数理解析研究所講究録
第2010巻 2016年 116-126

Inequivalent Weyl Representations of Canonical Commutation Relations in an Abstract Bose Field Theory

Asao Arai (新井朝雄)
Department of Mathematics, Hokkaido University
Sapporo, 060-0810, Japan

Abstract

Considered is a family of irreducible Weyl representations of canonical commutation relations with infinite degrees of freedom on the abstract boson Fock space over a complex Hilbert space. Theorems on equivalence or inequivalence of the representations are reported. As a simple application, the well known inequivalence of the time-zero field and conjugate momentum of different masses in a quantum scalar field theory is rederived with space dimension $d \geq 1$ arbitrary. Also a generalization of representations of the time-zero field and conjugate momentum is presented. Comparison is made with a quantum scalar field on a bounded space of $\mathbb{R}^d$. In the case of a bounded space with $d = 1, 2, 3$, the representations of different masses turn out to be mutually equivalent.

Keywords: Boson Fock space, canonical commutation relations, inequivalent representation, quantum field, time-zero field, Weyl representation.
Mathematics Subject Classification 2010: 81R10, 47L60.

1 Introduction

In the canonical formalism of quantum field theory (e.g, [1, Introduction]), a Bose field theory on the $(1+d)$-dimensional space-time $\mathbb{R} \times \mathbb{R}^d$ with $d \in \mathbb{N}$ being the space dimension is constructed from a representation of the canonical commutation relations (CCR) over $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^d)$ (the Schwartz space of real-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$) with the inner product of $L^2(\mathbb{R}^d)$ or a similar real inner product space, giving a time-zero field and its conjugate momentum which are quantum fields on $\mathbb{R}^d$ (for the definition of representation of CCR, see Subsection 2.1). On the other hand, there exist many representations of the CCR over a real inner product Hilbert space which are mutually inequivalent. If the time-zero field and its conjugate momentum in a Bose field theory are inequivalent to those in another Bose field theory, then these two Bose field theories are inequivalent. Therefore it is important to classify representations of the CCR over a real inner product space into mutually equivalent ones and inequivalent ones.

It is well known [6, §X.7] that the time-zero field $\phi_m(f)$ and conjugate momentum $\pi_m(f)$ of a free scalar field on the four-dimensional space-time $\mathbb{R}\times\mathbb{R}^3$ with mass $m>0$ ($f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)$) give an irreducible Weyl representation of the CCR over $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)$ (see Definition 1-(ii) below). Moreover, interestingly enough, the quantum fields of different masses are inequivalent, i.e., if $m_1\neq m_2$ ($m_1,m_2>0$), then there is no unitary operator U such that, for all $f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)$, $U\phi_{m_1}(f)U^{-1}=\phi_{m_2}(f)$ and $U\pi_{m_1}(f)U^{-1}=\pi_{m_2}(f)$ ([6, Theorem X.46]). This fact gives a representation theoretic characterization for boson mass. Namely the set of boson masses can be viewed as an index set of mutually inequivalent irreducible Weyl representations of the CCR over $\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)$. This is an example which shows physical importance of inequivalent representations of CCR.

The proof of the above fact given in [6, Theorem X.46] uses the Euclidean invariance of the operators $\phi_m(f)$ and $\pi_m(f)$. This comes from "the idea that Euclidean invariance is deeply connected with questions of inequivalence of representations of the CCR" [6, p.329]. But, in our intuition, there should be a general structure behind it. Indeed, in the previous paper [3], the author showed that this intuition is true by establishing an abstract theorem on inequivalence of representations of CCR on the abstract boson Fock space and rederiving the above fact as an application of the abstract theorem. This work clarifies a more essential and fundamental reason why the representations $\{\phi_{m_1}(f),\pi_{m_1}(f)|f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)\}$ and $\{\phi_{m_2}(f),\pi_{m_2}(f)|f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)\}$ ($m_1\neq m_2$) are inequivalent. Schematically speaking, the infiniteness of $\mathbb{R}^3$ implies the continuity of the energy spectrum of one free boson, which, in turn, implies the non-Hilbert-Schmidtness of an operator which makes the two representations inequivalent.

In [3], a generalization of the representation $\{\phi_m(f),\pi_m(f)|f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)\}$ also is presented in such a way that the energy function ω_m of a free relativistic boson with mass m is replaced by a general function and the space $\mathbb{R}^3$ is replaced by $\mathbb{R}^d$ with $d\in\mathbb{N}$ arbitrary, and a theorem on equivalence of the representations in the generalized family is proved.

