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Density of states and level statistics for 1d Schrödinger operators

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Abstract

We report our work on the fluctuation of IDS and level statistics for the 1d Schrödinger operators with (1) random decaying potential, and (2) decaying coupling constants, some part of which is the joint work with Prof. S. Kotani.

1 Introduction

1d Schrödinger operators with random decaying potential have rich spectral properties depending on the decay rate and have been studied by many researchers(e.g. [12, 8] and references therein). Recently, there are growing interests and discussions on the level statistics problem on these operators, mainly on the context of the random matrix theory. This manuscript is a survey on our works [9, 10, 17, 19] studying the fluctuation of the integrated density of states (IDS) and the level statistics problem for 1d Schrödinger operators with random decaying potentials (Section 2) and random stationary potential with decaying coupling constants (Section 3).

2 Decaying potential model

In this section we consider

$$H := -\frac{d^2}{dt^2} + a(t)F(X_t) \quad \text{on } L^2(\mathbf{R})$$

where $a \in C^\infty(\mathbf{R})$, $a(-s) = a(s)$, non-increasing for $s > 0$, and

$$a(s) = s^{-\alpha}(1 + o(1)), \quad s \to \infty, \quad \alpha > 0.$$

$F \in C^\infty(M)$ for a torus M satisfying

$$\langle F \rangle := \int_M F(x)dx = 0.$$

$(X_t)_{t\in\mathbf{R}}$ is a Brownian motion on M. Since this potential is compact with respect to $-\frac{d^2}{dt^2}$, $\sigma_{ess}(H) = [0, \infty)$ which is [12] (i) $(\alpha > \frac{1}{2})$ absolutely continuous, (ii) $(\alpha < \frac{1}{2})$ pure point, (iii) $(\alpha = \frac{1}{2})$ pure point on $[0, E_c]$ and singular continuous on $[E_c, \infty)$ for some $E_c \geq 0$.

2.1 Fluctuation of IDS

The results in this section are in [19]. Let $H_L := H|_{[0,L]}$ be the local Hamiltonian of H restricted on $[0, L]$ with Dirichlet boundary condition. And let

$$N(E) := \lim_{L\to\infty} \frac{1}{L} \sharp\{ \text{ eigenvalues of } H_L \leq E\}, \quad E > 0$$

be the integrated density of states (IDS) of H. Since the potential vanishes at infinity, we have $N(E) = N_0(E) := \pi^{-1}\sqrt{E}$ which is the IDS of the free Laplacian, so that

$$N_n(\kappa_1, \kappa_2) := \sharp\{ \text{ eigenvalues of } H_n \text{ in } (\kappa_1^2, \kappa_2^2)\}, \quad 0 < \kappa_1 < \kappa_2$$

satisfies

$$N_n(\kappa_1, \kappa_2) = \frac{n}{\pi}(\kappa_2 - \kappa_1)(1 + o(1)), \quad n \to \infty.$$

The purpose of this section is to study the 2nd term asymptotics. This problem is often studied in the context of the random matrix theory(e.g., [6]). In what follows, we state our results which are divided into three parts, viz. $\alpha > \frac{1}{2}$, $\alpha = \frac{1}{2}$, and $\alpha < \frac{1}{2}$.

(1) **Super-critical decay** $(\alpha > \frac{1}{2})$ **:** we need to take suitable subsequences.

Assumption A
A subsequence $\{n_k\}_{k=1}^{\infty}$ satisfies $\lim_{k\to\infty} n_k = \infty$ and

$$\{\kappa_j n_k\}_{\pi} = \gamma_j + o(1), \quad j = 1, 2, \quad k \to \infty$$

where $\gamma_j \in [0, \pi)$, $[x]_\pi := \lfloor x/\pi \rfloor$, and $\{x\}_\pi := x - \lfloor x/\pi \rfloor \cdot \pi$.

Theorem 2.1 ($\alpha > \frac{1}{2}$) *Assume Assumption A. Then we can find random constants* $C(\kappa_1, \kappa_2)$, ϕ_1, ϕ_2, $M_\infty(\kappa_1, \kappa_2)$ *such that*

$$N_{n_k}(\kappa_1, \kappa_2) - \frac{n_k}{\pi}(\kappa_2 - \kappa_1) \overset{a.s.}{\to} \frac{1}{\pi}\Big(C(\kappa_1, \kappa_2) + \phi_1 - \phi_2 + M_\infty(\kappa_1, \kappa_2)\Big).$$

as $k \to \infty$.

