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INVERSE PROBLEM ON ISOMORPHISM THEOREM OF $A^p(G)$-ALGEBRAS $1\leq p\leq 2$

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Abstract

Let G_1 and G_2 be locally compact Hausdorff groups, $E(G_1)$ and $E(G_2)$ the function spaces (Banach algebras or Banach spaces) on G_1 and G_2 respectively. Then it is known that if $G_1\simeq G_2$, implies $E(G_1)$ and $E(G_2)$ are isomorphic. Naturally, an inverse problem arises that

(P) Whether an algebaic isomorphism $\Phi : E(G_1)\longrightarrow E(G_2)$ could deduce $G_1\simeq G_2$?

In this paper, we would solve Problem (P) for $A^p(G)-$algebras, $1\leq p\leq 2$.

1. Preliminaries

(1) 1948, Y. Kawada [7] solved this problem under bipositive isomorphism $\Phi : L^1(G_1)\longrightarrow L^1(G_2)$.

(2) 1952, Wendel [13] proved (P) under the isomorphism Φ from the algebra $L^1(G_1)$ onto $L^1(G_2)$ by assuming Φ is a norm nonincreasing.

(3) 1965, Edwards [2] considered the groups $G_i (i=1,2)$ are compact, and if there exists a bipositive isomorphism of $L^p(G_1)$ onto $L^p(G_2)$ to get ,then $G_1\simeq G_2$. He asked whether the compact groups G_1 and G_2 are necessarily homeomorphic, if bipositive is replaced by isometry?

(4) 1966, The affirmative answer to this question in [2] by positive replaced isometry was given by Strichartz [12].

(5) 1968, Further, Parrott [11] proved the question in Edwards [2] for general locally compact groups G_1 and G_2 if there is an isomertic transformation of $L^p(G_1)$ onto $L^p(G_2)$ $(1\leq p<\infty,\ p\neq 2)$.

Remark : The Lebesgue space $L^p(G)$ need not be an algebra if G is not compact.

(6) 1973, Lai/Lien [10] solved the problem (P) by assume that if there exists an injective bipositive linear mapping from the Banach space $L^p(G_1)$ onto Banach space $L^p(G_2)$, then $G_1 \simeq G_2$ is deduced.

(7) Some other isomorphism problems were solved by Johnson [6], Gaudry [4] and Figa Talamanca [3] in different view points.

(8) In this article we would solve problem (P) on the Banach algebra $A^p(G)$, $1 \le p \le 2$.

2. $A^p(G)$–ALGEBRAS, $1 \le p < \infty$

In this paper, we would consider the isomorphism theorem for $A^p(G) - algebras$.

Let G be a LCA group with dual group $\widehat{G}$. The space $A^p(G)$ is defined by

$$A^p(G) = \{f \in L^1(G) \ ; \ \text{Fourier transform } \widehat{f} \in L^p(\widehat{G})\}, 1 \le p < \infty. \tag{1}$$

Then $A^p(G)$ is a commutative Banach algebra under convolution product with the norm given by

$$\| f \|^p = \| f \|_1 + \| \widehat{f} \|_p, \text{ for each } \quad p, \ 1 \le p < \infty \text{ for } f \in A^p(G). \tag{2}$$

The norm $\| \ \|^p$ is equivalent to $\max(\| f \|_1, \| \widehat{f} \|_p)$.

Since $f \in A^p(G) \Longrightarrow \widehat{f} \in L^P(\widehat{G}) \cap C_0(\widehat{G})$, thus $\widehat{f} \in L^r(\widehat{G})$ for $r > p > 1$, but such $f \notin A^r$ for $1 \le p \le 2 \le q < r < \infty$.

By this fact, we know that $A^p(G)$ can not include all Fourier transforms of $C_c(G) \cap A^r$. And $A^1(G) \supset A^p(G) \supset A^2(G) \supset A^q(G) \supset C_0(G)$, where $A^1(G) = \bigcup_{1<p\le 2} A^p(G)$ is the closure of such union sets.

