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測地距離空間における近似点列の計算誤差

Calculation errors of the iterative sequence in a geodesic space

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1 Introduction

Let X be a metric space and let T be a mapping of X into itself. The fixed point problem for T is to find a point $z\in X$ satisfying that $z=Tz$. The study of approximation methods for a fixed point of a mapping is a central topic of fixed point theory as well as the study of existence of a fixed point.

In particular, the shrinking projection method, which was proved by Takahashi, Takeuchi, and Kubota [12] in 2008, was proposed as a new projection method for the approximation of a common fixed point of a family of nonexpansive mappings defined on a Hilbert space.

Since then, there have been a number of generalization. For example, Takahashi and Zembayashi [13] applied this method to solve an equilibrium problem defined on a Banach space. Plubtieng and Ungchittrakool [10] proposed an approximation method for a finite family of relatively nonexpansive mappings using their convex combination. Qin, Cho, and Kang [11] shows an approximate sequence converging to a common solution to equilibrium problems and fixed point problems.

The technique to prove this type of convergence theorem was improved by two papers, Kimura, Nakajo, and Takahashi [5], Kimura and Takahashi [8]. In these papers, they use the concept of Mosco convergence [9] and obtain the convergence theorem for a mapping defined on a reflexive Banach space with certain differentiability and convexity conditions of the norm.

The shrinking projection method has also been applied to complete geodesic spaces; to Hadamard spaces by Kimura [2] and to complete CAT(1) spaces by Kimura and Satô [6].

In this short note, we discuss the shrinking projection method containing calculation errors, which has been studied by the author. We survey the recent results proved with various types of underlying spaces. Further, by investigating from a different point of view, we propose several deduced results from these theorems.

2 Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $c : [0, d(x, y)] \to X$ is called a geodesic with endpoints x, y if $c(0) = x$, $c(l) = y$, and $d(c(t), c(s)) = |t - s|$ for every $t, s \in [0, l]$. Let $r \in]0, \infty]$. We say X is r-geodesic if a geodesic with endpoints x, y exists for every $x, y \in X$ with $d(x, y) < r$. If such a geodesic is unique for each pair of points, then X is said to be r-uniquely geodesic.

In this paper, we only consider that every geodesic between two points is unique.

A geodesic segment joining x and y is defined as the image of a geodesic c with endpoints $x, y \in X$ We denote it by $[x, y]$. A subset C of a r-uniquely geodesic space X is said to be r-convex if for every $x, y \in C$ with $d(x, y) < r$, a geodesic segment $[x, y]$ is included in C. If C is r-convex for every $r > 0$, we say that C is convex.

For $x, y, z \in X$, a geodesic triangle $\triangle(x, y, z)$ is a subset of X defined by the union of $[y, z]$, $[z, x]$, and $[x, y]$.

For $\kappa \in \mathbb{R}$, we define the two-dimensional model space M_κ^2 with the curvature κ by

$$M_\kappa^2 = \begin{cases} \dfrac{1}{\sqrt{-\kappa}}\mathbb{H}^2 & (\kappa < 0), \\ \mathbb{R}^2 & (\kappa = 0), \\ \dfrac{1}{\sqrt{\kappa}}\mathbb{S}^2 & (\kappa > 0), \end{cases}$$

where $\mathbb{R}^2$ is the Euclidean space with the metric induced from the Euclidean norm, $\mathbb{S}^2$ is the two-dimensional unit sphere in $\mathbb{R}^3$ whose metric is a length of a minimal great arc joining each two points, and $\mathbb{H}^2$ is the two-dimensional hyperbolic space with the metric defined by a usual hyperbolic distance.

The diameter of M_κ^2 is denoted by D_κ, that is, $D_\kappa = \infty$ if $\kappa \le 0$, and $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$. We know that M_κ^2 is a D_κ-uniquely geodesic space for any $\kappa \in \mathbb{R}$.

For $\triangle(x, y, z)$ in a geodesic space X satisfying that $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, there exist points $\overline{x}, \overline{y}, \overline{z} \in M_\kappa^2$ such that

$$d(x, y) = d_{M_\kappa^2}(\overline{x}, \overline{y}), \ d(y, z) = d_{M_\kappa^2}(\overline{y}, \overline{z}), \text{ and } d(z, x) = d_{M_\kappa^2}(\overline{z}, \overline{x}).$$

We call the triangle $\triangle(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa^2$ a comparison triangle of $\triangle(x, y, z)$. It is unique up to an isometry of M_κ^2. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\overline{x}, \overline{p})$.

Let X be a D_κ-geodesic space for $\kappa \in \mathbb{R}$. If for any $x, y, z \in X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, for any $p, q \in \triangle(x, y, z)$, and for their comparison points $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$, the inequality $d(p, q) \le d_{M_\kappa^2}(\overline{p}, \overline{q})$ holds, then X is called a CAT(κ) space.