Since infinity in space may give rise to inequivalence of representations $\{\phi_m(f),\pi_m(f)|f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}^3)\}$, also a quantum field on a bounded space of $\mathbb{R}^d$ is discussed in [3] for comparison. In this case with $d=1,2,3$, representations of time-zero fields of different masses are mutually equivalent, in contrast to the case of the infinite space $\mathbb{R}^d$. This may be an interesting phenomenon to note.

The present article is a short summary of some results in [3].

2 Preliminaries

2.1 Representations of the CCR over a real inner product space

We first recall concepts of representation of the CCR over a real inner product space. For a Hilbert space $\mathscr{H}$, we denote its inner product and norm by $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ (linear in the right variable and anti-linear in the left variable if $\mathscr{H}$ is a complex Hilbert space) and $\|\cdot\|_{\mathscr{H}}$ respectively.

Definition 1 Let $\mathscr{F}$ be a complex Hilbert space, $\mathscr{F}_0$ be a dense subspace in $\mathscr{F}$ and $\mathscr{V}$ be a real inner product space. Suppose that, for each $f\in\mathscr{V}$, closed symmetric operators

$q(f)$ and $p(f)$ on $\mathscr{F}$ are given.

(i) The triple $(\mathscr{F}, \mathscr{F}_0, \{q(f), p(f) | f \in \mathscr{V}\})$ is called a *Heisenberg representation of the CCR over* $\mathscr{V}$ if, for all $f \in \mathscr{V}$, $\mathscr{F}_0 \subset D(q(f)) \cap D(p(f))$ and $q(f)$ and $p(f)$ leave $\mathscr{F}_0$ invariant, satisfying the CCR

$$[q(f), p(g)] = i \langle f, g \rangle_{\mathscr{V}}, \quad [q(f), q(g)] = 0, \quad [p(f), p(g)] = 0, \quad f, g \in \mathscr{V}, \tag{1}$$

on $\mathscr{F}_0$.

(ii) Assume that, for each $f \in \mathscr{V}$, $q(f)$ and $p(f)$ are self-adjoint. Then $(\mathscr{F}, \{e^{iq(f)}, e^{ip(f)} | f \in \mathscr{V}\})$ is called a *Weyl representation of the CCR over* $\mathscr{V}$ if the Weyl relations

$$e^{iq(f)} e^{ip(g)} = e^{-i\langle f, g \rangle_{\mathscr{V}}} e^{ip(g)} e^{iq(f)}, \tag{2}$$

$$e^{iq(f)} e^{iq(g)} = e^{iq(g)} e^{iq(f)}, \quad e^{ip(f)} e^{ip(g)} = e^{ip(g)} e^{ip(f)}, \quad f, g \in \mathscr{V}, \tag{3}$$

hold.

The Weyl representation $(\mathscr{F}, \{e^{iq(f)}, e^{ip(f)} | f \in \mathscr{V}\})$ is said to be *irreducible* if there is no non-trivial closed subspace left invariant by all $e^{iq(f)}$ and $e^{ip(f)}$, $f \in \mathscr{V}$ (i.e., if a closed subspace $\mathscr{M}$ of $\mathscr{F}$ satisfies that, for all $f \in \mathscr{V}$, $e^{iq(f)} \mathscr{M} \subset \mathscr{M}$ and $e^{ip(f)} \mathscr{M} \subset \mathscr{M}$, then $\mathscr{M} = \{0\}$ or $\mathscr{F}$).

(iii) Let $\rho := (\mathscr{F}, \mathscr{F}_0, \{q(f), p(f) | f \in \mathscr{V}\})$ and $\rho' = (\mathscr{F}', \mathscr{F}'_0, \{q(f)', p(f)' | f \in \mathscr{V}\})$ be Heisenberg representations of the CCR over $\mathscr{V}$. Then ρ and ρ' are equivalent if there exists a unitary operator $U : \mathscr{F} \to \mathscr{F}'$ such that $Uq(f)U^{-1} = q(f)'$, $Up(f)U^{-1} = p(f)'$ for all $f \in \mathscr{V}$.

(iv) Let $\rho := (\mathscr{F}, \{e^{iq(f)}, e^{ip(f)} | f \in \mathscr{V}\})$ and $\rho' = (\mathscr{F}', \{e^{iq(f)'}, e^{ip(f)'} | f \in \mathscr{V}\})$ be Weyl representations of the CCR over $\mathscr{V}$. Then ρ and ρ' are equivalent if there exists a unitary operator $U : \mathscr{F} \to \mathscr{F}'$ such that $Uq(f)U^{-1} = q(f)'$, $Up(f)U^{-1} = p(f)'$ for all $f \in \mathscr{V}$.

