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<td>Kim, Jong Kyu; Salahuddin</td>
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Kyoto University
Iterative Algorithms for a System of Random Nonlinear Equations in Hilbert Spaces with Fuzzy Mappings

Jong Kyu Kim¹ and Salahuddin²

¹Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr
²Department of Mathematics, Jazan University,
Jazan, Kingdom of Saudi Arabia
e-mail: salahuddin12@mailcity.com

Abstract

The purpose of this paper, by using the resolvent operator technique associated with randomly \((A, \eta, m)\)-monotone operator, is to establish an existence and convergence theorem for a class of system of random nonlinear equations with fuzzy mappings in Hilbert spaces. Our works are improvements and generalizations of the corresponding well-known results.

Keywords: system of random nonlinear equations, relaxed cocoercive operators, randomly \((A, \eta, m)\)-proximal operator equations, fuzzy mappings, Hilbert spaces.

AMS Mathematics Subject Classification: 49J40, 47H06.

1 Introduction

The fuzzy sets theory is an extension of a crisp set by enlarging the truth valued set \(\{0, 1\}\) to the real unit interval \([0, 1] ([27])\). A fuzzy set is characterized and identified by a mapping called a membership grade function from the whole set into \([0, 1]\). Heilpern [13] introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mapping
which is a fuzzy analogue of Nadler's fixed point theorem for multi-valued mappings.

In 1989, Chang and Zhu [6] first introduced and studied a class of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities, quasi variational inequalities and complementarity problems with fuzzy mappings were considered by Agarwal et al. [1], Chang and Huang [8], Ding et al. [9], Huang [10], Lee et al. [21], Salahuddin [25] in the setting of Hilbert spaces and Banach spaces.

Lan [20] introduced a new concepts of \((A, \eta)\)-monotone operator which generalizes the \((H, \eta)\)-monotonicity and \(A\)-monotonicity in Hilbert spaces and studied some properties of \((A, \eta)\)-monotone operators and applied resolvent operators associated with \((A, \eta)\)-monotone operators to approximate the solution of a new class of nonlinear \((A, \eta)\)-monotone operator inclusion problems with relaxed cocoercive operators in Hilbert spaces. Recently Kim et al. [16] introduced the \((A, \eta, m)\)-proximal operator to study the system of equations in Hilbert spaces.

Recently some systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their close relation to Nash equilibrium problems. Huang and Fang [11] introduced a system of order complementarity problems and established some existence results for these using fixed point theory. Kim and Kim [18] introduced and studied some system of variational inequalities and developed some iterative algorithms for approximately the solutions of system of variational inequalities.

On the other hand, random variational inequality problems, random quasi variational inequality problems and random variational inclusions and complementarity problems have been studied by Chang [5], Chang and Huang [7], Huang [10], Khan and Salahuddin [15] and Bharucha-Red [3], etc.

The concepts of random fuzzy mapping was first introduced by Huang [10]. Subsequently the random variational inclusion problems for random fuzzy mappings is studied by Anastassiou et al. [2], Salahuddin [25], Zhang and Bi [28].

Inspired and motivated by the works [2, 12, 14, 17, 23, 26], we establish the existence and convergence theorem for system of random nonlinear equations with fuzzy mapping in Hilbert spaces by using random \((A, \eta, m)\)-proximal operator equations.

2 Preliminaries

Throughout this paper, \((\Omega, \Sigma)\) is a measurable space with a set \(\Omega\) and a \(\sigma\)-algebra \(\Sigma\) of a subset of \(\Omega\), \(H\) is a real separable Hilbert space endowed with a norm \(\| \cdot \|\) and inner product
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\langle \cdot, \cdot \rangle. Notations \( B(H), 2^H \) and \( CB(H) \) denote the class of Borel \( \sigma \)-fields in \( H \), the family of all nonempty subsets of \( H \) and the family of all nonempty closed bounded subset of \( H \), respectively.