Remark 2.1 *(1) Let* $\theta_t(\kappa)$ *be the Prüfer angle defined later. Writing* $\theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa)$, *the following limit exists [12] :* $\tilde{\theta}_\infty(\kappa) := \lim_{t\to\infty} \tilde{\theta}_t(\kappa)$. *Then we have* $\phi_j = \{\tilde{\theta}_\infty(\kappa_j) + \gamma_j\}_\pi$, $j = 1, 2$. *Moreover* $C(\kappa_1, \kappa_2)$ *can be explicitly written down using* $a(s)$, $F(X_s)$, *and* $\theta_s(\kappa)$.
(2) $M_\infty(\kappa_1, \kappa_2)$ *is given by the* $t \to \infty$ *limit of a martingale* M_t *so that it has the same distribution as a time change of a Brownian motion.*

(2) **Critical decay** ($\alpha = \frac{1}{2}$) **:**

Theorem 2.2 ($\alpha = \frac{1}{2}$)
Let $\{G(\kappa)\}_{\kappa>0}$, G *be the mutually independent Gaussian fields with*

$$\begin{aligned}
&Cov\,(G(\kappa), G(\kappa')) = \frac{1}{2}\delta_{\kappa,\kappa'}\langle [g_\kappa, \overline{g}_\kappa]\rangle, \quad \kappa, \kappa' > 0\\
&Cov\,(G, G) = \langle [g, g]\rangle\\
&g_\kappa = (L + 2i\kappa)^{-1}F, \quad g := L^{-1}(F - \langle F\rangle),\\
&[f, g] := \nabla f \cdot \nabla g.
\end{aligned}$$

Then in the sense of weak convergence of processes on $(\kappa_1, \kappa_2) \in (0, \infty)^2$,

$$\left\{N_n(\kappa_1, \kappa_2) - \frac{n}{\pi}(\kappa_2 - \kappa_1) - Re\left(\frac{C_1(\kappa_2)}{2\pi\kappa_2} - \frac{C_1(\kappa_1)}{2\pi\kappa_1}\right)\int_0^n a(s)^2 ds\right\}\frac{1}{\sqrt{\log n}}$$
$$\overset{d}{\to} \frac{1}{2\pi\kappa_2}G(\kappa_2) - \frac{1}{2\pi\kappa_1}G(\kappa_1) - \left(\frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1}\right)G$$

as $n \to \infty$, *where* $C_1(\kappa) := -\frac{i}{2\kappa}\langle F g_\kappa\rangle$.

Remark 2.2 *Killip [6] studies the same problem for CMV matrices. But they do not have constant term of the order of* $\log n$ *due to the rotational invariance of the system.*

(3) **Subcritical - decay** ($\alpha < \frac{1}{2}$)

Theorem 2.3 ($\alpha < \frac{1}{2}$)
Set $D := \min\{d \in \mathbf{N} \,|\, \frac{1}{2\alpha} < d+1\}$. *Let* $\{G_t(\kappa)\}_{t\in[0,1],\,\kappa>0}$, $\{G_t\}_{t\in[0,1]}$ *be the independent Gaussians such that*

$$Cov\,(G_t(\kappa), G_s(\kappa')) = \frac{1}{2}\delta_{\kappa,\kappa'}\frac{\langle[g_\kappa, \overline{g}_\kappa]\rangle}{1-2\alpha}(t\wedge s)^{1-2\alpha}$$
$$Cov\,(G_t, G_s) = \frac{\langle[g,g]\rangle}{1-2\alpha}(t\wedge s)^{1-2\alpha}.$$