Remark 2 (i) In our definition, the operators forming a Heisenberg representation are not necessarily self-adjoint.

(ii) A Weyl representation $(\mathscr{F}, \{e^{iq(f)}, e^{ip(f)} | f \in \mathscr{V}\})$ is a Heisenberg representation $(\mathscr{F}, \mathscr{F}_0, \{q(f), p(f) | f \in \mathscr{V}\})$ for a suitable $\mathscr{F}_0$. But *the converse is not true.* This situation already occurs in the case where $\mathscr{V}$ is finite dimensional (see [2, Chapter 3] and references therein).

(iii) In the case where $\mathscr{V}$ is finite dimensional, all irreducible Weyl representations of the CCR over $\mathscr{V}$ are mutually equivalent (von Neumann's uniqueness theorem [5]). But, as for Heisenberg representations, von Neumann's uniqueness theorem does not hold in general.

2.2 Boson Fock space and Fock representation of CCR

Let

$$\mathscr{F}_{\mathrm{b}}(\mathscr{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathscr{H}$$

be the *boson Fock space over a complex Hilbert space* $\mathscr{H}$, where $\otimes_{\mathrm{s}}^{n}\mathscr{H}$ denotes the n-fold symmetric tensor product Hilbert space with $\otimes_{\mathrm{s}}^{0}\mathscr{H} := \mathbb{C}$, and $A(f)$ be the *annihilation operator* with test vector $f \in \mathscr{H}$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, i.e., it is a densely defined closed linear operator on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ such that, for all $\Psi \in D(A(f)^*)$, $(A(f)^*\Psi)^{(0)} = 0$ and

$$(A(f)^*\Psi)^{(n)} = \sqrt{n}S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

where S_n is the symmetrization operator on the n-fold tensor product Hilbert space $\otimes^n\mathscr{H}$. The adjoint $A(f)^*$ of $A(f)$ is called the *creation operator* with test vector f.[1]

The subspace

$$\mathscr{F}_0(\mathscr{H}) := \{\Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \mid \Psi^{(n)} \in \otimes_{\mathrm{s}}^{n}\mathscr{H}, n \geq 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \Psi^{(n)} = 0\},$$

called the *finite particle subspace*, is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. It is easy to see that, for all $f \in \mathscr{H}$, $\mathscr{F}_0(\mathscr{H}) \subset D(A(f)) \cap D(A(f)^*)$ and $A(f)$ and $A(f)^*$ leave $\mathscr{F}_0(\mathscr{H})$ invariant, satisfying

$$[A(f), A(g)^*] = \langle f, g\rangle_{\mathscr{H}}, \quad [A(f), A(g)] = 0, \quad [A(f)^*, A(g)^*] = 0 \quad (f, g \in \mathscr{H}) \tag{4}$$

on $\mathscr{F}_0(\mathscr{H})$.

A natural operator constructed from $A(f)$ and $A(f)^*$ is the *Segal field operator*:

$$\Phi(f) := \frac{1}{\sqrt{2}}\overline{(A(f)^* + A(f))}, \quad f \in \mathscr{H},$$

It is shown that $\Phi(f)$ is a self-adjoint operator on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$ and is essentially self-adjoint on $\mathscr{F}_0(\mathscr{H})$. It follows from (4) that, for all $f, g \in \mathscr{H}$,

$$[\Phi(f), \Phi(g)] = i\Im \langle f, g\rangle_{\mathscr{H}} \tag{5}$$

on $\mathscr{F}_0(\mathscr{H})$.

The operator

$$\Pi(f) := \Phi(if), \quad f \in \mathscr{H}$$

is called the *conjugate momentum* of $\Phi(f)$. By (5), we have

$$[\Phi(f), \Pi(g)] = i\Re \langle f, g\rangle_{\mathscr{H}}, \quad f, g \in \mathscr{H}.$$

Let C be a conjugation on $\mathscr{H}$, i.e., C is an anti-linear mapping on $\mathscr{H}$ such that $C^2 = I$ (identity) and $\|Cf\|_{\mathscr{H}} = \|f\|_{\mathscr{H}}$, $f \in \mathscr{H}$. Then the subset

$$\mathscr{H}_C := \{f \in \mathscr{H} | Cf = f\}$$

[1] As a general reference for the theory on boson Fock space, we refer the reader to [1, Chapter 4].

is a real Hilbert space with the inner product of $\mathscr{H}$. It is easy to see that each $f \in \mathscr{H}$ is uniquely written as

$$f = \Re f + i\Im f$$

with

$$\Re f := \frac{f + Cf}{2} \in \mathscr{H}_C, \quad \Im f := \frac{f - Cf}{2i} \in \mathscr{H}_C$$

Let

$$\phi_C(f) := \Phi(f), \quad \pi_C(f) = \Pi(f), \quad f \in \mathscr{H}_C.$$

Then one can show that $(\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \{e^{i\phi_C(f)}, e^{i\pi_C(f)} | f \in \mathscr{H}_C\}$ is an irreducible Weyl representation of the CCR over $\mathscr{H}_C$ [6, Theorem X.43 and Appendix to X.7]. This representation is called the *Fock representation of the CCR over $\mathscr{H}_C$*.