We then conclude that

$$1 \le p \le 2, \ C_c \cap A^p(G) \text{ is dense in } A^p(G) \text{ with respect to the } A^p\text{-norm.}$$

Thus,

$$\text{if } f \in A^p(G), \text{ then } \widehat{f} \in L^p(\widehat{G}) \text{ and } \widehat{f} \in L^q(\widehat{G}) \text{ for } p \le 2 < q, \ f \notin A^q(G),$$

$$\text{and so } \forall p, \ 1 \le p \le 2 \le q < \infty, \ \tfrac{1}{p} + \tfrac{1}{q} = 1, \ A^p(G) \cap A^q(G) = \emptyset.$$

Hence the index p, only taken in the interval $1 \le p \le 2$ could get $T(A^p) \subset A^p$ by a continuous linear operator T. So we can discuss the multipliers T on $A^p(G)$ only taken

$1 \leq p \leq 2$ which could get $T(A^p) \subset A^p$. Therefore in later part, all $A^p(G)$ we discuss will take $1 \leq p \leq 2$.

3. Multipliers of $A^p(G)$

A multiplier T of $A^p(G)$ is a continuous linear mapping of $A^p(G), 1 \leq p \leq 2$ into itself, such that

$$T(f * g) = T(f) * g = f * T(g), \text{ for all } f, g \in A^p(G).$$

In order to solve problem (P) on $A^p(G)-$algebras. We use a technique by passing the multiplier of $A^p(G)$, thus we subscrip the definition of $A^p(G)$, as follows. Let $\mathfrak{L}(A^p)$ be the space of all bounded linear operator of $A^p(G)$, $1 \leq p \leq 2$ into itself.

Definition 1. *An operator $T \in \mathfrak{L}(A^p(G))$ is said to be a multiplier of $A^p(G)$ if*

$$T(f * g) = Tf * g = f * Tg \text{ for } f, g \in A^p(G). \tag{3}$$

The concept of multiplier T, one can consult Lai/ Lee / Liu [9, Theorem 1.1]. It deduces the space $\mathfrak{M}(A^p)$ of multipliers of $A^p(G)$ is isometrically isomorphic to $M(G)$, the space of all regular measures of G, that is

$$\mathfrak{M}(A^p) \cong \mathfrak{M}(L^1) \cong M(G), \ 1 \leq p \leq 2. \tag{4}$$

On the other hand, it is known that $A^p(G)$ is essential $L^1(G)$-module, since $L^1(G)$ has bounded approximate identity of norm 1 [9, Theorem 2.1]. It is remarkable that $A^p(G)$ has no A^p-uniform bounded approximate identity [8, p.574].

$$A^p * L^1 = A^p, \text{ and } ||f * g||^p \leq ||f||^p ||g||, \text{ for } f \in A^p, g \in L^1. \tag{5}$$

Thus the space $\mathfrak{M}(A^p, L^1)$ of multiplier A^p into L^1 is identical to $\mathfrak{M}(A^p)$. Hence there exists a unique $\mu \in M(G)$ such that

$$Tf = \mu * f \text{ for all } f \in A^p(G) \tag{6}$$

for any $T \in \mathfrak{M}(A^p, L^1) \cong \mathfrak{M}(A^p)$. By the property of $A^p(G)-$algebras, we will show the Isomorphism Theorem of $A^p(G)-$algebras can be stated as the following:

Theorem 2. *Let G_1 and G_2 be locally compact abelian groups and Φ an algebaic isomorphism of $A^P(G_1)$ onto $A^p(G_2)$, $1 \le p \le 2$. Suppose that one of $\widehat{G_1}$ and $\widehat{G_2}$ is connected, then Φ induces a topological isomorphism τ carrying G_2 onto G_1. Furthermore,*

$$\Phi f(x) = c\widehat{x}(x) f(\tau x) \text{ for } f \in A^p(G_1), \text{ and } x \in G_2,$$

where $\widehat{x}(x)$ is a fixed character on G_2 and c a constant depending only on the choice of Haar measure in G_2.

Outline of the proof for the main Theorem is given as follows:

Since the isomorphism

$$
\begin{aligned}
&\Phi : A^p(G_1) \overset{onto}{\longrightarrow} A^p(G_2),\\
\Longrightarrow\ &\Phi \text{ maps the Maximal ideal spaces } \mathfrak{Max}(A^p(G_1)) \text{ of } A^p(G_1)\\
&\text{on to } \mathfrak{Max}(A^p(G_2)) \text{ of } A^p(G_2),\\
\Longrightarrow\ &\Phi : \mathfrak{Max}(A^p(G_1)) \longrightarrow \mathfrak{Max}(A^p(G_2))\\
&\qquad\qquad \| \qquad\qquad\qquad\qquad \| \\
\Longrightarrow\ &\Phi : \qquad \widehat{G_1} \quad \overset{\text{onto}}{\longrightarrow} \quad \widehat{G_2}
\end{aligned}
\tag{7}
$$

The reason of (7) is that since $A^p(G)$ is a semisimple commutative Banach algebra, then the space $\mathfrak{Max}(A^p(G))$ is characterized by $\widehat{G}$.