**Definition 2.1.** A mapping \( u : \Omega \rightarrow H \) is said to be measurable if for any \( B \in B(H) \), \( u^{-1} = \{ t \in \Omega, u(t) \in B \in \Sigma \} \).

**Definition 2.2.** A mapping \( f : \Omega \times H \rightarrow H \) is called a random mapping if for each fixed \( u \in H \), a mapping \( f(\cdot, u) : \Omega \rightarrow H \) is measurable. A random mapping \( f \) is said to be continuous if for each fixed \( t \in \Omega \), a mapping \( f(t, \cdot) : H \rightarrow H \) is continuous.

**Definition 2.3.** A multi-valued mapping \( T : \Omega \rightarrow 2^H \) is said to be measurable if for any \( B \in B(H) \), \( T^{-1}(B) = \{ t \in \Omega : T(t) \cap B \neq \emptyset \} \in \Sigma \).

**Definition 2.4.** A mapping \( u : \Omega \rightarrow H \) is called a measurable selection of a measurable multi-valued mapping \( T : \Omega \rightarrow 2^H \), if \( u \) is measurable and for any \( t \in \Omega \), \( u(t) \in T_{t}(u(t)) \).

**Definition 2.5.** A mapping \( T : \Omega \times H \rightarrow 2^H \) is called a random multi-valued mapping if for each fixed \( x \in H \), \( T(x) \) is a measurable multi-valued mapping. A random multi-valued mapping \( T : \Omega \times H \rightarrow CB(H) \) is said to be \( \mathfrak{D} \)-continuous if for each fixed \( t \in \Omega \), \( T(t, \cdot) : \Omega \times H \rightarrow 2^H \) is randomly continuous with respect to the Hausdorff metric \( \mathfrak{D} \).

**Definition 2.6.** A multi-valued mapping \( T : \Omega \times H \rightarrow 2^H \) is called random if for any \( x \in H, T(\cdot, x) \) is measurable (denoted by \( T_{t,x} \) or \( T_t \)).

Let \( \Omega \) be a set and \( \mathfrak{F}(H) \) be a collection of fuzzy sets over \( H \). A mapping \( \tilde{F} : \Omega \times H \rightarrow \mathfrak{F}(H) \) is called a fuzzy mapping on \( H \). If \( \tilde{F} \) is a fuzzy mapping on \( H \) then for any \( t \in \Omega \), \( \tilde{F}(t) \) (denote it by \( \tilde{F}_t \) in the sequel) is a fuzzy mapping on \( H \) and \( \tilde{F}_t(x) \) is the membership grade of \( x \) in \( \tilde{F}_t \). Let \( A \in \mathfrak{F}(H), \alpha \in (0,1] \). Then the set

\[
A_{\alpha} = \{ x \in H : A(x) \geq \alpha \}
\]

is called an \( \alpha \)-cut of \( A \).

**Definition 2.7.** A fuzzy mapping \( \tilde{F} : \Omega \times H \rightarrow \mathfrak{F}(H) \) is said to be measurable, if for any \( \alpha \in (0,1], (\tilde{F}(\cdot))_{\alpha} : \Omega \rightarrow 2^H \) is a measurable multi-valued mapping.

**Definition 2.8.** A fuzzy mapping \( \tilde{F} : \Omega \times H \rightarrow \mathfrak{F}(H) \) is a random fuzzy mapping if for any \( x \in H, \tilde{F}(\cdot, x) : \Omega \times H \rightarrow \mathfrak{F}(H) \) is a measurable fuzzy mapping (denoted by \( \tilde{F}_{t,x} \) short down \( \tilde{F}_t(x) \)).
Let $\tilde{T} : \Omega \times H \rightarrow \mathfrak{F}(H)$ be a random fuzzy mapping satisfying the following condition: 

