数理解析研究所講究録
第2011巻 2016年 42-46

不動点定理とニューロンモデルに関係する非整数階微分方程式

Fixed point theorem and fractional differential equations related with a neuron model

川崎敏治[‡] 佐々木寛[*] 豊田昌史[‡]
Toshiharu Kawasaki Hiroshi Sasaki Masashi Toyoda

[‡] 玉川大学工学部マネジメントサイエンス学科, [*] 玉川大学工学部ソフトウエアサイエンス学科
194–8610 東京都町田市玉川学園 6–1–1
Tamagawa University, 6–1–1 Tamagawa-gakuen, Machida-shi, Tokyo 194–8610.

1 Introduction

In this paper we show a fixed point theorem (Theorem 1). Using Theorem 1, we show the existence and uniqueness of solutions for fractional differential equations with multiple delays (Theorem 2). Using Theorem 2, we discuss the fractional chaos neuron model.

2 Information processing mechanisms in nervous system and neuron models

Mammalian behaviour is thought to be controlled by nervous system including brain. Neuron, which is structural and functional unit of nervous system, at the sensory receptor plays a key role in transforming physical information from the outside into the signals which can be treated in nervous system. The nervous signals are transferred to central nervous system and are processed, producing the output toward effector organ like muscles.

Neuron itself can be considered as the system with multi-input and one output. Neuron forms the unique special structure called synapse to contact with other neurons, composing the neural networks in nervous system. One neuron receives inputs as postsynaptic potential of either positive or negative value from other neurons through these synapses and summate them, generating output as an action potential when summed input exceed the threshold value.

Key points of information processing in neuron, therefore, are the summation of positive/negative inputs and threshold effect in input-output transformation, resulting in all-or-none characteristic of output.

Representative basic mathematical neuron model was reported by McCulloch and Pitts in 1943 [4]. It relatively faithfully followed the neural information processing in mammalian neurons especially for the all-or-none output characteristic. This model is expressed as follows:

$$z = s\left(\sum_{i=1}^{n} w_i x_i - \theta\right)$$

where n is a number of inputs, x_i is an i-th input , w_i is a synaptic weight of an i-th input, z is an output, θ is a threshold and s is a step function

$$s(t) = \begin{cases} 1 & (t \geq 0), \\ 0 & (t < 0). \end{cases}$$

Other than this model, variety of mathematical neuron models were suggested for not only the basic research in neuroscience but also the engineering research to develop novel system based on neural information processing mechanisms. As for the engineering purpose, a model does not need to precisely follow the neural behaviour of neuron in mammals, e.g. the neuron model without all-or-none output characteristic. One of such models is the fractional chaos neuron model in which its dynamics are exhibited by the fractional differential equation [3]. Example 3 in Section 4, we discuss the fractional chaos neuron model.

3 Fixed point theorem

Let I be an arbitrary interval, let J be an interval with $I \subset J$. Let $(E, \|\cdot\|_E)$ be a Banach space, let $BC(I, E)$ be the Banach space consisting all bounded continuous mappings from I into E with the norm $\|u\| = \sup\{\|u(t)\|_E \mid t \in I\}$ for any $u \in BC(I, E)$. Let F be a nonempty closed subset of $BC(I, E)$ and let ϕ be a mapping from $J \setminus I$ into E. Define a mapping u_ϕ by $u_\phi = u$ if on I, $u_\phi = \phi$ if on $J \setminus I$ for any $u \in F$. We say F satisfies $(*)$ for ϕ if $(*)$ $u_\phi \in BC(J, E)$ holds for any $u \in F$.

We obtain the following fixed point theorem. For the proof of Theorem 1, see [2].

Theorem 1. *Let I be an arbitrary finite or infinite interval, let J_0, J be intervals with $I \subset J_0 \subset J$, let $(E, \|\cdot\|_E)$ be a Banach space and let F be a nonempty closed subset of $BC(I, E)$. Suppose that there exists a mapping ϕ from $J \setminus I$ into E such that F satisfies $(*)$ for ϕ. Let A be a mapping from F into itself. Suppose that there exist $\beta \in [0, 1)$, a mapping G from $I \times J_0$ into $[0, \infty)$ integrable with respect to the second variable for any the first variable, mappings γ, δ from I into J_0 with $\gamma \leq \delta$, $n \in \mathbb{N}$, and mappings $\eta_i \in C(J_0, J)$ for any $i = 1, \ldots, n$ such that*