Then in the sense of weak convergence of processes on $(\kappa_1, \kappa_2, t) \in (0,\infty)^2 \times [0,\infty)$, *we have*

$$\left\{N_{nt}(\kappa_1,\kappa_2) - \frac{nt}{\pi}(\kappa_2-\kappa_1) - \sum_{j=1}^{D} Re\left(\frac{C_j(\kappa_2)}{2\pi\kappa_2} - \frac{C_j(\kappa_1)}{2\pi\kappa_1}\right)\int_0^{nt} a(s)^{j+1}ds\right\}\frac{1}{n^{\frac{1}{2}-\alpha}}$$
$$\stackrel{d}{\to} \frac{1}{2\pi\kappa_2}G_t(\kappa_2) - \frac{1}{2\pi\kappa_1}G_t(\kappa_1) - \left(\frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1}\right)G_t$$

where $C_j(\kappa)$, $j = 1, 2, \cdots, D$ *are deterministic constants.*

Remark 2.3 *For any fixed* κ_1, κ_2, *RHS above has the same distribution as the linear combination of Brownian motions :*

$$\frac{1}{2\pi\kappa_2}G_t(\kappa_2) - \frac{1}{2\pi\kappa_1}G_t(\kappa_1) - \left(\frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1}\right)G_{0,t}$$
$$\stackrel{d}{=} \frac{1}{2\pi\kappa_2}\sqrt{\frac{1}{2}\frac{\langle[g_{\kappa_2}, \overline{g}_{\kappa_2}]\rangle}{1-2\alpha}}B^{(2)}_{t^{1-2\alpha}} - \frac{1}{2\pi\kappa_1}\sqrt{\frac{1}{2}\frac{\langle[g_{\kappa_1}, \overline{g}_{\kappa_1}]\rangle}{1-2\alpha}}B^{(1)}_{t^{1-2\alpha}}$$
$$-\left(\frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1}\right)\sqrt{\frac{\langle[g,g]\rangle}{1-2\alpha}}B^{(0)}_{t^{1-2\alpha}}.$$

Remark 2.4 *If* $a(s)$ *satisfies* $a(s) = s^{-\alpha}$, $s \geq R$ *for some* $R > 0$, *we obtain the following asymptotic expansion of* $N_{nt}(\kappa_1, \kappa_2)$.

$$N_{nt}(\kappa_1,\kappa_2) \sim \frac{nt}{\pi}(\kappa_2-\kappa_1) + C_2(nt)^{1-2\alpha} + C_3(nt)^{1-3\alpha}$$
$$+\cdots + C_D(nt)^{1-(D+1)\alpha} + n^{\frac{1}{2}-\alpha}(Gaussian)$$

Remark 2.5 *To summarize Theorems 2.1, 2.2, 2.3, 2nd order asymptotics of* $N_{nt}(\kappa_1,\kappa_2)$ *is (i) (supercritical)* $O(1)$, *(ii) (critical)* $O(\log n)$, *(iii) (subcritical)* $O(n^{1-2\alpha})$, *which grows as* α *tends to* 0 *reflecting the fact that IDS for* $\alpha = 0$ *is entirely different from that of free Laplacian.*

To describe the idea of proof, let x_t be the solution to the equation $H_L x_t = \kappa^2 x_t$, $x_0 = 0$ which we write using the Prüfer coordinate :

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = r_t \begin{pmatrix} \sin\theta_t \\ \cos\theta_t \end{pmatrix}, \quad \theta_0 = 0.$$

Then by the Sturm oscillation theorem, $N_{nt}(\kappa_1, \kappa_2)$ can be represented in terms of θ_{nt}. Then it suffices to study the behavior of θ_t as $t \to \infty$ by using the method introduced in [12].

2.2 Level statistics

The results in this section are from [9, 17, 10]. Let $\{E_k(L)\}_{k \ge k_0}$ be the set of positive eigenvalues of H_L. Take $E_0 > 0$ arbitrary as the reference energy. To study the behavior of eigenvalues near E_0, we set the point process

$$\xi_L := \sum_{k \ge k_0} \delta_{L(\sqrt{E_k(L)} - \sqrt{E_0})}.$$

Our problem is to study the behavior of ξ_L as $L \to \infty$. As for the known results, Killip-Stoiciu [11] studied this problem for CMV matrices and showed that $\xi_\infty = \lim_{L\to\infty} \xi_L$ is equal to the clock process ($\alpha > \frac{1}{2}$), Poisson process ($\alpha < \frac{1}{2}$), and the scaling limit of the circular β-ensemble ($\alpha = \frac{1}{2}$). The motivation of our work is to study the analogue of that for H. For the discrete Schrödinger operators, Avila-Last-Simon [2] and Mallik-Dolai [15] showed the convergence to the clock process for $\alpha > \frac{1}{2}$ and Kritchevski-Valkó-Virág [13] showed the convergence to the scaling limit of the Gaussian β - ensembles for $\alpha = \frac{1}{2}$.

