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ACUTE POINTS, WEAK AND STRONG CONVERGENCE THEOREMS FOR NONLINEAR MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce the concept of k-acute points of a mapping for $k \in [0,1]$. We study some properties of k-acute points and relations among k-acute points, attractive points and fixed points. Then, We prove some convergence theorems by using these concepts.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. For a mapping $T : C \to H$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [9] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1975, Baillon [3] proved the following first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, $S_n x = \dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T (see also [8]).

Recently, Kocourek, Takahashi and Yao [4] introduced a wide class of nonlinear mappings called generalized hybrid which containing nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem. Motivated by Baillon [3], and Kocourek, Takahashi and Yao [4], Takahashi and Takeuchi [9] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for a generalized hybrid mapping.

In this paper, we introduce the concept of k-acute points of a mapping for $k \in [0,1]$. We study some properties of k-acute points and relations among k-acute points, attractive points and fixed points. Then, We prove some convergence theorems by using these concepts.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the metric projection of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [8] for more details. The following result is well-known; see also [8].

Lemma 2.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

Let C be a subset of a Banach space E and let T be a mapping of C into E. A mapping T is said to be L-Lipschitzian if $\|Tx - Ty\| \leq L\|x - y\|$ for any $x, y \in C$, where $L \in [0, \infty)$. In particular, T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. Usually, T is said to be quasi-nonexpansive if

$$(1)\, F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\| \leq \|x - v\| \quad \text{for} \quad x \in C, v \in F(T).$$

Let C be a subset of a Hilbert space H. Let T be a mapping of C into H and I be the identity mapping on C. T is said to be pseudo-contractive if, for any $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2.$$

Assume $F(T) \neq \emptyset$ and set $y = v \in F(T)$. Then we have that, for any $x \in C$,

$$\|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2.$$

Usually, T is said to be hemi–contractive if

$$(1) \quad F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2 \quad \text{for} \quad x \in C,\ v \in F(T).$$

Let $k \in [0, 1)$. T is said to be k–strictly pseudo–contractive if, for any $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2.$$

Assume $F(T) \neq \emptyset$ and set $y = v \in F(T)$. Then we have that, for any $x \in C$,

$$\|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2.$$

Usually, T is said to be k-demi-contractive if

$$(1) \quad F(T) \neq \emptyset, \quad (2) \quad \|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2 \quad \text{for} \quad x \in C,\ v \in F(T).$$

We call T a strictly pseudo–contraction if T is a k–strictly pseudo–contraction for some $k \in [0, 1)$. We also call T a demi–contraction if T is a k–demi–contraction for some $k \in [0, 1)$. Assume $F(T) \neq \emptyset$.

Let $k \in [0, 1]$. We define the set of k–acute points $\mathrm{A}_k(T)$ of T by

$$\mathrm{A}_k(T) = \{\ v \in H : \ \|Tx - v\|^2 \leq \|x - v\|^2 + k\|x - Tx\|^2 \quad \text{for all} \ \ x \in C\ \}.$$

We denote $\mathrm{A}_0(T)$ by $A(T)$ because $\mathrm{A}_0(T)$ and attractive points set of T are the same. We denote $\mathrm{A}_1(T)$ by $\mathrm{A}(T)$, that is,

$$\mathrm{A}(T) = \{\ v \in H :\ \|Tx - v\|^2 \leq \|x - v\|^2 + \|x - Tx\|^2 \quad \text{for all} \quad x \in C\ \}.$$

For details, see [2].

3. Acute points and convergence theorems

In this section, we prove convergence theorems by using the concept of k-acute points of a mapping for $k \in [0,1]$.

Let C be a subset of a Hilbert space H and let T be a mapping of C into H. Recall $\mathrm{A}_k(T) \subset \mathrm{A}(T)$ for all $k \in [0,1]$. Sometimes, we do not distinguish $\mathrm{A}(T)$ from $\mathrm{A}_k(T)$ with $k \in (0,1)$ strictly. For details, see [2].

Let C be a subset of a Hilbert space H and S be a mapping of C into H. Under the condition $A(S) \neq \emptyset$, we prove some convergence theorems (see [2]).

Theorem 3.1 ([2]). *Let $\{a_n\}$ be a sequence in $[a,b] \subset (0,1)$. Let C be a compact subset of a Hilbert space H. Let S be a continuous self-mapping on C such that $F(S) \subset \mathrm{A}(S)$ and $A(S) \neq \emptyset$. Suppose there is a sequence $\{u_n\}$ in C such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for} \quad n \in N.$$

Then, $\{u_n\}$ converges strongly to some $u \in F(S)$.

We also get the following theorem by Theorem 3.1 (see [2]).

Theorem 3.2 ([2]). *Let $\{a_n\}$ be a sequence in $[a,b] \subset (0,1)$. Let C be a compact subset of a Hilbert space H. Let T be a continuous self-mapping on C. Assume that one of the followings holds.*

(1) T is hemi–contractive with $A(T) \neq \emptyset$. S is the mapping defined by $S = T$.
(2) T is k–demi–contractive. S is the mapping defined by $S = kI + (1-k)T$.
(3) T is quasi–nonexpansive. S is the mapping defined by $S = T$.

Suppose there is a sequence $\{u_n\}$ in C such that

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for} \quad n \in N.$$

Then, $\{u_n\}$ converges strongly to some $u \in F(T)$.