(H_1) *for any $u, v \in F$ and for any $t \in I$*

$$\|Au(t) - Av(t)\|_E \leq \beta \|u(t) - v(t)\|_E + \int_{\gamma(t)}^{\delta(t)} G(t, s) \sum_{i=1}^{n} \|u_\phi(\eta_i(s)) - v_\phi(\eta_i(s))\|_E ds;$$

(H_2) *there exist* $\alpha \in [0,\infty)$, $K \in [0,\infty)$, $m, M \in (0,\infty)$ *with* $m \leq M$ *and* $y \in BC(J,[m,M])$ *such that*

(1) $\beta + nK\alpha \in [0,1)$;

(2) $y(\eta_i(t)) \leq Ky(t)$ *for any* $t \in J_0$ *and for any* $i = 1,\ldots,n$;

(3) $\displaystyle\int_{\gamma(t)}^{\delta(t)} G(t,s)y(s)ds \leq \alpha y(t)$ *for any* $t \in I$.

Then A *has a unique fixed point in* F.

4 Fractional differential equations related with a neuron model

In this section, using Theorem 1, we show the existence and uniqueness of solutions for fractional differential equations with multiple delays. Throughout this paper the fractional derivative means the Caputo-Riesz derivative ${}^cD^\alpha$ defined by

$$ {}^cD^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} u(s)ds $$

for any $\alpha \in (0,\infty)$ and for any function u of $(0,\infty)$ into $\mathbb{R}$, where Γ is the gamma function and n is a natural number with $n-1 \leq \alpha < n$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function u of $(0,\infty)$ into $\mathbb{R}$ is defined by

$$ I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds. $$

For the fractional derivative and integral, for instance, see [1].

Using Theorem 1, we obtain the following.

Theorem 2. *Let* $(E, \|\cdot\|_E)$ *be a Banach space, let* $C([0,T]\times E\times E, E)$ *be the space consisting of all continuous mappings from* $[0,T]\times E\times E$ *into* E *and let* $f \in C([0,T]\times E\times E, E)$ *satisfying*

(H_f) *there exist* $L_1, L_2 \in [0,T]$ *such that*

$$ \|f(t,x_1,x_2) - f(t,y_1,y_2)\|_E \leq \|x_1 - y_1\|_E + \|x_2 - y_2\|_E $$

for any $t \in [0,T]$ *and for any* $x_1, x_2, y_1, y_2 \in E$.

Let $C([0,T],E)$ *be the Banach space consisting of all continuous mappings from* $[0,T]$ *into* E, *let* $C([0,T],[0,\infty))$ *be the space consisting of all continuous mappings from* $[0,T]$ *into* $[0,\infty)$ *and let* $C((-\infty,0],E)$ *be the space consisting of all continuous mappings from* $(-\infty,0]$ *into* E. *Then the following fractional differential equation with multiple delays*

$$ {}^cD^\alpha u(t) = f(t,u(t),u_\phi(t-\tau(t))) \qquad (t \in [0,T]), \tag{1} $$

where $\alpha \in (0,1]$, $\tau \in C([0,T],[0,\infty))$ *and* $\phi \in C((-\infty,0],E)$, *has a unique solution in* $\{u \mid u \in C([0,T],E) \text{ and } u(0) = \phi(0)\}$.

Proof. Put $I = J_0 = [0,T]$, $\tau_0 = \inf\{t - \tau(t) \mid t \in I\}$, $J = [\tau_0, T]$ and

$$F = \{u \mid u \in C(I,E) \text{ and } u(0) = \phi(0)\}.$$

Then F is closed. Since $\phi \in C([\tau_0, 0), E)$ and $u(0) = \phi(0)$ for any $u \in F$, we obtain $u_\phi \in C(J,E)$ for any $u \in F$. Therefore F satisfies $(*)$ for ϕ. Then $u \in C(I,E)$ is a solution of the equation (1) if and only if it is a solution of the integral equation

$$u(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u_\phi(s - \tau(s)))ds. \tag{2}$$

In fact, if u satisfies the equation (1), then we have

$$I^{\alpha c}D^\alpha u(t) = I^\alpha f(t, u(t), u_\phi(t - \tau(t))).$$

Since $I^{\alpha c}D^\alpha u(t) = u(t) - u(0)$ ([1, Lemma 2.22]) and $u(0) = \phi(0)$, we have the equation (2). If u satisfies the equation (2), then we have

$${}^cD^\alpha u(t) = {}^cD^\alpha \phi(0) + {}^cD^\alpha I^\alpha f(t, u(t), u_\phi(t - \tau(t))).$$

Since ${}^cD^\alpha \phi(0) = 0$ and ${}^cD^\alpha I^\alpha = I$ ([1, Lemma 2.21]), we have the equation (1).