\((*) : \) there exists a function $\alpha : H \rightarrow (0, 1]$ such that for all $(t, x) \in \Omega \times H$, we have $(\tilde{T}_t x(t))_{\alpha(x(t))} \in CB(H)$, where $T_{tx}$ denotes the value of $T$ at $(t, x)$. Induced multi-valued random mapping $\tilde{T}_t$ from $T$ as follows:

$$
T : \Omega \times H \rightarrow CB(H), T_t = \tilde{T}(t, x(t))_{\alpha(x(t))}, (t, x) \rightarrow T_{tx}, \forall (t, x) \in \Omega \times H.
$$

In this paper we consider the following random $(A_t, \eta_t, m_t)$-proximal operator equation system with fuzzy mappings, we consider for each fixed $t \in \Omega$ finding $(x(t), y(t)), (z(t), w(t)) \in H_1 \times H_2, u(t) \in T_t(x(t))$ and

$$
E_t((t, y(t)) + \rho^{-1} R^{M_{t}(x(t))}_{\rho, A_{1,t}}(z(t))) = 0,
$$

$$
G_t((u(t), y(t)) + \rho^{-1} R^{N_{t}(y(t))}_{\rho, A_{2,t}}(w(t))) = 0
$$

where $T : H_1 \times \Omega \rightarrow \mathfrak{F}(H_1)$ is a fuzzy mapping, $E : H_1 \times H_2 \times \Omega \rightarrow H_1, G : H_1 \times H_2 \times \Omega \rightarrow H_2, \eta : H_1 \times H_2 \times \Omega \rightarrow H_1 \times H_2 \times \Omega \rightarrow H_2$ and $\eta : H_2 \times H_2 \times \Omega \rightarrow H_2$ are nonlinear random single-valued mappings, $A_1 : H_1 \times \Omega \rightarrow H_1, A_2 : H_2 \times \Omega \rightarrow H_2, M : H_1 \times H_1 \times \Omega \rightarrow 2^{H_1}$ and $N : H_2 \times H_2 \times \Omega \rightarrow 2H_2$ are any nonlinear operators such that for all $(z(t), t) \in H_1 \times \Omega, M_t(\cdot, z(t)) : H_1 \rightarrow H_1$ is a randomly $(A_{1,t}, \eta_{1,t}, m_{1,t})$-monotone operator with $f_t(H_1) \cap dom(M_t(\cdot, z(t))) \neq \emptyset$ and for all $(w, t) \in H_2 \times \Omega, N_t(\cdot, w(t)) : H_2 \rightarrow 2H_2$ is a randomly $(A_{2,t}, \eta_{2,t}, m_{2,t})$-monotone operator with $g_t(H_2) \cap dom(N_t(\cdot, w(t))) \neq \emptyset, R^{M_{t}(x(t))}_{\rho, A_{1,t}} = I - A_{1,t} J^{M_{t}(x(t))}_{\rho, A_{1,t}}, R^{N_{t}(y(t))}_{\rho, A_{2,t}} = I - A_{2,t} J^{N_{t}(y(t))}_{\rho, A_{2,t}}, I$ is the identity mapping, $A_{1,t}(J^{M_{t}(x(t))}_{\rho, A_{1,t}}(z(t))) = A_{1,t} J^{M_{t}(x(t))}_{\rho, A_{1,t}}(z(t))$, $A_{2,t}(J^{N_{t}(y(t))}_{\rho, A_{2,t}}(w(t))) = A_{2,t} J^{N_{t}(y(t))}_{\rho, A_{2,t}}(w(t))$, for all $(x(t), z(t)) \in H_1, (y(t), w(t)) \in H_2$ and $\rho, \rho : \Omega \rightarrow (0, 1)$ are measurable mappings.

For appropriate and suitable choice of $T, E, G, M, N, f, g, A_t, \eta_t$ and $H_t$ for $i = 1, 2$ we see that (2.1) is generalized version of some problems which include the system (random) variational inclusions, (random) generalized quasi variational inequalities and (random) implicit quasi variational inequalities for fuzzy mappings, see [17, 18].