3 A Family of Irreducible Weyl Representations of CCR

Let T be an injective self-adjoint operator on $\mathscr{H}$ (not necessarily bounded) such that

$$CT \subset TC.$$

Then it is easy to see that, for all $f \in D(T)$, $\Re f$ in $\mathscr{H}_C \cap D(T)$ and $\Re(Tf) = T\Re f$. Moreover, $D(T^{\pm 1}) \cap \mathscr{H}_C$ is dense in $\mathscr{H}_C$ and $T^{\pm 1}(D(T^{\pm 1}) \cap \mathscr{H}_C) \subset \mathscr{H}_C$.

We introduce new fields:

$$\Phi_T(f) := \phi_C(T^{-1}f), \quad f \in D(T^{-1}) \cap \mathscr{H}_C,$$
$$\Pi_T(f) := \pi_C(iTf), \quad f \in D(T) \cap \mathscr{H}_C.$$

Let $\mathscr{V}$ be a dense subspace in $\mathscr{H}_C$ and $\mathsf{S}_{\mathscr{V}}(\mathscr{H})$ be the set of injective self-adjoint operators T on $\mathscr{H}$ satisfying the following conditions:

(T.1) $CT \subset TC$

(T.2) $\mathscr{V} \subset D(T) \cap D(T^{-1})$ and $T^{\pm 1}\mathscr{V}$ are dense in $\mathscr{H}_C$.

Theorem 3 $\{e^{i\Phi_T(f)}, e^{i\Pi_T(f)} | f \in \mathscr{V}\}$ *is an irreducible Weyl representation of the CCR over $\mathscr{V}$:*

$$e^{i\Phi_T(f)}e^{i\Pi_T(g)} = e^{-i\langle f,g\rangle_{\mathscr{H}}}e^{i\Pi_T(g)}e^{i\Phi_T(f)},$$
$$e^{i\Phi_T(f)}e^{i\Phi_T(g)} = e^{i\Phi_T(g)}e^{i\Phi_T(f)}, \quad e^{i\Pi_T(f)}e^{i\Pi_T(g)} = e^{i\Pi_T(g)}e^{i\Pi_T(f)}, \quad f, g \in \mathscr{V}.$$

4 Main Theorems

Theorem 4 *Let $T_1, T_2 \in \mathsf{S}_{\mathscr{V}}(\mathscr{H})$ such that the following* (a) *and* (b) *hold:*

(a) $D(T_1^{-1}T_2^2T_1^{-1}) \cap D(T_1T_2^{-2}T_1)$ *and* $D(T_2^{-1}T_1^2T_2^{-1}) \cap D(T_2T_1^{-2}T_2)$ *are dense in $\mathscr{H}$.*

(b) $T_2^{-1}T_1$ *and* $T_2T_1^{-1}$ *are bounded with* $\mathscr{V} \subset D(T_2^{-1}T_1) \cap D(T_2T_1^{-1}) \cap D(T_1^{-1}T_2) \cap D(T_1T_2^{-1})$.

Then $\{e^{i\Phi_{T_1}(f)}, e^{i\Pi_{T_1}(f)} | f \in \mathscr{V}\}$ *is equivalent to* $\{e^{i\Phi_{T_2}(f)}, e^{i\Pi_{T_2}(f)} | f \in \mathscr{V}\}$ *if and only if* $T_2^{-1}T_1 - T_2T_1^{-1}$ *is Hilbert-Schmidt.*

Remark 5 The conditions for T_1 and T_2 in Theorem 4 are related to an equivalence relation in a subset of $\mathsf{S}_{\mathscr{V}}(\mathscr{H})$. Let

$$\mathsf{S}_{\mathscr{V}}(\mathscr{H})^{\times} := \{T \in \mathsf{S}_{\mathscr{V}}(\mathscr{H}) | T \text{ is surjective}\}.$$

Then, for all $T \in \mathsf{S}_{\mathscr{V}}(\mathscr{H})^{\times}$, $T^{-1} \in \mathfrak{B}(\mathscr{H})$. For $T_1, T_2 \in \mathsf{S}_{\mathscr{V}}(\mathscr{H})^{\times}$, we write $T_1 \sim T_2$ if $T_2T_1^{-1}, T_1T_2^{-1} \in \mathfrak{B}(\mathscr{H})$ and $T_2^{-1}T_1 - T_2T_1^{-1}$ is Hilbert-Schmidt. It is easy to see that the relation $\sim$ is an equivalent relation in $\mathsf{S}_{\mathscr{V}}(\mathscr{H})^{\times}$.

