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Brezis-Gallouet-Wainger type inequalities and blow-up criteria for Navier-Stokes equations in bounded domains

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1 Introduction

This note is a survey of our work [10]. Let Ω be a 3-dimensional bounded domain with $\partial\Omega\in C^{\infty}$. The motion of a viscous incompressible fluid in Ω is governed by the Navier-Stokes equations:

$$\text{(N-S)}\left\{\begin{array}{ll} \partial_{t}u-\Delta u+u\cdot\nabla u+\nabla p=0, & \text{div } u=0 \qquad t\in(0,T), \quad x\in\Omega, \\ \qquad\qquad\qquad u|_{\partial\Omega}=0, & u|_{t=0}=u_{0}, \end{array}\right.$$

where $u=(u^{1}(x,t),u^{2}(x,t),u^{3}(x,t))$ and $p=p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t)\in\Omega\times(0,T)$ and u_{0} is the given initial velocity vector field. In this note, we consider Beale-Kato-Majda type blow-up criteria of classical solutions to (N-S).

Beale-Kato-Majda [1] and Kato-Ponce [7] showed that the L^{∞}-norm of the vorticity $\omega=\text{rot } u$ controls the breakdown of smooth solutions to the Euler and Navier-Stokes equations if $\Omega=\mathbb{R}^{n}$. To be precise, if the smooth solution u in $C([0,T);W^{s,p}(\mathbb{R}^{n})(s>n/p+1)$ breaks down at a finite time $t=T$, then $\int_{0}^{t}\|\omega(\tau)\|_{L^{\infty}(\Omega)}d\tau\nearrow\infty$ as $t\nearrow T$. Chemin [4] and Kozono-Ogawa-Taniuchi [8] proved similar blow-up criteria with $\|\omega\|_{\infty}$ replaced by $\|u\|_{B_{\infty,\infty}^{1}}$ and $\|\omega\|_{\dot{B}_{\infty,\infty}^{0}}$. In the case where Ω is a bounded domain, for the 3-D

Euler equations, Shirota-Yanagisawa [13] and Ferrari [6] proved the same result as Beale-Kato-Majda. See also Zajaczkowski [15]. Ogawa-Taniuchi [11] proved a similar blow-up criterion with $\|\omega\|_{L^\infty(\Omega)}$ replaced by $\|\omega\|_{bmo(\Omega)}$. However, in [11], the blow-up criterion via $\|\omega\|_{bmo(\Omega)}$ was proven only for 3-D Euler equations. In this note, we prove the same criterion for 3-D Navier-Stokes equations in bounded domains with smooth boundary.

In order to prove the above-mentioned results, the following Brezis-Gallouet-Wainger type inequalities play important roles:

$$(BGW)_\beta \qquad \|f\|_{L^\infty} \le C(1 + \|f\|_X \log^\beta(e + \|f\|_Y)).$$

Brezis-Gallouet-Wainger [2, 3] proved $(BGW)_\beta$ with $\beta = 1 - 1/p$, $X = W^{n/p,p}(\mathbb{R}^n)$, $Y = W^{n/q+\alpha,q}(\mathbb{R}^n)(\subset \dot{C}^\alpha)(\alpha > 0)$. Engler [5] proved the same inquality for general domains Ω without using the Fourier transform. We note that Ozawa [12] proved that $\|f\|_{L^q(\mathbb{R}^n)} \le C(p,n)q^{1-1/p}\|(-\Delta)^{n/2p}f\|_p^{1-p/q}\|f\|_p^{p/q}$ holds for all $q \in [p,\infty)$ and that this estimate directly yields $(BGW)_\beta$. When Ω is a bounded domain, in [11], $(BGW)_\beta$ was proved for $\beta = 1$, $X = bmo(\Omega)$ and $Y = \dot{C}^\alpha(\Omega)$ by using the method given in [12]. We note that in [1, 4, 5, 6, 7, 8, 9, 12, 13, 15] several inequalities of Brezis-Gallouet-Wainger type were established. Then, we have one question. What is the largest normed space X that satisfies $(BGW)_\beta$ with $Y = \dot{C}^\alpha(\Omega)$? In this note, we also consider this problem.

2 Function spaces

We first introduce Banach spaces of Morrey type and Besov type which are wider than L^∞.