**Lemma 2.9.** [3] Let $M : \Omega \times H \rightarrow CB(H)$ be a $\mathcal{D}$-continuous random multi-valued mapping. Then for a measurable mapping $x : \Omega \rightarrow H$, the mapping $M(\cdot, x(\cdot)) : \Omega \rightarrow CB(H)$ is measurable.

**Lemma 2.10.** [4] Let $M, V : \Omega \rightarrow CB(H)$ be two measurable multi-valued mappings and $\epsilon > 0$ be a constant and $x : \Omega \rightarrow H$ be a measurable selection of $M$. Then there exists a
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measurable selection $y : \Omega \to H$ of $V$ such that for all $t \in \Omega$

\[ \|x(t) - y(t)\| \leq (1 + \epsilon) \mathcal{D}(M(t), V(t)). \]

**Lemma 2.11.** [22] Let $(H, d)$ be a complete metric space. Suppose that $G : H \to CB(H)$ satisfies

\[ \mathcal{D}(G(x), G(y)) \leq \omega d(x, y), \forall x, y \in H, \]

where $\omega \in (0, 1)$ is a constant. Then the mapping $G$ has a fixed point in $H$.

**Definition 2.12.** Let $x, y, w : \Omega \to H$ be random mappings and $t \in \Omega$. A random mapping $T : \Omega \times H \times H \to H$ is said to be:

(i) randomly monotone in the first argument if

\[ \langle T_{t}(x(t), w(t)) - T_{t}(y(t), w(t)), x(t) - y(t) \rangle \geq 0, \]

for all $x(t), y(t) \in H$.

(ii) randomly strictly monotone in the first argument if $T_t$ is monotone and

\[ \langle T_{t}(x(t), w(t)) - T_{t}(y(t), w(t)), x(t) - y(t) \rangle = 0 \]

if and only if $x(t) = y(t)$;

(iii) randomly $r_t$-strongly monotone in the first argument if there exists a measurable function $r_t : \Omega \to (0, \infty)$ such that

\[ \langle T_{t}(x(t), w(t)) - T_{t}(y(t), w(t)), x(t) - y(t) \rangle \geq r_t \|x(t) - y(t)\|^2, \]

for all $x(t), y(t) \in H$.

(iv) randomly $m_t$-relaxed monotone in the first argument if there exists a measurable function $m_t : \Omega \to (0, \infty)$ such that

\[ \langle T_{t}(x(t), w(t)) - T_{t}(y(t), w(t)), x(t) - y(t) \rangle \geq -m_t \|x(t) - y(t)\|^2, \]

for all $x(t), y(t) \in H$.

(v) randomly $s_t$-cocoercive in the first argument if there exists a measurable function $s_t : \Omega \to (0, \infty)$ such that

\[ \langle T_{t}(x(t), w(t)) - T_{t}(y(t), w(t)), x(t) - y(t) \rangle \geq s_t \|T_{t}(x(t), w(t)) - T_{t}(y(t), w(t))\|^2, \]

for all $x(t), y(t), w(t) \in H \times H \times H$. 
(vi) randomly $\gamma_t$-relaxed cocoercive with respect to $A_t$ in the first argument if there exists a measurable function $\gamma_t \rightarrow (0, \infty)$ such that
\[
\langle T_t(x(t), w(t)) - T_t(y(t), w(t)), A_t(x(t)) - A_t(y(t)) \rangle \geq -\gamma_t \|T_t(x(t), w(t)) - T_t(y(t), w(t))\|^2,
\]
for all $x(t), y(t), w(t) \in H \times H \times H$.

(vii) randomly $(\gamma_t, \alpha_t)$-relaxed cocoercive with respect to $A_t$ in the first argument if there exist measurable functions $\gamma_t, \alpha_t : \Omega \rightarrow (0, \infty)$ such that
\[
\langle T_t(x(t), w(t)) - T_t(y(t), w(t)), A_t(x(t)) - A_t(y(t)) \rangle \geq -\gamma_t \|T_t(x(t), w(t)) - T_t(y(t), w(t))\|^2 + \alpha_t \|x(t) - y(t)\|^2,
\]
for all $x(t), y(t), w(t) \in H \times H \times H$.