Hence if one of $\widehat{G_1}$ and $\widehat{G_2}$ is connected, then both of $\widehat{G_1}$ and $\widehat{G_2}$ are connected. Therefore G_1 and G_2 are non-compact.

Since the theorem in [3] is applicable, we note that operator T commutes with convolution on $A^p(G_1)$ is represented uniquely by $\mu \in M(G_1)$

$$
\begin{aligned}
&Tf = \mu * f = 0 \text{ for all } f \in A^p(G_1)\\
&\Longrightarrow \mu = 0.
\end{aligned}
\tag{8}
$$

Thus we take $\nu \in M(G_2)$ for any $f \in A^p(G_1)$, it can define this operator

$$T : A^p(G_1) \longrightarrow A^p(G_2)$$

by

$$\mu * f = \Phi^{-1}(\nu * \Phi f) = Tf. \tag{9}$$

It is well-defined by (8) since $A^p(G_1)$ is semisimple, and by Loomis [book : p.76 Theorem], one sees that Φ is bicontinuous and hence T is a multiplier of $A^p(G_1)$, thus $\exists!\ \mu \in M(G_1)$ such that

$$\mu * f = \Phi^{-1}(\nu * \Phi f) = Tf.$$

This μ is uniquely determined by ν, we define a mapping Ψ of $M(G_2)$ into $M(G_1)$ by

$$\Psi\nu * f = \Phi^{-1}(\nu * \Phi f).$$

It is not hard to prove that Ψ is an isomorphism of $M(G_2)$ onto $M(G_1)$. Since both measure algebras $M(G_1)$ and $M(G_2)$ are semi-simple and commutative, Ψ is bicontinuous and one can show that

$$\Psi|_{A^p(G_2)} \text{ on the algebra } A^p(G_2) \text{ is dense in } L^1(G_2),$$

hence $\Psi|L^1(G_2)$ becomes an isomorphism of $L^1(G_2)$ onto $L^1(G_1)$ [See Rudin's book Theorem 6.6.4]. Hence by Helsen [5], the theorem is complete. $\square$

The full paper about Isomorphism Theorem of $A^p(G)$-algebras will appear in elsewhere.

References

[1] B. Brainerd and R. E. Edwards, Linear operator which commutes with translation, *The Journal of Australian Math. Soc.*, **6** (1966), 289-327.

[2] R. E. Edwards, Bipositive and isometric isomorphism of certain convolution algebras, *Canad. J. Math.*, **17** (1965), 839-846 .

[3] A. Figa-Talamanca and G. I. Gaudry, Multipliers and sets of uniqueness of L^p, *Michigan Math. J.*, **17** (1970), 170-191.

[4] G. I. Gaudry, Isomorphisms of Multipliers algebras, *Canad. J. Math.*, **20** (1969), 1165-1172.

[5] H. Helsen, Isomorphism of abelian group algebras, *Ark. Math. Bd.*, **2** (1953), 475-487.

[6] B. E. Johnson, Isometric isomorphisms of Measure algebras, *Proc. Amer. Math. Soc.*, **15** (1964), 186-188.

[7] Y. Kawada, On the group ring of a topological group, *Math. Japonica.*, **1** (1948), 1-5.

[8] Hang-Chin Lai, On some properties of $A^p(G)$-algebras, *Proc. Japan Acad.*, **45** (1969), 572-576.

[9] Hang-Chin Lai, Jin-Chirng Lee and Cheng-Te Liu, Multipliers of Banach-valued function spaces on LCA group, *J. Nonlinear Convex Anal.*, **16** (2015), 1949-1963.

[10] Hang-Chin Lai and Ming-Chao Lien, *Isomorphism of the spaces $L^p(G)$*, *Chinese J. Math.* **1** (1973) 2, 167-173.

[11] S. K. Parrott, Isometric multiplier, *Pacific J. Math.*, **5** (1968), 158-166.

[12] R. S. Strichartz, Isomorphism of group algebrasp, *Proc. Amer. Math. Soc.*, **17** (1966), 858-862.

[13] I. G. Wendel, Left centralizers and isomorphism of group algebras, *Pacific J. Math.*, **2** (1952), 251-261.

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