<u>DEFINITION.</u> (1) $M_\beta(\Omega) = \{f \in L^1(\Omega); \|f\|_{M_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{M_\beta(\Omega)} := \sup_{x\in\mathbb{R}^n,\ 0<t<1} \frac{1}{|B(x,t)|\log^\beta(e+\frac{1}{t})} \int_{B(x,t)} E_0|f(y)|dy$$

where E_0 is the 0-extension operator from functions defined on Ω to functions on $\mathbb{R}^n$ and $B(x,t) := \{y \in \mathbb{R}^n;\ |y-x| < t\}$.

(2) (Modified Vishik's space). Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a spherical symmetric function with $\hat{\psi}(\xi) = 1$ in $B(0,1)$ and $\hat{\psi}(\xi) = 0$ in $B(0,2)^c$. $V_\beta(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{V_\beta} < \infty\}$ is introduced by the norm

$$\|f\|_{V_\beta} := \sup_{N=1,2,\cdots} \frac{\|\psi_N * f\|_\infty}{N^\beta}, \quad \text{where } \psi_N(x) := 2^{nN}\psi(2^N x).$$

We note that the space V_β is a modified version of spaces introduced by Vishik[14]. We also note that $M_1(\Omega) \supset bmo(\Omega) \supset L^\infty(\Omega)$ and $V_1(\mathbb{R}^n) \supset B^0_{\infty,\infty}(\mathbb{R}^n) \supset bmo(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n)$.

Let $C^\infty_{0,\sigma}(\Omega) = C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector fields $\phi = (\phi^1, \cdots, \phi^n)$ with compact support in Ω such that div $\phi = 0$. Then L^r_σ, $1 < r < \infty$, is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$. Note that very often we will simply write L^r and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. The symbol $(\cdot,\cdot)$ denotes the L^2-inner product and the duality pairing between L^p and $L^{p'}$, where $1/p + 1/p' = 1$.

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L^r_\sigma \oplus G_r$ $(1 < r < \infty)$, where $G_r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$, P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W^{1,r}_0 \cap L^r_\sigma$. Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ $(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u, P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u, A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively.

In this paper, we denote by C various constants.

3 Main Theorems

Now our results read as follows:

Theorem 1. *Let $\Omega(\subset \mathbb{R}^n)$ be a bounded domain with $\partial\Omega \in C^\infty$.*

(1) *For any $\alpha \in (0,1)$ and $\beta > 0$, there exists a constant $C(\Omega,\alpha,\beta) > 0$ such that*

$$(3.1)\quad \|f\|_{L^\infty(\Omega)} \le C\Big(1 + \|f\|_{M_\beta(\Omega)} \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\Omega)}\right)\Big) \text{ for all } f \in \dot{C}^\alpha(\Omega) \cap M_\beta(\Omega).$$

(2) *Let $\beta > 0$ and X be a normed space. Assume that X satisfies the following conditions:*

$$(A)\begin{cases} (i) & L^\infty \hookrightarrow X \subset L^1(\Omega), \\ (ii) & X \text{ is a translation invariant space, i.e.,} \\ & \quad \|f(\cdot - y)\|_X = \|f\|_X \text{ if } f \text{ and } y \in \mathbb{R}^n \text{ satisfy } \operatorname{supp} f(\cdot - y), \operatorname{supp} f \subset \bar{\Omega}, \\ (iii) & \|f\|_X \le \|g\|_X \text{ if } |f(x)| \le |g(x)| \text{ a.e. } x \in \Omega, \\ (iv) & \text{there exist constants } \alpha \in (0,1) \text{ and } C > 0 \text{ such that} \\ & \quad \|f\|_{L^\infty(\Omega)} \le C\Big(1 + \|f\|_X \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\Omega)}\right)\Big) \text{ for all } f \in \dot{C}^\alpha(\Omega) \cap X. \end{cases}$$

Then, X is continuously imbedded in $M_\beta(\Omega)$.

Remarks. Since $M_\beta(\Omega)$ satisfies (A), Theorem 1 implies that M_β is the largest normed space that satisfies conditions (A).

When we do not assume $(A)(iii)$, there is a normed space wider than M_β if $\Omega = \mathbb{R}^n$ as below.