(viii) randomly $\mu_t$-Lipschitz continuous in the first argument if there exists a measurable function $\mu_t : \Omega \rightarrow (0, \infty)$ such that
\[
\|T_t(x(t), w(t)) - T_t(y(t), w(t))\| \leq \mu_t \|x(t) - y(t)\|,
\]
for all $x(t), y(t), w(t) \in H \times H \times H$.

In a similar way, we can define a randomly Lipschitz continuity of the operator $T(\cdot, \cdot, \cdot)$ in the second argument.

**Definition 2.13.** Let $T : H \times \Omega \rightarrow 2^H$ be a random multi-valued mapping. Then $T$ is said to be randomly $\tau_t$-\(\widetilde{D}\)-Lipschitz continuous in the first argument if there exists a measurable mapping $\tau : \Omega \rightarrow (0, 1)$ such that
\[
\widetilde{D}(T_t(x(t)), T_t(y(t))) \leq \tau_t \|x(t) - y(t)\|,
\]
for all $x(t), y(t) \in H, t \in \Omega$, where $\widetilde{D} : 2^H \times 2^H \rightarrow (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff metric i.e.,
\[
\widetilde{D}(A, B) = \max \left\{ \sup_{x(t) \in A} \inf_{y(t) \in B} \|x(t) - y(t)\|, \sup_{x(t) \in B} \inf_{y(t) \in A} \|x(t) - y(t)\| \right\}, \forall A, B \in 2^H.
\]

In a similar way we can define randomly $\tilde{D}$-Lipschitz continuity of the $T(\cdot, \cdot)$ in the second argument.
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**Lemma 2.14.** Let \((H,d)\) be a complete metric space and \(T_1, T_2 : H \rightarrow CB(H)\) be two set-valued contractive mappings with same contractive constant \(t \in (0,1)\) i.e.,

\[
\tilde{D}(T_i(x), T_i(y)) \leq td(x, y), \forall x, y \in H, i = 1, 2.
\]

Then

\[
\tilde{D}(F(T_i), F(T_i)) \leq \frac{1}{1-t} \sup_{x \in H} \tilde{D}(T_1(x), T_2(x)),
\]

where \(F(T_1)\) and \(F(T_2)\) are the sets of fixed points of \(T_1\) and \(T_2\), respectively.

**Definition 2.15.** Let \(A: H \times \Omega \rightarrow H, \eta: H \times H \times \Omega \rightarrow H\) be two random single-valued mappings. The set-valued mapping \(M: H \times H \times \Omega \rightarrow 2^H\) is said to be random \((A_t, \eta_t, m_t)\)-monotone if

1. \(M\) is randomly \(m_t\)-relaxed \(\eta_t\)-monotone mapping;
2. \((A_t + \rho_t M_t)(H) = H\), where \(\rho: \Omega \rightarrow (0,1)\) is a measurable mapping.

**Definition 2.16.** Let \(A : \Omega \times H \rightarrow H\) be a randomly \(\tau_t\)-Lipschitz continuous mapping and \(M : \Omega \times H \rightarrow 2^H\) be a randomly \((A_t, \eta_t, m_t)\)-monotone mapping. Then for any given measurable mapping \(\rho : \Omega \rightarrow (0,1)\), the random resolvent operator \(J_{\rho_t, A_t}^{\eta_t, M_t} : H \rightarrow H\) is defined by

\[
J_{\rho_t, A_t}^{\eta_t, M_t}(x(t)) = (A_t + \rho_t M_t)^{-1}(x(t)), \forall t \in \Omega, x(t) \in H.
\]