Let C be a nonempty closed D_κ-convex subset in a complete CAT(κ) space X. Then, for $x \in X$ satisfying that $d(x,C) = \inf_{y\in C} d(x,y) < D_\kappa/2$, there exists a unique $y_x \in C$ such that $d(x,y_x) = d(x,C)$. We define a mapping $P_C : X \to C$ by $P_C x = y_x$ for $x \in X$ and we call it the metric projection of X onto C. It is known that P_C is quasinonexpansive, that is, $d(P_C x, z) \le d(x,z)$ for every $x \in X$ and $z \in C$.

A mapping $T : X \to X$ is said to be nonexpansive if $d(Tx,Ty) \le d(x,y)$ for every $x,y \in X$. It is easy to see that if X is CAT(κ) space with $d(u,v) < D_\kappa/2$ for every $u,v \in X$, then $F(T)$ is closed and convex. For such X, a metric projection $P_C : X \to X$ is nonexpansive whenever $\kappa \le 0$. On the other hand, P_C is not necessarily nonexpansive if $\kappa > 0$.

For more details about CAT(κ) spaces, see [1].

3 Approximate sequences on complete geodesic spaces

We begin with the following result proved by the author [4].

Theorem 1 (Kimura [4]). *Let X be a complete* CAT(0) *space and suppose that a subset $\{z \in X : d(v,z) \le d(u,z)\}$ is convex for every $u,v \in X$. Let $T : X \to X$ be a nonexpansive mapping such that the set $F(T)$ of fixed points is nonempty. Let $\{\epsilon_n\}$ be a sequence of nonnegative numbers and $\epsilon_0 = \limsup_{n\to\infty} \epsilon_n$. For a given point $x_0 \in X$, generate a sequence $\{x_n\}$ as follows: $C_1 = X$, $x_1 \in C_1$, and*

$$C_{n+1} = \{z \in X : d(Tx_n, z) \le d(x_n, z)\} \cap C_n,$$
$$x_{n+1} \in C_{n+1} \text{ such that } d(x_0, x_{n+1})^2 \le d(x_0, C_{n+1})^2 + \epsilon_{n+1}$$

for each $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} d(x_n, Tx_n) \le 2\sqrt{\epsilon_0}.$$

Moreover, if $\epsilon_0 = 0$, then $\{x_n\}$ converges to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection of X onto $F(T)$.

This theorem shows that the iterative scheme still has sufficient property to approximate a fixed point even if calculation errors occur for each time to compute the values of metric projections. Moreover, we do not assume any summability conditions of the error terms, which is a very important property for numerical experiments by the computer.

On the other hand, this theorem can be applied to another type of shrinking projection method, which has a perturbation at the anchor point x_0.

Theorem 2. *Let X be a bounded* CAT(0) *space with the diameter $D \ge 0$ and let T and x_0 be the same as in Theorem 1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences with $\alpha_0 = \limsup_{n\to\infty} \alpha_n$ and $\beta_0 = \limsup_{n\to\infty} \beta_n$. Let $\{u_n\}$ be a sequence in X such that $d(x_0, u_n) \le \alpha_n$ for $n \in \mathbb{N}$. Generate an iterative sequence $\{y_n\} \subset X$ as follows:*

$C_1 = X$, $y_1 \in C_1$ and

$$\begin{aligned}
&C_{n+1} = \{z \in X : d(Ty_n, z) \le d(y_n, z)\} \cap C_n, \\
&y_{n+1} \in C_{n+1} \text{ such that } d(u_{n+1}, y_{n+1})^2 \le d(u_{n+1}, C_{n+1})^2 + \beta_{n+1}^2,
\end{aligned}$$

for each $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} d(y_n, Ty_n) \le 2\sqrt{(2D + \alpha_0 + \beta_0)(\alpha_0 + \beta_0)}.$$

Moreover, if $\alpha_0 = \beta_0 = 0$, then $\{y_n\}$ converges to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection of X onto $F(T)$.

Proof. To apply Theorem 1 with the iterative scheme $\{y_n\}$, we show that each y_{n+1} satisfies

$$d(x_0, y_{n+1})^2 \le d(x_0, C_{n+1})^2 + \epsilon_{n+1}.$$

for some ϵ_n. For $\tau \in]0, 1[$ and $n \in \mathbb{N}$, let $w_n = \tau y_n \oplus (1-\tau)P_{C_n}u_n \in C_n$. Then we have that

$$\begin{aligned}
d(u_n, P_{C_n}u_n)^2 &\le d(u_n, w_n)^2 \\
&\le \tau d(u_n, y_n)^2 + (1-\tau)d(u_n, P_{C_n}u_n)^2 - \tau(1-\tau)d(y_n, P_{C_n}u_n)
\end{aligned}$$

and thus, for $n \in \mathbb{N} \setminus \{1\}$, we have

$$(1-\tau)d(y_n, P_{C_n}u_n)^2 \le d(u_n, y_n)^2 - d(u_n, P_{C_n}u_n)^2 \le \beta_n^2.$$

