a BT-set on Y_4^*, we have that

$$(3.4) \qquad \begin{cases} \beta_{42}^{(i+1)} & = v\beta_{42}^{(i-1)} \beta_{43}^{(i-1)^{-1}} \beta_{43}^{(i)} \quad (\ mod\ N_4'\ ) \ \text{ for } \ i \geq 0, \\ \beta_{42}^{(i-1)} & = \beta_{42}^{(i+1)} \beta_{43}^{(i-1)} \beta_{43}^{(i)^{-1}} \quad (\ mod\ N_4'\ ) \ \text{ for } \ i < 0. \end{cases}$$

Similarly if $i \geq 0$, under $mod\ N_4'$, we have

$$(3.5) \qquad \begin{aligned} \beta_{41}^{(i)} & = \beta_{41}^{(i-1)} \delta_4^{(i-1)^{-1}} \delta_4^{(i)^{-1}}, \\ \beta_{43}^{(i+1)} & = \beta_{41}^{(i-2)^{-1}} \beta_{41}^{(i-1)} \beta_{42}^{(i-1)^{-1}} \beta_{42}^{(i+1)} \beta_{43}^{(i-2)} \delta_4^{(i-1)} \delta_4^{(i)}, \\ \delta_4^{(i+1)} & = \beta_{41}^{(i-2)^{-1}} \beta_{41}^{(i-1)^2} \beta_{41}^{(i)} \beta_{42}^{(i-1)^{-1}} \beta_{42}^{(i+1)} \beta_{43}^{(i)} \beta_{43}^{(i+1)^{-1}} \delta_4^{(i-1)}. \end{aligned}$$

Then the first equation in (3.5) implies $\beta_{41}^{(0)} \in N_4^*$. Since $\beta_{42}^{(1)} \beta_{42}^{(-1)^{-1}} \in N_4^*$ by (3.4), the second equation in (3.5) implies $\beta_{43}^{(1)} \in N_4^*$, and so the last equation in (3.5) implies $\delta_4^{(1)} \in N_4^*$. That is $\{\beta_{41}^{(0)}, \beta_{43}^{(1)}, \delta_4^{(1)}\} \subseteq N_4^*$. By induction on i, we have that

$$\{\beta_{41}^{(i)}, \beta_{43}^{(i+1)}, \delta_4^{(i+1)} \mid i \geq 0\} \subseteq N_4^*.$$

Similarly, if $i < 0$, it is verified that all of $\delta_4^{(i-1)}$, $\beta_{41}^{(i-2)}$ and $\beta_{43}^{(i-2)}$ are in N_4^*. We have thus seen that $\{\beta_{41}^{(i)}, \beta_{43}^{(i)}, \delta_4^{(i)} \mid i \in \mathbb{Z}\} \subseteq N_4^*$. Since $G_4' = \langle \alpha_4 \rangle N_4$ and $\alpha_4^j \beta_{4t}^{(i)} \alpha_4^{-j} = \beta_{4t}^{(i+j)}$ for each $i, j \in \mathbb{Z}$, it follows that N_4^* is a normal subgroup of G_4', and also that $\beta_{42}^{(i)} N_4^* = \beta_{42}^{(i+2)} N_4^*$ for each $i \in \mathbb{Z}$ by (3.4). Hence, if we set $a = \alpha_4 N_4^*$, $b = \beta_{42}^{(-1)} N_4^*$ and $c = \beta_{42}^{(0)} N_4^*$, then G_4'/N_4^* is isomorphic to the group $\langle a, b, c \mid aba^{-1} = c, aca^{-1} = b, [b, c] = 1 \rangle$.

Finally, let $n \geq 5$. Recall that $Y_n^{(3)} = \{\varepsilon^{(i)} \mid \varepsilon \in X_n^{(3)\prime}, i \in \mathbb{Z}\}$ and $Y_n^{(3)} \neq \emptyset$, where $X_n^{(3)\prime} = X_n^{(3)} \setminus \{\varepsilon_{i0} \mid 0 \leq i \leq n-1\}$. Let $\varepsilon_0 \in X_n^{(3)\prime}$, and set $Y_n^{(3)\varepsilon_0} = Y_n^{(3)} \setminus \{\varepsilon_0^{(i)} \mid i \in \mathbb{Z}\}$, $N_{n1}^* = \langle \varepsilon_0^{(i)} \varepsilon_0^{(i+2)} \mid i \in \mathbb{Z} \rangle [N_n, N_n]$ and $N_{n2}^* = \langle \varepsilon^{(i)} \mid \varepsilon^{(i)} \in Y_n^{(3)\varepsilon_0} \rangle [N_n, N_n]$. Since $G_n' = \langle \alpha_n \rangle N_n$ and $\alpha_n^j \varepsilon^{(i)} \alpha_n^{-j} = \varepsilon^{(i+j)}$ for each $\varepsilon^{(i)} \in Y_n^{(3)}$ and each $j \in \mathbb{Z}$, it is verified that both of N_{n1}^* and N_{n2}^* are normal subgroup of G_n' and so is $N_n^* = N_{n1}^* N_{n2}^*$. Moreover, if we set $a = \alpha_n N_n^*$, $b = \varepsilon_0^{(0)} N_n^*$ and $c = \varepsilon_0^{(1)} N_n^*$, then G_n'/N_n^* is isomorphic to the group $\langle a, b, c \mid aba^{-1} = c, aca^{-1} = b, [b, c] = 1 \rangle$. □