**Proposition 2.18.** [19] Let \(H\) be a Hilbert space and \(\eta: \Omega \times H \times H \rightarrow H\) be a randomly \(\tau_t\)-Lipschitz continuous mapping, \(A : \Omega \times H \rightarrow H\) be a randomly \(\tau_t\)-strongly \(\eta_t\)-monotone mapping and \(M : \Omega \times H \rightarrow 2^H\) be a randomly \((A_t, \eta_t, m_t)\)-monotone mapping. Then the random resolvent operator \(J_{\rho_t, A_t}^{\eta_t, M_t} : H \rightarrow H\) is a randomly \((\frac{\tau_t}{\tau_t - \rho_t m_t})\)-Lipschitz continuous mapping i.e.,

\[
\|J_{\rho_t, A_t}^{\eta_t, M_t}x(t) - J_{\rho_t, A_t}^{\eta_t, M_t}y(t)\| \leq \frac{\tau_t}{\tau_t - \rho_t m_t}\|x(t) - y(t)\|,
\]

where \(\rho_t \in (0, \tau_t)\) is a real-valued random variable for all \(t \in \Omega\).

In connection with a randomly \((A_t, \eta_t, m_t)\)-proximal operator equation system \((2.1)\), we consider the system of random nonlinear equation with fuzzy mappings for finding measurable
mappings $x, u : \Omega \rightarrow H_1, y : \Omega \rightarrow H_2$ such that for all $t \in \Omega$ and each fixed $\tilde{T}_t(x(t))(u(t)) \geq \alpha(x(t))$ and

\[ 0 \in E_t(x(t), y(t)) + M_t(x(t), x(t)), \]
\[ 0 \in G_t(u(t), y(t)) + N_t(y(t), y(t)). \] (2.2)

**Lemma 2.19.** For $t \in \Omega$, $x, u : Q \rightarrow H_1$ and $y : \Omega \rightarrow H_2$, $(x(t), y(t), u(t))$ is a solution of problem (2.2) if and only if $(x(t), u(t)) \in H_1, y(t) \in H_2$ such that

\[
\begin{align*}
x(t) &= J_{A_{1,t}, \varphi_t}^{M_t(\cdot, x(t))} [A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))] \\
y(t) &= J_{A_{2,t}, \varphi_t}^{N_t(\cdot, y(t))} [A_{2,t}(y(t)) - \rho_t G_t(u(t), y(t))]
\end{align*}
\] (2.3)

where $J_{A_{1,t}, \varphi_t}^{M_t(\cdot, x(t))} = (A_{1,t} + \rho_t M_t(\cdot, x(t)))^{-1}$ and $J_{A_{2,t}, \varphi_t}^{N_t(\cdot, y(t))} = (A_{2,t} + \rho_t N_t(\cdot, y(t)))^{-1}$ are corresponding random resolvent operator in the first argument of a random $(A_{1,t}, \eta_{1,t})$-monotone operator $M_t(\cdot, \cdot)$, random $(A_{2,t}, \eta_{2,t})$-monotone operator $N_t(\cdot, \cdot)$, respectively, $A_{i,t}$ is a randomly $r_{i,t}$-strongly monotone operator for $i = 1, 2$ and $\rho, \varphi : \Omega \rightarrow (0, 1)$ are measurable mappings.

Now we prove that problem (2.1) is equivalent to problem (2.3).

**Lemma 2.20.** For $t \in \Omega$ the problem (2.1) has a solution $(x(t), y(t), u(t))$ with $u(t) \in \tilde{T}_t(x(t))$ if and only if the problem (2.3) has a solution $(x(t), y(t), u(t))$ with $u(t) \in \tilde{T}_t(x(t))$, where

\[
\begin{align*}
x(t) &= J_{A_{1,t}, \varphi_t}^{M_t(\cdot, x(t))}(z(t)), \\
y(t) &= J_{A_{2,t}, \varphi_t}^{N_t(\cdot, y(t))}(w(t))
\end{align*}
\] (2.4)

and

\[
\begin{align*}
z(t) &= A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t)), \\
w(t) &= A_{2,t}(y(t)) - \rho_t G_t(u(t), y(t)),
\end{align*}
\]

where $\rho, \varphi : \Omega \rightarrow (0, 1)$ are measurable mappings.