Let

$$\rho_T := \{e^{i\Phi_T(f)}, e^{i\Pi_T(f)} | f \in \mathscr{V}\}.$$

and $T_1, T_2 \in \mathsf{S}_{\mathscr{V}}(\mathscr{H})^{\times}$. Then ρ_{T_1} is equivalent to ρ_{T_2} if and only if $T_1 \sim T_2$ and condition (a) holds.

In the case where at least one of $T_2^{-1}T_1$ and $T_2T_1^{-1}$ is unbounded, the proof of Theorem 4 is not valid any more. In this case, we need a separate consideration. To state a theorem in such a case, we need a lemma.

Lemma 6 *For all* $T_1, T_2 \in \mathsf{S}_{\mathscr{V}}(\mathscr{H})$,

$$T_+ := T_2^{-1}T_1 + T_2T_1^{-1} \tag{6}$$

is injective.

Let

$$T_- := T_2^{-1}T_1 - T_2T_1^{-1}. \tag{7}$$

Theorem 7 *Let* T_1, $T_2 \in \mathsf{S}_{\mathscr{V}}(\mathscr{H})$ *such that the following* (a) –(c) *hold:*

(a) $(T_1\mathscr{V}) \cap (T_1^{-1}\mathscr{V})$ *is dense in* $\mathscr{H}_{\mathbb{C}}$.

(b) $\{T_+f | f \in (T_1\mathscr{V}) \cap (T_1^{-1}\mathscr{V})\}$ *is dense in* $\mathscr{H}_{\mathbb{C}}$.

(c) $T_-T_+^{-1}$ *is bounded and its closure* $\overline{T_-T_+^{-1}}$ *is not Hilbert-Schmidt.*

Then ρ_{T_1} *is inequivalent to* ρ_{T_2}.

Remark 8 In Theorem 7, the $T_2^{-1}T_1$ and $T_2T_1^{-1}$ are not necessarily bounded.

5 Application—Inequivalence of Time-Zero Fields and Conjugate Momenta of Different Masses in Any Space Dimension

We denote by $\mathbb{R}^d_x = \{x = (x_1, \ldots, x_d) | x_j \in \mathbb{R}, j = 1, \ldots, d\}$ the d-dimensional position space and by $\mathbb{R}^d_k = \{k = (k_1, \ldots, k_d) | k_j \in \mathbb{R}, j = 1, \ldots, d\}$ the d-dimensional momentum space[2]. Let $\mathscr{F}_d : L^2(\mathbb{R}^d_x) \to L^2(\mathbb{R}^d_k)$ be the Fourier transform:

$$(\mathscr{F}_d f)(k) := \hat{f}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d_x} e^{-ik\cdot x} f(x) dx, \quad f \in L^2(\mathbb{R}^d_x)$$

in the L^2-sense, where $kx := \sum_{j=1}^d k_j x_j$. We consider the boson Fock space $\mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d_k))$ over $L^2(\mathbb{R}^d_k)$ and we denote the annihilation operator on this Fock space by $a(f)$ ($f \in L^2(\mathbb{R}^d_k)$). Then the time-zero field $\phi_m(f)$ ($f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d_x)$) and its conjugate momentum $\pi_m(f)$ for the standard neutral scalar field with mass $m \geq 0$ is defined by

$$\phi_m(f) := \frac{1}{\sqrt{2}} \overline{\left(a(\omega_m^{-1/2} \hat{f})^* + a(\omega_m^{-1/2} \hat{f}) \right)},$$
$$\pi_m(f) := \frac{i}{\sqrt{2}} \overline{\left(a(\omega_m^{1/2} \hat{f})^* - a(\omega_m^{1/2} \hat{f}) \right)},$$

where $\omega_m(k) := \sqrt{k^2 + m^2}$, $k \in \mathbb{R}^d_k$.