(1) **Super-critical decay ($\alpha > \frac{1}{2}$) :**

Theorem 2.4 *Assume Assumption A for a subsequence $\{n_k\}$ and $\gamma_0 \in [0, \pi)$. Then we can find a probability measure μ_{γ_0} on $[0, \pi)$, such that*

$$\lim_{k\to\infty} \mathbf{E}[e^{-\xi_{n_k}(f)}] = \int_0^\pi d\mu_{\gamma_0}(\phi) \exp\left(-\sum_{n \in \mathbf{Z}} f(n\pi - \phi) \right).$$

μ_{γ_0} is the distribution of $\phi = \{\tilde{\theta}_\infty(\kappa_0) + \gamma_0\}_\pi$ which also appeared in Theorem 2.1. ϕ is uniformly distributed on $[0, \pi)$ for $\alpha \le \frac{1}{2}$, while it generically is not for $\alpha > \frac{1}{2}$.

(2) **Critical decay** ($\alpha = \frac{1}{2}$) **:**
We first introduce two β- ensembles in the random matrix theory.
(i) circular β-ensemble : consider the n-points $\theta_1, \theta_2, \cdots, \theta_n$ on $T \simeq (-\pi, \pi]$ according to a probability distribution proportional to $|\triangle(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n})|^\beta$ and let

$$\zeta_\beta^C := \lim_{n\to\infty} \sum_{j=1}^n \delta_{n\theta_j}$$

be the scaling limit of that. Killip-Stoiciu [11] gave the characterization of ζ_β^C as follows.

$$\mathbf{E}[e^{-\zeta_\beta^C(f)}] = \mathbf{E}\left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n\in\mathbf{Z}} f\left(\Psi_1^{-1}(2n\pi + \theta)\right)\right)\right]$$

where $\{\Psi_t(\cdot)\}_{t\geq 0}$ is the increasing function-valued process such that $\{\Psi_t(\lambda)\}_{t>0}$ is the solution to the following SDE.

$$\begin{aligned} d\Psi_t(\lambda) &= \lambda\, dt + \frac{2}{\sqrt{\beta t}} Re\left\{(e^{i\Psi_t(\lambda)} - 1)dZ_t\right\}, \qquad (2.1) \\ \Psi_0(\lambda) &= 0 \end{aligned}$$

where Z_t is the complex Brownian motion.
(ii) Gaussian β-ensemble : this is the ensemble of n points $\lambda_1, \lambda_2, \cdots, \lambda_n$ on $\mathbf{R}$ distributed proportional to $\exp\left(-\frac{\beta}{4}\sum_{k=1}^n \lambda_k^2\right) |\triangle(\lambda_1, \cdots, \lambda_n)|^\beta$. Let

$$\zeta_\beta^G := \lim_{n\to\infty} \sum_{j=1}^n \delta_{\Lambda_j}, \quad \Lambda_j := \sqrt{4n - \mu_n^2}(\lambda_j - \mu_n)$$

be the scaling limit of that. Assuming $n^{1/6}(2\sqrt{n} - |\mu_n|) \to \infty$ (i.e., apart from Tracy-Widom region), Valkó-Virág [20] gave the characterization of ζ_β^G as follows. Let

$$N(\lambda) := \sharp\{\text{ points of } \zeta_\beta^G \text{ in } [0, \lambda]\}$$

be the counting function of ζ_β^G. Then we have

$$N(\lambda) \stackrel{d}{=} \frac{1}{2\pi}\Psi_{1-}(\lambda)$$

where Ψ_t, $t \in [0, 1)$ is the solution to the following SDE.

$$\begin{aligned} d\Psi_t(\lambda) &= \lambda dt + \frac{D(E_0)}{\sqrt{1-t}} Re\left[(e^{i\Psi_t(\lambda)} - 1)dZ_t\right], \qquad (2.2) \\ \Psi_0(\lambda) &= 0. \end{aligned}$$

($\lim_{t\uparrow 1} \Psi_t(\lambda) \in 2\pi\mathbf{Z}$, a.s.) ζ_β^G is called the Sine$_\beta$-process.