### 3 Main Results

In this section, we first discuss the existence theorem. And then we developed an algorithm for the problem and proved the convergence of the random sequence generated by given algorithm.

**Theorem 3.1.** Let $(\Omega, \sigma)$ be a measurable space. Let $A_t : H_t \times \Omega \rightarrow H_t$ be a randomly $r_{i,t}$-strongly monotone and randomly $s_{i,t}$-Lipschitz continuous mapping for each $i = 1, 2$, $T : H_1 \times \Omega \rightarrow \mathcal{I}(H_1)$ be a fuzzy mapping induced by a set-valued mapping $\tilde{T} : H_1 \times \Omega \rightarrow H_1$, and
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\[ \alpha : H_1 \to (0,1) \text{ and } \tilde{T}_{T,t}(x(t)) \geq \alpha(x(t)) \] satisfying the condition (*). Let \( \tilde{T} : H_1 \times \Omega \to H_1 \) be the randomly \( \kappa \tilde{T} - \tilde{D} \)-Lipschitz continuous mapping induced by \( T \), where \( \tilde{D} \) is the Hausdorff pseudo metric on \( 2^H_i \), for each \( i = 1,2 \). Let \( M : H_1 \times H_1 \times \Omega \to 2^{H_1} \) be a randomly \((A_{1,t}, \eta_{1,t})\)-monotone mapping with measurable mapping \( m_1 : \Omega \to (0,1) \) in the first variable and \( N : H_2 \times H_2 \times \Omega \to 2^{H_2} \) be a randomly \((A_{2,t}, \eta_{2,t})\)-monotone mapping with measurable mapping \( m_2 : \Omega \to (0,1) \) in the first variable. Let \( \eta_1 : H_1 \times H_1 \Omega \to H_1 \) be a randomly \( \tau_{2,t} \)-Lipschitz continuous mapping, \( \eta_2 : H_2 \times H_2 \times \Omega \to H_2 \) be randomly \( \tau_{2,t} \)-Lipschitz continuous mapping, \( E : H_1 \times H_2 \times \Omega \to H_1 \) be the randomly Lipschitz continuous mapping with respect to first variable with measurable mapping \( \beta : \Omega \to (0,1) \), and second argument with respect to the measurable mapping \( \xi : \Omega \to (0,1) \) and randomly \((\gamma_1, \alpha_1)\)-relaxed cocoercive with respect to \( A_{1,t} \) and first variable of \( E_t \) with measurable mappings \( \gamma, \alpha : \Omega \to (0,1) \). Let \( G : H_1 \times H_2 \times \Omega \to H_2 \) be the randomly Lipschitz continuous with respect to first and second variables with measurable mappings \( \mu, \zeta : \Omega \to (0,1) \), respectively. Let \( G \) be a randomly \((\gamma_2, \alpha_2)\)-relaxed cocoercive mapping with respect to \( A_{2,t} \) with measurable mappings \( \gamma_2, \alpha_2 : \Omega \to (0,1) \), respectively. If in addition \( \rho : \Omega \to (0, \frac{\tau_{1,t}}{m_{1,t}}) \) and \( \rho : \Omega \to (0, \frac{\tau_{2,t}}{m_{2,t}}) \) are measurable mappings and

\[ \|J_{\mu_1,A_{1,t}}(z(t)) - J_{\mu_1,A_{1,t}}(y(t))\| \leq \nu_{1,t}\|x(t) - y(t)\| \],

for all \((x(t), y(t), z(t), t) \in H_1 \times H_1 \times \Omega \),

\[ \|J_{\mu_2,A_{2,t}}(z(t)) - J_{\mu_2,A_{2,t}}(y(t))\| \leq \nu_{2,t}\|x(t) - y(t)\| \],

for all \((x(t), y(t), z(t), t) \in H_2 \times H_2 \times \Omega \), where \( x, u : \Omega \to H_1 \) and \( y : \Omega \to H_2 \) are measurable mappings, then problem (2.1) has a random solution \((x^*(t), y^*(t), u^*(t))\).