Consider the compact and convex set $C = \{(x_1, x_2) \in R^2 : x_1, x_2 \in [0,1],\ x_1 + x_2 \leq 1\}$ of 2-dimensional Euclidean space R^2. Let T be the self–mapping on C defined by

$$T(x_1, x_2) = \left(\tfrac{1}{2}(1 + x_1 - x_2)\,,\, x_2\right) \quad \text{for} \quad (x_1, x_2) \in C.$$

Let $u_1 \in C$ and $\{u_n\}$ be the sequence generated by $u_{n+1} = (u_n + Tu_n)/2$ for $n \in N$.

Under this setting, we can easily verify $F(T) = \{(x_1, x_2) \in C : x_1 + x_2 = 1\}$,

$$A(T) = \{(x_1, x_2) \in R^2 : x_1 \geq 1\}, \qquad A(T) \cap C = \mathrm{A}(T) \cap C = \{(1,0)\}.$$

Since $F(T) \not\subset \mathrm{A}(T)$, T is not hemi–contractive. However, it is obvious that $\{u_n\}$ converges to a fixed point. For such mappings, we did not have convergence theorems. Here, we give a convergence theorem [2] for such mappings.

Theorem 3.3 ([2]). *Let $\{a_n\}$ be a sequence in $[a,b] \subset (0,1)$. Let C be a compact and convex subset of a Hilbert space H. Let T be a continuous self-mapping on C with $A(T) \neq \emptyset$. Let $u_1 \in C$ and $\{u_n\}$ be a sequence defined by*

$$u_{n+1} = a_n u_n + (1 - a_n) T u_n \qquad \text{for } n \in N.$$

Suppose $F(T) \subset P_C(A(T))$, where P_C is the metric projection of H onto C. Then, $\{u_n\}$ converges strongly to some $u \in F(T)$.

We consider weak convergence theorems in the case $A(T) \neq \emptyset$ and $F(T) \subset A(T)$. To have the following results, we have to assume demiclosedness at 0 of $I - T$ (see [2]).

Theorem 3.4 ([2]). *Let $\{a_n\}$ be a sequence in $[a,b] \subset (0,1)$. Let C be a weakly compact subset of a Hilbert space H. Let S be a self-mapping on C such that $F(S) \subset A(S)$, $A(S) \neq \emptyset$ and $I - S$ is demiclosed at 0. Suppose there is a sequence in C such that*

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(S)$.

We also have the following theorem by Theorem 3.4 (see [2]).

Theorem 3.5 ([2]). *Let $\{a_n\}$ be a sequence in $[a,b] \subset (0,1)$. Let C be a weakly compact subset of a Hilbert space H and let T be a self-mapping on C such that $I - T$ is demiclosed at 0. Assume one of the followings.*

(1) T is hemi-contractive with $A(T) \neq \emptyset$. Define the mapping S by $S = T$.
(2) T is k-demi-contractive. Define the mapping S by $S = kI + (1-k)T$.
(3) T is quasi-nonexpansive. Define the mapping S by $S = T$.

Suppose there is a sequence $\{u_n\}$ in C such that

$$u_{n+1} = a_n u_n + (1 - a_n) S u_n \qquad \text{for } n \in N.$$

Then, $\{u_n\}$ converges weakly to some $u \in F(T)$.

4. Nonlinear ergodic theorems

We begin this section with presenting Theorem 4.1 due to Takahashi and Takeuchi [9]. Then, we also have Theorem 4.2 (see [2]).

Theorem 4.1. *Let C be a non-empty bounded subset of a Hilbert space H. Let S be a nonexpansive self-mapping on C. Let $v_1 \in C$ and let $\{v_n\}$, $\{b_n\}$ be sequences defined by*

$$v_{n+1} = S v_n, \quad b_n = \tfrac{1}{n} \textstyle\sum_{t=1}^{n} v_t \qquad \text{for } n \in N.$$

Then the followings hold.

(1) $A(S)$ is non-empty closed and convex.
(2) $\{b_n\}$ converges weakly to some $u \in A(S)$.

We also have the following theorem.

Theorem 4.2 ([2]). *Let $k \in [0,1)$. Let C be a non–empty bounded subset of a Hilbert space H. Let T be a k–strictly pseudo-contractive self-mapping on C. Let S be the mapping defined by $Sx = (kI + (1-k)T)x$ for $x \in C$. Assume that S is a self mapping on C. Let $v_1 \in C$ and let $\{v_n\}$, $\{b_n\}$ be sequences defined by*

$$v_{n+1} = Sv_n, \quad b_n = \frac{1}{n}\sum_{t=1}^{n} v_t \qquad \text{for} \quad n \in N.$$

Then the followings hold.

(1) $\mathrm{A}_k(T)$ is non-empty closed and convex.

(2) $\{b_n\}$ converges weakly to some $u \in \mathrm{A}_k(T)$.

Further, if C is weakly closed then the followings hold.

(3) $F(T)$ is non-empty and weakly closed.

(4) $\{b_n\}$ converges weakly to $u \in F(T)$.

Remark 4.3. *In Theorem 4.2, convexity of C is not always necessary. We give an example. Let Q be the set of rational numbers, $k \in [0,1) \cap Q$ and $C = (0,1) \cap Q$. For any self–mapping T on C, S is also a self–mapping. However, C is not convex.*

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