Let

$$\mathscr{V}_{d,m} := \begin{cases} \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d_x) & \text{for } m > 0 \\ \{f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d_x) | \operatorname{supp} \hat{f} \subset \mathbb{R}^d \setminus \{0\}\} & \text{for } m = 0 \end{cases}.$$

Then

$$\tau_m := \{e^{i\phi_m(f)}, e^{i\pi_m(f)} | f \in \mathscr{V}_{d,m}\} \tag{8}$$

is an irreducible Weyl representation of the CCR over $\mathscr{V}_{d,m}$.

Theorem 9 *Let $m_1, m_2 \geq 0$ with $m_1 \neq m_2$. Then τ_{m_1} is inequivalent to τ_{m_2}.*

Proof (outline). We denote by T_m the multiplication operator on $L^2(\mathbb{R}^d_k)$ by the function $\omega_m^{-1/2}$. We define a mapping $C : L^2(\mathbb{R}^d_k) \to L^2(\mathbb{R}^d_k)$ by

$$(Cu)(k) := u(-k)^*, \quad u \in L^2(\mathbb{R}^d_k(, \text{ a.e.} k \in \mathbb{R}^d_k.$$

Then it is easy to see that C is a conjugation and $CT_m \subset T_m C$. Morever,

$$\mathscr{F}_d \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d_x) = \{\hat{f} | f \in \mathscr{S}_{\mathbb{R}}(\mathbb{R}^d_x), C\hat{f} = \hat{f}\}.$$

We note that

$$\phi_m(f) = \Phi_{T_m}(\hat{f}), \quad \pi_m(f) = \Pi_{T_m}(\hat{f}),$$

[2] We work with the physical unit system where the light speed c and the Planck constant $\hbar$ divided by 2π are equal to 1.

where $\Phi(\cdot)$ and $\Pi(\cdot)$ are resepectively the Segal field operator and its conjugate momentum on $\mathscr{F}_{\rm b}(L^2(\mathbb{R}^d_k))$. Hence it follows that τ_{m_1} is equivalent to τ_{m_2} if and only if $\rho_{T_{m_1}}$ is equivalent to $\rho_{T_{m_2}}$.

Let $m_1, m_2 > 0$. Then

$$T_{m_1}T_{m_2}^{-1} = \frac{\sqrt{\omega_{m_2}}}{\sqrt{\omega_{m_1}}}$$

is bounded and

$$T_{m_1}T_{m_2}^{-1} - T_{m_2}T_{m_1}^{-1} = \frac{m_2^2 - m_1^2}{\sqrt{\omega_{m_1}\omega_{m_2}}(\omega_{m_2} + \omega_{m_1})}.$$

This is Hilbert-Schmidt if and only if $m_1 = m_2$. Hence, if $m_1 \neq m_2$, then, by Theorem 4, $\rho_{T_{m_1}}$ is inequivalent to $\rho_{T_{m_2}}$. The case where one of m_1 and m_2 is equal to zero can be similarly treated. ∎

Remark 10 The method of proof of Theorem 9 presented here is different from that used in the proof of [6, Theorem X.46] (Theorem 9 with $d = 3$) and simpler.

Remark 11 As is seen, the non-Hilbert-Schmidt property of $T_{m_1}T_{m_2}^{-1} - T_{m_2}T_{m_1}^{-1}$ in the case $m_1 \neq m_2$ comes from that the spectrum of the one-particle Hamiltonian ω_m is continuous. The continuity of the spectrum of ω_m is due to that the one-particle momentum operator is purely (absolutely) continuous. On the other hand, the continuity of the spectrum of the momentum operator comes from that the position space is $\mathbb{R}^d$. Thus the inequivalence between τ_{m_1} and τ_{m_2} comes from that the position space in which bosons exist is $\mathbb{R}^d$.

6 A General Family of Inequivalent Representations of CCR on $\mathscr{F}_{\rm b}(L^2(\mathbb{R}^d_k))$

As an application of Theorems 4 and 7, one can construct a general family of inequivalent representations of CCR on $\mathscr{F}_{\rm b}(L^2(\mathbb{R}^d_k))$ including $\{\tau_m | m \geq 0\}$.

Let $v : \mathbb{R}^d \to \mathbb{R}$ such that

$$v(k) = v(-k), \quad 0 < |v(k)| < \infty, \quad \text{a.e.} k \in \mathbb{R}^d_k,$$

and $\nabla := (-iD_1, \dots, -iD_d)$, where D_j is the generalized partial differential operator in x_j. The operator

$$v(-i\nabla) := \mathscr{F}_d^{-1} v \mathscr{F}_d$$

acting in $L^2(\mathbb{R}^d_x)$ is self-adjoint, injective and

$$C_d v(-i\nabla) \subset v(-i\nabla) C_d,$$

where $C_d f := f^*$, $f \in L^2(\mathbb{R}^d_x)$.