### 4 Iterative algorithms and convergence analysis

In this section, based on Lemma 2.20 and Nadler results [23], we shall construct a new class of iterative algorithms for solving problems (2.1) and discuss the convergence analysis of the algorithms.

**Algorithm 4.1.** Assume that \( H_1, A_i, \eta_i, M, N, E, G, T, \tilde{T} \) are same as in the problem (2.1) for each \( i = 1, 2 \) and \( x_0 : \Omega \to H_1, y_0 : \Omega \to H_2 \) are measurable mappings. For \( \alpha : H_2 \to (0,1) \), \( n \geq 0 \) and the random element \((x(t), y(t), u(t)) \in H_1 \times H_2 \times H_1 \), we define the iterative sequences \( \{x_n(t)\}, \{y_n(t)\}, \{u_n(t)\} \) by

\[ x_{n+1}(t) = (1 - \lambda_n(t))x_n(t) + \lambda_n(t)\left[J_{\mu_1,A_{1,t}}(A_{1,t}(x_n(t)) - \rho_t E_t(x_n(t), y_n(t)))\right] + p_n(t) \]
y_{n+1}(t) = (1 - \lambda_n(t))y_n(t) + \lambda_n(t) \left[ J_{\rho_{t},A_{2,t}}^{N_{t}(\cdot,y_{n}(t))}(A_{2,t}(y_{n}(t)) - \rho_{t}G_{t}(u_{n}(t), y_{n}(t))) \right] + q_{n}(t), \quad (4.2)

\tilde{T}_{t,x(t)}(u_{n}(t)) \geq a(x_{n}(t)), \|u_{n}(t) - u(t)\| \leq (1 + \iota)\overline{D}(\tilde{T}_{t}(x_{n}(t)),\tilde{T}_{t}(x(t))), \quad (4.3)

where \rho, \rho: O \rightarrow (0,1) are measurable, \{\lambda_n(t)\} is a measurable sequence in (0,1], and \(p_n(t), q_n(t)\) are two random error sequences satisfying the same conditions in \(H_1\) and \(H_2\), respectively.

**Lemma 4.2.** [24] Let \{a_n\}, \{b_n\} and \{c_n\} be three sequences of nonnegative real numbers satisfying the following conditions:

(i) \(0 \leq b_n < 1, n = 0, 1, 2, \ldots \) and \(\lim \sup_n b_n < 1\);

(ii) \(\sum_{n=0}^{\infty} c_n < +\infty\);

(ii) \(a_{n+1} \leq b_n a_n + c_n, n = 0, 1, 2, \ldots\).

Then \(\lim_{n \to \infty} a_n = 0\).

**Theorem 4.3.** Let \(H_1, H_2, T_t, \tilde{T}_t, \eta_{1,t}, \eta_{2,t}, A_{1,t}, A_{2,t}, M_t, N_t, E_t, G_t\) be the same as in Theorem 3.1. Assume that all the conditions of Theorem 3.1 hold and

\[ \lim \sup_n \lambda_n(t) < 1, \sum_{n=0}^{\infty} (\|p_n(t)\| + \|q_n(t)\|) < +\infty. \quad (4.4) \]

Then the random iterative sequences \((x_n(t), y_n(t))\) with \(u_n(t) \in \tilde{T}_t(x(t))\) defined by Algorithm 4.1, converges strongly to the random solution \((x^*(t), y^*(t), u^*(t))\) of (2.1).

**References**


A system of random nonlinear equations in Hilbert spaces


Jong Kyu Kim and Salahuddin