Define a mapping A by

$$Au(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u_\phi(s - \tau(s)))ds$$

for any $u \in F$. Since $Au(0) = \phi(0)$, we obtain $Au \in F$. We show A has a unique fixed point. Indeed we have

$$\begin{aligned}
&\|Au(t) - Av(t)\|_E \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(L_1 \|u_\phi(s - \tau_1(s)) - v_\phi(s - \tau_1(s))\|_E \right. \\
&\qquad\qquad \left. + L_2 \|u_\phi(s - \tau_2(s)) - v_\phi(s - \tau_2(s))\|_E\right) ds \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\|u_\phi(\eta_1(s)) - v_\phi(\eta_1(s))\|_E + \|u_\phi(\eta_2(s)) - v_\phi(\eta_2(s))\|_E\right) ds,
\end{aligned}$$

where $\tau_1(t) = 0$, $L = \max\{L_1, L_2\}$ and $\eta_i(t) = t - \tau_i(t)$ $(i = 1,2)$. Put $\beta = 0$,

$$G(t,s) = \begin{cases} \dfrac{L}{\Gamma(\alpha)}(t-s)^{\alpha-1} & \text{if } 0 \leq s < t, \\ 0 & \text{if } t \leq s, \end{cases}$$

$\gamma(t) = 0$ and $\delta(t) = t$. Then the condition (H_1) holds. Take α with $0 < n\alpha < 1$ and take c with $c^\alpha \geq \frac{L}{\alpha}$. Put $K = 1$, $m = e^{c\tau}$, $M = e^{cT}$ and $y(t) = e^{ct}$. Then the conditions (1) and (2) of (H_2) hold. Moreover, since

$$\int_{\gamma(t)}^{\delta(t)} G(t,s)y(s)ds = \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{cs} ds = \frac{Le^{ct}}{c^\alpha \Gamma(\alpha)} \int_0^{ct} s^{\alpha-1} e^{-s} ds \leq \frac{L}{c^\alpha} e^{ct} \leq \alpha y(t),$$

the condition (3) of (H_2) holds. Therefore by Theorem 1, A has a unique fixed point in F. □

Using Theorem 2, we discuss the fractional chaos neuron model [3].

Example 3. *We consider the following fractional differential equation with delay*

$$^{c}D^{\alpha}u(t) = -\beta u(t) + \sin\frac{\pi u_{\phi}(t-\tau)}{2T_0} \qquad (t \in [0,T]),$$

where $\alpha \in (0,1]$, $\beta, \tau \in [0,\infty)$, $T_0 \in (0,\infty)$ *and* $\phi \in C([-\tau,0],\mathbb{R})$. *In this equation,* $u(t)$ *is an internal state of the neuron at time* t, β *is a dissipative parameter and* τ *is delay time. Moreover we use a sinusoidal function with a periodic parameter* T_0 *as an activation to be related to the output of the neuron. This equation is called the fractional chaos neuron model* [3]. *Put* $E = \mathbb{R}$, $\tau(t) = \tau$ *and* $f(t,x_1,x_2) = -\beta x_1 + \sin\frac{\pi x_2}{2T_0}$. *Since*

$$\begin{aligned} |f(t,x_1,x_2) - f(t,y_1,y_2)| &\leq |\beta||x_1-y_1| + \left|\sin\frac{\pi x_2}{2T_0} - \sin\frac{\pi y_2}{2T_0}\right| \\ &\leq |\beta||x_1-y_1| + \frac{\pi}{2T_0}|x_2-y_2|, \end{aligned}$$

f *satisfies* (H_f) *for* $L_1 = |\beta|$ *and* $L_2 = \frac{\pi}{2T_0}$. *Therefore by Theorem 2 the equation above has a unique solution in* $\{u \mid u \in C([0,T],\mathbb{R})$ *and* $u(0) = \phi(0)\}$.

References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier, Amsterdam, 2006.

[2] T. Kawasaki and M. Toyoda, *Fixed point theorem and fractional differential equations with multiple delays related with chaos neuron models*, Applied Mathematics, Vol.6, No.13 (2015), 2192–2198.

[3] T. Matsuzaki and M. Nakagawa, *A chaos neuron model with fractional differential equation*, Journal of the Physical Society of Japan, **72** (2003), 2678–2684.

[4] W. S. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, **5** (1943), 115–133.