Let $\mathscr{D}_d$ be a dense subspace in $L^2_{\mathbb{R}}(\mathbb{R}^d_x) := \{f \in L^2(\mathbb{R}^d_x) | C_d f = f\}$ satisfying the following conditions:

(i) $\mathscr{D}_d \subset D(v(-i\nabla)) \cap D(v(-i\nabla)^{-1})$.

(ii) $v(-i\nabla)\mathscr{D}_d$ and $v(-i\nabla)^{-1}\mathscr{D}_d$ are dense in $L^2_{\mathbb{R}}(\mathbb{R}^d_x)$.

We introduce operators $\phi_v(f)$ and $\pi_v(f)$ $(f \in \mathscr{D}_d)$ as follows:

$$\phi_v(f) := \Phi(v(-iD)^{-1}f), \quad \pi_v(f) := \Pi(v(-iD)f), \quad f \in \mathscr{D}_d, \tag{9}$$

where $\Phi(\cdot)$ and $\Pi(\cdot)$ are respectively the Segal field operator and its conjugate momentum on $\mathscr{F}_{\mathrm{b}}(L^2(\mathbb{R}^d_x))$.

Lemma 12 *$\{e^{i\phi_v(f)}, e^{i\pi_v(f)} | f \in \mathscr{D}_d\}$ is an irreducible Weyl representation of the CCR over $\mathscr{D}_d$.*

Lemma 13 *Let v_1 and v_2 be functions on $\mathbb{R}^d$ having the same properties as those of v described above. Suppose that v_1/v_2 and v_2/v_1 are essentially bounded. Then*

$$W := \overline{(v_2(-i\nabla)^{-1}v_1(-i\nabla) - v_2(-i\nabla)v_1(-i\nabla)^{-1})}.$$

is bounded. Moreover, W is Hilbert-Schmidt if and only if $v_1 = v_2$.

Theorem 14 *Let v_1 and v_2 be functions having the same properties as those of v described above. Suppose that v_1/v_2 and v_2/v_1 are essentially bounded. Then $\{e^{i\phi_{v_1}(f)}, e^{i\pi_{v_1}(f)} | f \in \mathscr{D}_d\}$ and $\{e^{i\phi_{v_2}(f)}, e^{i\pi_{v_2}(f)} | f \in \mathscr{D}_d\}$ are inequivalent if and only if $v_1 \neq v_2$.*

In the case where v_1/v_2 and v_2/v_1 are not necessarily essentially bounded, we have the following theorem.

Theorem 15 *Let v_1 and v_2 be functions having the same properties as those of v described above and $v_1 \neq v_2$. Let*

$$\mathscr{D}_{d,v_1} := (v_1(-i\nabla)\mathscr{D}_d) \cap (v_1(-i\nabla)^{-1}\mathscr{D}_d)$$

and

$$T_{d,\pm} := v_2(-i\nabla)^{-1}v_1(-i\nabla) \pm v_2(-i\nabla)v_1(-i\nabla)^{-1}.$$

Suppose that the following (a) *and* (b) *hold:*

(a) *$\mathscr{D}_{d,v_1}$ is dense in $L^2_{\mathbb{R}}(\mathbb{R}^d_x)$.*

(b) *$T_{d,+}\mathscr{D}_{d,v_1}$ is dense in $L^2_{\mathbb{R}}(\mathbb{R}^d_x)$.*

Then $\{e^{i\phi_{v_1}(f)}, e^{i\pi_{v_1}(f)} | f \in \mathscr{D}_d\}$ is inequivalent to $\{e^{i\phi_{v_2}(f)}, e^{i\pi_{v_2}(f)} | f \in \mathscr{D}_d\}$

7 Quantum Fields on a Bounded Space

In view of Remark 11, it may be interesting to consider quantum fields on bounded space in $\mathbb{R}_x^d$.

Let M be a bounded connected open set in $\mathbb{R}_x^d$ and $\Delta_0 := \sum_{j=1}^d \partial^2/\partial x_j^2$ with domain $D(\Delta_0) := C_0^\infty(M)$, acting in $L^2(M)$. Let $\Delta^{(M)}$ be any self-adjoint extension of Δ_0 such that

(i) $\Delta^{(M)} \leq 0$;

(ii) $C_M \Delta^{(M)} \subset \Delta^{(M)} C_M$, where C_M is the complex conjugation on $L^2(M)$;