Roughly speaking, these two SDE's correspond to solving the same SDE from the opposite side : (2.1) has a singularity at $t = 0$, but the solution Ψ_t^{KS} is continuious for $t > 0$, on the other hand, (2.2) does not have singularity but its solution approaches to an element of $2\pi\mathbf{Z}$. We showed

Theorem 2.5
(1) $\xi_L \stackrel{L\to\infty}{\to} \zeta_\beta^C$, *(2)* $\xi_L \stackrel{L\to\infty}{\to} \zeta_\beta^G$ *with* $\beta = \beta(E_0) = 8\kappa_0^2/C(\kappa_0) = \gamma(E_0)^{-1}$. *where* $\gamma(E)$ *is a "Lyapunov exponent" in the sense that the solution to* $H\psi = E\psi$ *satisfies* $\psi(x) \sim |x|^{-\gamma(E)}$ *for large* $|x|$.

Then we have $\beta(E) < 2$ (resp. $\beta(E) > 2$) for $E < E_c$ (resp. $E > E_c$) and $\beta(E_c) = 2$ (Figure 1).

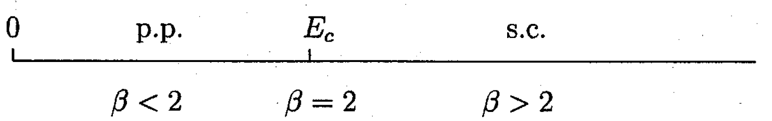

Figure 1: Spectrum and corresponding β.

As for the related works, Dumitriu and Edelman [5] considered the the following random Jacobi matrices

$$\mathcal{H} := \frac{1}{\sqrt{\beta}} \begin{pmatrix} N_0 & \chi_\beta & & \\ \chi_\beta & N_1 & \chi_{2\beta} & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

where $N_j = N(0,2)$, and χ_t is the chi square distribution of freedom t. They found that the eigenvalues of $\mathcal{H}_n := \mathcal{H}|_{\{1,2,\cdots,n\}}$ obey Gaussian β-ensemble. Breuer, Forrester, and Smilansky [3] showed that the spectrum of $\mathcal{H}$ is pure point ($\beta < 2$) and singular continuous ($\beta \geq 2$). Similar result also holds for CMV matrices. This is also consistent with the general belief that the level repulsion is weaker (resp. stronger) on point spectrum (resp. continuous spectrum). Moreover $\beta(E_0)$ is smooth with respect to E_0 and $\lim_{E_0\downarrow 0} \beta(E_0) = 0$, $\lim_{E_0\uparrow\infty} \beta(E_0) = \infty$ so that all β's are realized as E_0 ranges over $(0, \infty)$. Therefore as a corollary, we have

Corollary 2.6 *The limits of C_β-ensemble and G_β-ensemble are equal :*

$$\zeta_\beta^C \stackrel{d}{=} \zeta_\beta^G$$

for all $\beta > 0$.

Remark 2.6 *(1) This fact had previously been known for specific β's, e.g., $\beta = 1, 2, 4$.*
(2) Valkó-Virág have a direct proof of this fact (private communication).

Remark 2.7 *Valkó-Virág [20] showed that Sine$_\beta$-process has a "phase transition" between at $\beta = 2$:*
(i) For $\beta < 2$, $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from below a.s.
(ii) For $\beta > 2$, $\Psi_t(\lambda)$ approaches to $2\pi\mathbf{Z}$ from above with positive probability.

Remark 2.8 *(1) As $\beta \uparrow \infty$, Sine$_\beta \stackrel{d}{\to}$ Clock process (μ uniform on $[0, 2\pi]$) [14, 18]*
(2) Allez - Dumaz [1] showed that as $\beta \downarrow 0$, Sine$_\beta \stackrel{d}{\to}$ Poisson $((2\pi)^{-1}d\lambda)$, where Poisson (μ) is the Poisson process with intensity measure μ.
They are also consistent with the observation in Figure 1.