Theorem 2. (1) *For any $\alpha \in (0,1)$ and $\beta > 0$, there exists a constant $C(\alpha,\beta) > 0$ such that*

$$(3.2)\quad \|f\|_{L^\infty(\mathbb{R}^n)} \le C\Big(1 + \|f\|_{V_\beta(\mathbb{R}^n)} \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\right)\Big) \text{ for all } f \in \dot{C}^\alpha(\mathbb{R}^n) \cap V_\beta.$$

(2) *Let $\beta > 0$ and X be a normed space. Assume that X satisfies the following conditions:*

$$(B)\begin{cases} (i) & X \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \\ (ii) & X \text{ is a translation invariant space, i.e.,} \\ & \quad \|f(\cdot - y)\|_X = \|f\|_X \text{ for all } y \in \mathbb{R}^n, \\ (iii) & \|\rho * f\|_X \le \|\rho\|_{L^1(\mathbb{R}^n)}\|f\|_X \text{ for all } \rho \in \mathcal{S}' \\ (iv) & \text{there exist constants } \alpha \in (0,1) \text{ and } C > 0 \text{ such that} \\ & \quad \|f\|_{L^\infty(\mathbb{R}^n)} \le C\Big(1 + \|f\|_X \log^\beta\left(e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}\right)\Big) \text{ for all } f \in BC^\infty(\mathbb{R}^n). \end{cases}$$

Then, X is continuously imbedded in V_β.

Remark. (i) Since $V_\beta(\mathbb{R}^n)$ satisfies (B), Theorem 2 implies that $V_\beta(\mathbb{R}^n)$ is the largest normed space that satisfies conditions (B).

(ii) Since $E_0 M_\beta(\Omega) \subset V_\beta(\mathbb{R}^n)$, $V_\beta(\mathbb{R}^n)$ can be regarded as a space wider than M_β.

Theorem 3. *Let $\Omega(\subset \mathbb{R}^3)$ be a bounded domain with $\partial\Omega \in C^\infty$, $p \ge 3$ and u be a solution to (N-S) on $(0,T)$ in the class*

$$S_p(0,T) := C([0,T); L^p_\sigma) \cap C^1((0,T); L^p_\sigma) \cap C((0,T); W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)).$$

Assume that $T < \infty$ and T is maximal, i.e., u cannot be continued to the solution of (N-S) in the class $S_p(0,T')$ for any $T' > T$. Then,

$$\int_s^t \|\omega(\tau)\|_{M_1(\Omega)} d\tau \nearrow \infty \quad \text{as} \quad t \nearrow T \quad \text{for any } s \in (0,T).$$

Remark. When $\Omega = \mathbb{R}^3$, under the same assumptions, we have $\int_s^t \|\omega(\tau)\|_{V_1(\mathbb{R}^3)} d\tau \nearrow \infty$ as $t \nearrow T$ for any $s \in (0,T)$.

4 Proof of main theorems

Proof of Theorem 1(1). For the proof of Theorem 1(2), see our paper[10]. Here, we prove only Theorem 1(1). To this end, we use an argument given in Ozawa [12]. See also [5] and [11]. For the sake of simplicity, we assume $n = 3$. Since $\partial\Omega \in C^\infty$, $\partial\Omega$ satisfies the interior cone condition. Namely there are $\delta \in (0,1)$ and $\theta \in (\pi/2, \pi)$ depending only on Ω with the following property: For any point $x \in \Omega$, there exists a spherical sector $C_\delta^\theta(x) = \{x + \xi \in \mathbb{R}^3; 0 < |\xi| \le \delta, \quad -|\xi| \le \kappa(x)\cdot\xi \le |\xi|\cos\theta\}$ having a vertex at x such that $C_\delta^\theta(x) \subset \Omega$, where $\kappa(x)$ is an appropriate unit vector from x. We note that for each $x \in \Omega$, $C_\delta^\theta(x)$ is congruent to $C_\delta^\theta \equiv \{\xi \in \mathbb{R}^3; 0 < |\xi| \le \delta, \quad -|\xi| \le \xi_3 \le |\xi|\cos\theta\}$. In particular, for any boundary point $x \in \partial\Omega$, $C_\delta^\theta(x)$ can be expressed as $C_\delta^\theta(x) \equiv \{x + \xi \in \mathbb{R}^3; 0 < |\xi| \le \delta, \quad -|\xi| \le \xi\cdot\nu(x) \le |\xi|\cos\theta\}$, where $\nu(x)$ denotes the unit outward normal at x.