4 Residually finiteness and primitivity

Theorem 1.1 says that the derived subgroup G_n' of G_n is a cyclic extension of a free group. Since we can see $\triangle(G) = 1$, by [11, Theorem 1], we have the following result:

Theorem 4.1. *For a positive integer n, let G_n be as described in Theorem 1.1. If $n > 3$ then the group algebra KG_n of G_n over a field K is primitive.*

Finally, by making use of Theorem 1.1, we shall prove residual finiteness of G_n.

Theorem 4.2. *If n is a positive integer and G_n is as described in Theorem 1.1, then G_n is residually finite.*

Proof. If $n \leq 3$, then G_n is finite by Theorem 1.1 (1), and so we may assume $n \geq 4$. Let G'_n be the derived subgroup of G_n and let $\gamma_i G'_n$ is the ith term of the lower central series of G'_n;thus $\gamma_1 G'_n = G'_n$ and $\gamma_{i+1} G'_n = [\gamma_i G'_n, G'_n]$.

First we shall show that G'_n is residually nilpotent, that is $\bigcap_{i=1}^{\infty} \gamma_i G'_n = 1$. By virtue of Theorem 1.1 (2), there exists a normal subgroup N^*_n of G'_n such that G'_n/N^*_n is isomorphic to the group $\langle a, b, c \mid aba^{-1} = c, aca^{-1} = b, [b, c] = 1\rangle$. Since $[aba^{-1}, b] = [[a, b], b]$, G'_n/N^*_n is isomorphic to the group $\overline{G'_n} = \langle a, b \mid a^2ba^{-2} = b, [[a, b], b] = 1\rangle$. Since the relation $a^2ba^{-2} = b$ implies $a[b, a]a^{-1} = [b, a]^{-1}$ and this implies $[[b, a], a] = [b, a]^2$, it is inductively verified that that $[b, a]_i = [b, a]^{2^{i-1}}$ for each $i > 0$ where $[b, a]_1 = [b, a]$ and $[b, a]_{i+1} = [[b, a]_i, a]$. Moreover, since $b[b, a]b^{-1} = [b, a]$, it follows that for each $i \geq 2$, the ith term $\gamma_i \overline{G'_n}$ of the lower central series of $\overline{G'_n}$ coincides with $\langle [b, a]^{2^{i-2}} \rangle$, the cyclic group generated by the element $[b, a]^{2^{i-2}}$. In particular, for each $i \geq 1$, $\gamma_i \overline{G'_n} \supset \gamma_{i+1} \overline{G'_n}$, a proper subgroup, and so $\gamma_{i+1} G'_n$ is a proper subgroup of $\gamma_i G'_n$ for each $i \geq 1$. Since $\gamma_2 G'_n$ is a subgroup of the free group N_n by Theorem 1.1 (2), $\gamma_2 G'_n$ is itself free. As is well known, any proper infinite descending chain of characteristic subgroups of a free group has trivial intersection, and so $\bigcap_{i=1}^{\infty} \gamma_i G'_n = 1$, as desired.

Now, let g be an arbitrary element in G_n with $g \neq 1$. To complete the proof, we require to find a normal subgroup, not containing g, and of finite index in G_n. Since G_n/G'_n is finite abelian, we may assume g in G'_n. As we sow in the above, $\bigcap_{i=1}^{\infty} \gamma_i G'_n = 1$, and so there exists a positive integer i_g such that $g \notin \gamma_{i_g} G'_n$. Moreover $\gamma_{i_g} G'_n$ is a normal subgroup of G_n, and therefore it suffices to show that $G_n/\gamma_{i_g} G'_n$ is residually finite. However, it is almost clear: In fact, $G'_n/\gamma_{i_g} G'_n$ is finitely generated nilpotent and so polycyclic. Hence $G_n/\gamma_{i_g} G'_n$ is also polycyclic, and the conclusion follows from residual finiteness of polycyclic groups. □

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