(3) **Sub-critical decay** $(\alpha < \frac{1}{2})$

Theorem 2.7

$$\xi_L \stackrel{d}{\to} Poisson(\pi^{-1}d\lambda).$$

(4) **Outline of Proof**
Let (r_t, θ_t) be the Prüfer coordinate introduced in Section 2.1 and let

$$\Psi_L(\lambda) := \theta_L(\kappa_0 + \frac{\lambda}{L}) - \theta_L(\kappa_0), \quad \kappa_0 := \sqrt{E_0}$$

be the relative Prüfer phase. Then we have

$$\mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E}\left[\exp\left(-\sum_{n \geq n(L) - m(\kappa_0, L)} f\left(\Psi_L^{-1}(n\pi - \phi(\kappa_0, L))\right)\right)\right]$$

where $m(\kappa_0, L) := [\theta_L(\kappa_0, L)]_\pi$, $\phi(\kappa_0, L) := \{\theta_L(\kappa_0, L)\}_\pi$. Thus our task is to study the joint limit of $(\Psi_L, \phi(\kappa_0, L))$. We replace L by n, and consider

$$\Psi_t^{(n)}(\lambda) := \theta_{nt}(\kappa_\lambda) - \theta_{nt}(\kappa_0) \sim \lambda t + \frac{1}{2\kappa_0} Re \int_0^{nt} a(s) \left(e^{2i\theta_s(\kappa_\lambda)} - e^{2i\theta_s(\kappa_0)} \right) F(X_s) ds$$

where $\kappa_\lambda := \kappa_0 + \frac{\lambda}{n}$, $n > 0$, $t \in [0, 1]$. Here we use "Ito's formula" :

$$e^{2i\kappa s} F(X_s) ds = d(e^{2i\kappa s} g_\kappa(X_s)) - e^{2i\kappa s} \nabla g_\kappa(X_s) dX_s$$

where $g_\kappa := (L + 2i\kappa)^{-1} F$, and L is the generator of (X_t). We then have

$$\Psi_t^{(n)}(\lambda) \sim \lambda t + n^{\frac{1}{2}-\alpha} \frac{1}{2\kappa_0} Re \int_0^t s^{-\alpha} (e^{2i\Psi_s^{(n)}(\lambda)} - 1) \nabla g_\kappa dX_s$$

where $\frac{1}{2}$ (resp. $-\alpha$) in the exponent of n comes from the Brownian scaling (resp. decay rate of the potential). It is then natural to expect
(1) supercritical case $(\alpha > \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \to \lambda t$, a.s.
(2) critical case $(\alpha = \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to} \Psi_t(\lambda)$: solution to SDE,
(3) subcritical case $(\alpha < \frac{1}{2})$: $\Psi_t^{(n)}(\lambda) \stackrel{d}{\to}$ Poisson jump process.
for the proof of (3), we use the idea of Allez - Dumaz[1]. Moreover in the subcritical case (3), we also have $\lim_{n\to\infty} \Psi_t^{(n)}(\lambda) = \pi Poisson_{\mathbf{R}^2}([0, t] \times [0, \lambda])$, with intensity measure $\pi^{-1} 1_{[0,1]}(s) ds d\lambda$. This fact can be regarded as a counterpart of the fact that in the Anderson model, the pair of eigenvalues and eigenfunctions jointly converge to the multi-dimensional Poisson process [7, 16].

3 Decaying coupling constant model

In 1d, the localization length of $H = -\Delta + \lambda V$ is typically $O(\lambda^{-2})$ so that $H_L := H|_{[0,L]}$ is believed to be (i) extended if $L \ll \frac{1}{\lambda^2}$, (ii) localized if $L \gg \frac{1}{\lambda^2}$, (iii) critical if $L \sim \frac{1}{\lambda^2}$. Motivated by this observation, we consider

$$H_L := -\frac{d^2}{dt^2} + \lambda_L F(X_t), \quad \text{on } L^2[0, L], \quad \lambda_L := L^{-\alpha}, \quad \alpha > 0$$

with Dirichlet boundary condition. By the discussion above, we expect that the property of H_L would be different between $\alpha > \frac{1}{2}$ and $\alpha < \frac{1}{2}$. By the method described in Section 2, we can study the fluctuation of IDS and level statistics which we state below. We use the same notation as Section 2.

3.1 Fluctuation of IDS

The results in this section are from [19].