Let $0 < t \le \delta$ and $C_t^\theta(x) := C_\delta^\theta(x) \cap \overline{B(x,t)}$. For any fixed $x \in \Omega$ and $y \in C_t^\theta(x) \subset \Omega$,

$$|f(x)| \le |f(x) - f(y)| + |f(y)| \le \|f\|_{\dot{C}^\alpha}|x-y|^\alpha + |f(y)| \le \|f\|_{\dot{C}^\alpha} t^\alpha + |f(y)|$$

Integrating both sides of above inequality with respect to y over $C_t^\theta(x)$,

$$\begin{aligned} |f(x)||C_t^\theta(x)| &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + \int_{y\in C_t^\theta(x)} |f(y)|dy \\ &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + \int_{y\in B(x,t)\cap\Omega} |f(y)|dy \\ &\le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)}|C_t^\theta(x)| + |B(x,t)|\log^\beta(\frac{1}{t}+e)\|f\|_{M_\beta(\Omega)} \end{aligned} \tag{4.1}$$

Since $|B(x,t)|/|C_t^\theta(x)| (< \infty)$ is only depending on θ, we have

$$|f(x)| \le t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} + C\log^\beta(\frac{1}{t}+e)\|f\|_{M_\beta(\Omega)} \tag{4.2}$$

for all $0 < t < \delta$.

Then we optimize t by letting $t = (1/\|f\|_{\dot{C}^\alpha(\Omega)})^{1/\alpha}$ if $\|f\|_{\dot{C}^\alpha(\Omega)} \ge \delta^{-\alpha}$ and letting $t = \delta$ if $\|f\|_{\dot{C}^\alpha(\Omega)} \le \delta^{-\alpha}$ to obtain (3.1).

□

Proof of Theorem 2(1). Here, we prove Theorem 2(1). For the proof of Theorem 2(2), see our paper[10]. We first recall the Littlewood-Paley decomposition. Let ψ be the function given in Definition (2) and let $\phi_j \in \mathcal{S}$ be the functions defined by

$$\hat{\phi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi) \text{ and } \hat{\phi}_j(\xi) := \hat{\phi}(\xi/2^j) \text{ for } \xi \in \mathbb{R}^3.$$

Then, supp $\hat{\phi}_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\}$ and

$$(4.3) \qquad 1 = \hat{\psi}(\xi/2^N) + \sum_{j=N+1}^{\infty} \hat{\phi}(\xi/2^j) = \hat{\psi}_N(\xi) + \sum_{j=N+1}^{\infty} \hat{\phi}_j(\xi) \text{ for } \xi \in \mathbb{R}^3, N = 1, 2, \cdots .$$

Using (4.3), we decompose f into two parts such as

$$(4.4) \qquad f(x) = \psi_N * f(x) + \sum_{j=N+1}^{\infty} \phi_j * f(x).$$

We have by Definition (2)

$$(4.5) \qquad \|\psi_N * f\|_\infty \le N^\beta \|f\|_{V_\beta}.$$

Since $\dot{B}^\alpha_{\infty,\infty} = \dot{C}^\alpha$ for $0 < \alpha < 1$, we have

$$(4.6) \qquad \begin{aligned} \sum_{j=N+1}^{\infty} \|\phi_j * f\|_\infty \le& C \sum_{j=N+1}^{\infty} 2^{\alpha j} \|\phi_j * f\|_\infty 2^{-\alpha j} \\ \le& C\|f\|_{\dot{B}^\alpha_{\infty,\infty}} \sum_{j=N+1}^{\infty} 2^{-\alpha j} \le C\|f\|_{\dot{C}^\alpha} 2^{-\alpha N}. \end{aligned}$$

Gathering (4.5) and (4.6) with (4.4), we obtain

$$(4.7) \qquad \|f\|_\infty \le C(2^{-\alpha N}\|f\|_{\dot{C}^\alpha} + N^\beta \|f\|_{V_\beta}).$$

Now we take $N = \left[\frac{\log(\|f\|_{\dot{C}^\alpha} + e)}{\alpha \log 2}\right] + 1$, where $[\cdot]$ denotes Gauss symbol. Then we have the desired estimate (3.2) □