(iii) The spectrum of $-\Delta^{(M)}$ is purely discrete. The eigenvalues of $-\Delta^{(M)}$ are labeled as $\{\lambda_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\Gamma}$ with $\Gamma = \mathbb{N}^d$ or $(\{0\}\cup\mathbb{N})^d$, counting multiplicities, and, for some constants $c_1, c_2 > 0$ with $c_1 < c_2$,

$$c_1|\boldsymbol{n}|^2 \leq \lambda_{\boldsymbol{n}} \leq c_2|\boldsymbol{n}|^2, \quad \boldsymbol{n} \in \Gamma. \tag{10}$$

Example 16

(i) $\Delta^{(M)} = \Delta_{\mathrm{D}}$ (the Dirichlet Laplacian in M)

(ii) $\Delta^{(M)} := \Delta_{\mathrm{N}}$ (the Neumann Laplacian in M)

(iii) In the case where $M = (-L_1/2, L_1/2) \times \cdots \times (-L_d/2, L_d/2)$, $\Delta^{(M)} = \Delta_{\mathrm{P}}$ (the Laplacian with the periodic boundary condition).

The one-particle Hamiltonian with mass $m > 0$ in the present context is given by

$$h_m^M := (-\Delta^{(M)} + m^2)^{1/2}$$

acting in $L^2(M)$. This is a strictly positive self-adjoint operator with $h_m^M \geq m > 0$. It follows from the assumption of $\Delta^{(M)}$ that there exists a CONS $\{f_{\boldsymbol{n}} | \boldsymbol{n} \in \Gamma\}$ of $L^2(M)$ such that

$$-\Delta^{(M)} f_{\boldsymbol{n}} = \lambda_{\boldsymbol{n}} f_{\boldsymbol{n}}, \quad \boldsymbol{n} \in \Gamma,$$

and each $f_{\boldsymbol{n}}$ is a real-valued function. Let $\mathscr{V}_M$ be the real subspace algebraically spanned by $\{f_{\boldsymbol{n}} | \boldsymbol{n} \in \Gamma\}$. Then $\mathscr{V}_M$ is dense in the real Hilbert space $L^2_{\mathbb{R}}(M)$. For all $\alpha > 0$,

$$\mathscr{V}^M \subset D((h_m^M)^\alpha), \quad (h_m^M)^{\pm\alpha}\mathscr{V}^M = \mathscr{V}^M.$$

Hence conditions (T.1) and (T.2) with $T = (h_m^M)^{1/2}$ and $\mathscr{V} = \mathscr{V}_M$ are satisfied.

Let $\Phi^M(\cdot)$ be the Segal field operator on $\mathscr{F}_{\mathrm{b}}(L^2(M))$ and

$$\phi_m^M(f) := \Phi^M((h_m^M)^{-1/2} f), \quad \pi_m^M(f) := \Phi^M(i(h_m^M)^{1/2} f), \quad f \in \mathscr{V}^M.$$

Then

$$\rho_m^M := \{e^{i\phi_m^M(f)}, e^{i\pi_m^M(f)} | f \in \mathscr{V}^M\}$$

is an irreducible Weyl representation of the CCR over $\mathscr{V}^M$. One can prove the following theorems:

Theorem 17 *Let $m_1, m_2 > 0$ and $m_1 \neq m_2$. Then $\rho^M_{m_1}$ and $\rho^M_{m_2}$ are equivalent if and only if $d \leq 3$.*

Theorem 18 *Let $m > 0$ and $0 \notin \sigma(\Delta^{(M)})$. Then ρ^M_m is equivalent to ρ^M_0 if and only if $d \leq 3$.*

These theorems show that, in the case $d = 1, 2, 3$, the infiniteness of the space on which quantum fields exist is crucial for the inequivalence of time-zero fields and conjugate momenta of different masses.

Remark 19 Considerations similar to those given in the present paper can be done for quantum Dirac fields of different masses which are representations of the canonical anti-commutation relations over a complex Hilbert space. See [4] for details.

Acknowledgement

This work is supported by the Grant-In-Aid No.15K04888 for Scientific Research from Japan Society for the Promotion of Science (JSPS).

References

[1] A. Arai, *Fock Spaces and Quantum Fields I, II* (in Japanese), Nippon-hyoronsha, Tokyo, 2000.

[2] A. Arai, *Mathematical Principles of Quantum Phenomena*, Asakura-shoten, 2006.

[3] A. Arai, A Family of Inequivalent Weyl Representations of Canonical Commutation Relations with Applications to Quantum Field Theory, Hokkaido University Preprint Series # 1085, 2015.

[4] A. Arai, Inequivalence of Quantum Dirac Fields of Different Masses and General Structures Behind, Hokkaido University Preprint Series # 1086, 2015.

[5] C. R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer, Berlin, Heidelberg, 1967.

[6] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.