Theorem 3.1 *Let $\alpha > \frac{1}{2}$ and assume Assumption A. Then we have*

$$N_{n_k}(\kappa_1, \kappa_2) - \frac{n_k}{\pi}(\kappa_2 - \kappa_1) \stackrel{a.s.}{\to} \frac{1}{\pi}(\phi_1 - \phi_2).$$

Theorem 3.2 *Let $\alpha = \frac{1}{2}$ and assume Assumption A. Let $\{G(\kappa)\}_{\kappa>0}$, G be the Gaussian fields given in Theorem 2.2. Then we have*

$$\begin{aligned} N_{n_k}(\kappa_1, \kappa_2) &- \frac{n_k}{\pi}(\kappa_2 - \kappa_1) - Re\left(\frac{C_1(\kappa_2)}{2\pi\kappa_2} - \frac{C_1(\kappa_2)}{2\pi\kappa_2}\right) - \frac{1}{\pi}(\phi_2 - \phi_1) \\ &\stackrel{d}{\to} \frac{1}{2\pi\kappa_2}G(\kappa_2) - \frac{1}{2\pi\kappa_1}G(\kappa_1) - \left(\frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1}\right)G. \end{aligned}$$

Theorem 3.3 *Let $D := \min\{d \in \mathbf{N} \,|\, \frac{1}{2\alpha} < d + 1\}$. Let $\{G_t(\kappa)\}_{t\in[0,1],\kappa>0}$, $\{G_t\}_{t\in[0,1]}$ be mutually independent Gaussian fields such that*

$$\begin{aligned} Cov\,(G_t(\kappa), G_s(\kappa')) &= \frac{1}{2}\delta_{\kappa,\kappa'}\langle[g_\kappa, \overline{g}_\kappa]\rangle(t \wedge s)^{1-2\alpha} \\ Cov\,(G_t, G_s) &= \langle[g, g]\rangle(t \wedge s)^{1-2\alpha}. \end{aligned}$$

Then as the processes on $(\kappa_1, \kappa_2, t) \in (0, \infty)^2 \times [0, \infty)$,

$$\begin{aligned} &\left\{N_{nt}(\kappa_1, \kappa_2) - \frac{nt}{\pi}(\kappa_2 - \kappa_1) - \sum_{j=1}^{D} Re\left(\frac{C_j(\kappa_2)}{2\pi\kappa_2} - \frac{C_j(\kappa_1)}{2\pi\kappa_1}\right)(nt)^{1-(j+1)\alpha}\right\}\frac{1}{n^{\frac{1}{2}-\alpha}} \\ &\stackrel{d}{\to} \frac{1}{2\pi\kappa_2}G_t(\kappa_2) - \frac{1}{2\pi\kappa_1}G_t(\kappa_1) - \left(\frac{1}{2\pi\kappa_2} - \frac{1}{2\pi\kappa_1}\right)G_t. \end{aligned}$$

3.2 Level statistics

The results in this sections are from [17, 10].

Theorem 3.4 *Let $\alpha > \frac{1}{2}$, and assume Assumption A. Then we have*

$$\lim_{k\to\infty} \mathbf{E}[e^{-\xi_{n_k}(f)}] = \exp\left(-\sum_{n\in\mathbf{Z}} f(n\pi - \gamma)\right).$$

Theorem 3.5 *Let* $\alpha = \frac{1}{2}$ *and assume Assumption A. Then* $\zeta_\beta^{Sch} := \lim_{k\to\infty} \xi_{n_k}$ *satisfies*

$$\mathbf{E}[e^{-\zeta_\beta^{Sch}(f)}] = \mathbf{E}\left[\exp\left(-\sum_{n\in\mathbf{Z}} f\left(\Psi_1^{-1}(2n\pi - 2\gamma)\right)\right)\right]$$

where $\Psi_t(\lambda)$ *is the solution to*

$$d\Psi_t(\lambda) = (2x + C_0)\,dt + C_1 Re\, e^{i\Psi_t(\lambda)} dZ_t + C_2 dB_t,$$
$$\Psi_0(\lambda) = 0,$$

where Z_t, B_t *are independent. This is similar to* $(Sch)_\tau$ *studied by Krichevski-Valkó-Virág [13].*

Theorem 3.6 *Let* $\alpha < \frac{1}{2}$*. Then* $\xi_L \to Poisson(\pi^{-1}d\lambda)$.

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