Proof of Theorem 3. For the sake of simplicity, we prove Theorem 3 only in the case $p > 3$. Since $u \in C((0,T); D(A_p))$, without loss of generality, we may assume that $u_0 \in D(A_p)$. Since $P(u \cdot \nabla u) = P(\omega \times u)$, u satisfies the following integral equation:

$$(I.E.) \qquad u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} P(\omega \times u)(s) ds$$

for all $0 < t < T$. Since the local existence time T_* of L^p-strong solutions can be estimated from below as

$$T_* > C(p, \Omega)/\|u_0\|_p^{2p/(p-3)},$$

we observe that if we assume

$$(4.8) \qquad \sup_{0<t<T} \|u(t)\|_p < \infty,$$

then u can be continued to the solution of (N-S) in the class $S_p(0,T')$ for some $T' > T$. Hence, in order to prove Theorem 3, it suffices to show that if

$$\int_0^T \|\omega(t)\|_{M_1} dt < \infty, \tag{4.9}$$

then (4.8) holds. From now on we assume (4.9). By (I.E.) we have

$$\|u(t)\|_p \le C\|u_0\|_p + C\int_0^t \|\omega(s)\|_\infty \|u(s)\|_p ds,$$

which yields

$$\sup_{0<s<t} \|u(s)\|_p \le C\|u_0\|_p \exp\Big(C\int_0^t \|\omega(\tau)\|_\infty d\tau\Big) \tag{4.10}$$

for all $0 < t \le T$. Therefore, in order to show (4.8), it suffices to show

$$\int_0^T \|\omega(\tau)\|_\infty d\tau < \infty. \tag{4.11}$$

Letting $\alpha > 0$ and substituting $f = \frac{\omega(s)}{\epsilon\|\omega(s)\|_{\dot{C}^\alpha}}$ into the Brezis-Gallouet-Wainger type inequality (3.1) with $\beta = 1$, we obtain

$$\begin{aligned} \|\omega(s)\|_\infty \le & C\Big(\epsilon\|\omega(s)\|_{\dot{C}^\alpha} + \log(e + \frac{1}{\epsilon})\|\omega(s)\|_{M_1}\Big) \\ \le & C\Big(\epsilon\|u(s)\|_{C^{1+\alpha}} + \log(e + \frac{1}{\epsilon})\|\omega(s)\|_{M_1}\Big) \end{aligned} \tag{4.12}$$

for all $\epsilon > 0$, where C is a constant independent of s and ϵ. Let $0 < \alpha < 1 - 3/p$ and

$$\begin{aligned} h(t) &:= \sup_{0<\tau<t} \|u(\tau)\|_p, \\ g(t) &:= \int_0^t \|\omega(\tau)\|_\infty d\tau \end{aligned}$$

for $0 < t < T$. Then, from (4.12), for any positive function $\epsilon(s)$ on $(0,T)$ we see that

$$\begin{aligned} g(t) \le & C\int_0^t \|u(s)\|_{C^{1+\alpha}}\epsilon(s) ds + C\int_0^t \log(e + \frac{1}{\epsilon(s)})\|\omega(s)\|_{M_1} ds \\ =: & I_1(t) + I_2(t). \end{aligned} \tag{4.13}$$

Since

$$\|e^{-tA}f\|_{C^{1+\alpha}} \le C(1 + t^{-\frac{1+\alpha}{2} - \frac{3}{2p}})\|f\|_p,$$

from (I.E.) we obtain

$$
\begin{aligned}
\|u(s)\|_{C^{1+\alpha}} \leq & C\|u_0\|_{D(A_p)} + C\int_0^s (1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega\times u(\tau)\|_p d\tau \\
\leq & C\|u_0\|_{D(A_p)} + Ch(s)\int_0^s (1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega(\tau)\|_\infty d\tau.
\end{aligned}
\tag{4.14}
$$

Hence, for $0 < t < T$ we have

$$
\begin{aligned}
I_1(t) \leq & C\|u_0\|_{D(A_p)} T \sup_{0<s<T} \epsilon(s) \\
& + C\int_0^t h(s)\epsilon(s)\int_0^s (1+(s-\tau)^{-\frac{1+\alpha}{2}-\frac{3}{2p}})\|\omega(\tau)\|_\infty d\tau ds
\end{aligned}
\tag{4.15}
$$

We now choose $\epsilon(s)$ such that

$$
\epsilon(s) := \frac{\delta}{Ch(s)+1},
$$

where $\delta > 0$ is a sufficiently small constant. Then, by Fubini's Theorem we have

$$
I_1(t) \leq CT\delta\|u_0\|_{D(A_p)} + C_0(T)\delta\int_0^t \|\omega(\tau)\|_\infty d\tau. \tag{4.16}
$$

Since (4.10) yields

$$
\log(e+\frac{1}{\epsilon(s)}) = \log(e+\frac{Ch(s)+1}{\delta}) \leq C(\delta)\big(\log(e+\|u_0\|_p)+g(s)\big),
$$

we have

$$
I_2(t) \leq C(\delta)\int_0^t \|\omega(s)\|_{M_1}\big(\log(e+\|u_0\|_p)+g(s)\big)ds \tag{4.17}
$$

Gathering (4.16) and (4.17) with (4.13) we obtain

$$
g(t) \leq CT\delta\|u_0\|_{D(A_p)} + C_0(T)\delta g(t) + C(\delta)\int_0^t \|\omega(s)\|_{M_1}\big(\log(e+\|u_0\|_p)+g(s)\big)ds.
$$

Therefore, letting $\delta = 1/(2C_0(T))$, by the Gronwall lemma, we have

$$
g(t) \leq C(\|u_0\|_{D(A_p)}, T, \|u_0\|_p)\exp\Big(C(T)\int_0^T \|\omega(s)\|_{M_1} ds\Big)
$$

for all $0 < t < T$, which implies the desired estimate (4.11).

□

References

[1] Beale, J.T., Kato, T., Majda, A., *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Commun. Math. Phys. , **94** (1984), 61-66.

[2] Brezis, H., Gallouet, T., *Nonlinear Schrödinger evolution equations* , Nonlinear Anal. T.M.A. , **4** (1980), 677-681.

[3] Brezis, H., Wainger, S., *A note on limiting cases of Sobolev embedding and convolution inequalities.*, Comm. Partial Differential Equations , **5** (1987), 773-789.

[4] Chemin, J-Y., *Perfect incompressible Fluids*, Oxford Lecture Series in Mathematics and its applications, 14, Oxford Science Publications, 1998.

[5] Engler, H., *An alternative proof of the Brezis-Wainger inequality*, Comm. Partial Differential equations. , **14** no. 4 (1989), 541-544.

[6] Ferrari, A. B., *On the blow-up of solutions of 3-D Euler equations in a bounded domain*, Commun. Math. Phys. , **155** (1993), 277-294.

[7] Kato, T., Ponce, G., *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. , **41**, (1988) 891-907.

[8] Kozono, H., Ogawa, T., Taniuchi, Y., *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations*, Math. Z. **242** (2002) 251-278.

[9] Kozono, H., Taniuchi, Y., *Limiting case of the Sobolev inequality in BMO, with application to the Euler equations*, Commun. Math. Phys. **214** (2000) 191-200

[10] Nakao, K., Taniuchi, Y., *Brezis-Gallouet-Wainger type inequalities and blow-up criteria for Navier-Stokes equations in unbounded domains*, Preprint

[11] Ogawa, T., Taniuchi, Y., *On blow-up criteria of smooth solutions to the 3-D Euler equations in a bounded domain*, J. Differential. Equation., **190** (2003), 39-63.

[12] Ozawa, T., *On critical cases of Sobolev's inequalities*, J. Funct. Anal., **127** (1995), 259-269.

[13] Shirota, T., Yanagisawa, T., *A continuation principle for the 3-D Euler equations for incompressible fluids in a bounded domain*, Proc. Japan Acad., **69** Ser. A (1993) 77-82.

[14] Vishik, M., *Incompressible flows of an ideal fluid with unbounded vorticity*, Commun. Math. Phys. **213** (2000) 697-731.

[15] Zajaczkowski, W. M., *Remarks on the breakdown of smooth solutions for the 3-d Euler equations in a bounded domain*, Bull. Polish Acad. of Sciences Mathematics **37** (1989), 169-181.