Tending $\tau \downarrow 0$, we get $d(y_n, P_{C_n}u_n) \le \beta_n$. Since every metric projection onto a nonempty closed convex subset of a complete CAT(κ) space is nonexpansive, we have that

$$\begin{aligned}
d(x_0, y_{n+1}) &\le d(x_0, P_{C_{n+1}}x_0) + d(P_{C_{n+1}}x_0, P_{C_{n+1}}u_{n+1}) + d(P_{C_{n+1}}u_{n+1}, y_{n+1}) \\
&\le d(x_0, C_{n+1}) + d(x_0, u_{n+1}) + \beta_{n+1} \\
&\le d(x_0, C_{n+1}) + \alpha_{n+1} + \beta_{n+1}.
\end{aligned}$$

Thus, letting $\epsilon_n = \sqrt{(2D + \alpha_n + \beta_n)(\alpha_n + \beta_n)}$ for $n \in \mathbb{N}$, we have that

$$\begin{aligned}
d(x_0, y_{n+1})^2 &\le (d(x_0, C_{n+1}) + \alpha_{n+1} + \beta_{n+1})^2 \\
&\le d(x_0, C_{n+1})^2 + (2d(x_0, C_{n+1}) + \alpha_{n+1} + \beta_{n+1})(\alpha_{n+1} + \beta_{n+1}) \\
&\le d(x_0, C_{n+1})^2 + \epsilon_{n+1}^2.
\end{aligned}$$

Hence we obtain from Theorem 1 that

$$\begin{aligned}
\limsup_{n\to\infty} d(y_n, Ty_n) &\le 2\limsup_{n\to\infty} \epsilon_n \\
&= 2\sqrt{(2D + \alpha_0 + \beta_0)(\alpha_0 + \beta_0)}.
\end{aligned}$$

The remainder part of the theorem is also obtained by Theorem 1. □

For a complete CAT(1) space, we can prove the following result. The method for the proof is essentially obtained in [7].

Theorem 3 (Kimura-Satô [7]). *Let X be a complete* CAT(1) *space such that $D = \operatorname{diam} X < \pi/2$ and that a subset $\{z \in X : d(v,z) \leq d(u,z)\}$ is convex for every $u, v \in X$. Let $T : X \to X$ be a nonexpansive mapping such that the set of its fixed points $F(T)$ is nonempty. Let $\{\delta_n\}$ be a sequence in $[0,\infty[$ and let $\delta_0 = \limsup_{n\to\infty} \delta_n$. For a given point $x_0 \in X$, generate a sequence $\{x_n\}$ as follows: $C_1 = X$, $x_1 \in C_1$, and*

$$\begin{aligned} &C_{n+1} = \{z \in X : d(Tx_n, z) \leq d(x_n, z)\} \cap C_n, \\ &x_{n+1} \in C_{n+1} \text{ such that } d(u, x_{n+1}) \leq d(u, C_{n+1}) + \delta_{n+1}, \end{aligned}$$

for each $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} d(x_n, Tx_n) \leq 2 \arccos(e^{-\delta_0 \tan D}).$$

if $\delta_0 = 0$, then $\{x_n\}$ converges to $P_{F(T)}x_0 \in X$.

In a complete CAT(1) space X, a metric projection is not necessarily nonexpansive even if $d(u,v) < \pi/2$ for every $u, v \in X$. Therefore, some part of the technique in the proof of Theorem 1 is not valid. However, we can show the following convergence theorem by using the similar way as above.

Theorem 4. *Let X be a complete* CAT(1) *space and suppose the same conditions for X as in Theorem 3. Let T and x_0 be the same as in Theorem 3. Let $\{u_n\}$ be a sequence in X converging to x_0 and generate an iterative sequence $\{y_n\} \subset X$ as follows: $C_1 = X$, $y_1 \in C_1$, and*

$$\begin{aligned} C_{n+1} &= \{z \in X : d(Ty_n, z) \leq d(y_n, z)\} \cap C_n, \\ y_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for each $n \in \mathbb{N}$. Then $\{y_n\}$ converge to $P_{F(T)}x_0$.

4 Related results

An analogous iterative method shown in Theorems 1 and 3 can be applied with the case of Banach spaces. We omit to define several notions shown in the following theorem. For the details of their exact definitions, see [3].

Theorem 5 (Kimura [3]). *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty bounded closed convex subset of E and $r \in]0,\infty[$ such that $C \subset B_r$. Let $T : C \to E$ be such that $\phi(z, Tx) \leq \phi(z, x)$ for every $x \in C$ and $z \in F(T) \neq \emptyset$. Let $\{\delta_n\}$ be a bounded nonnegative real sequence and let $\delta_0 =$*

$\limsup_{n\to\infty} \delta_n$. *For a given point* $u \in E$, *generate a sequence* $\{x_n\}$ *by the following way:* $x_1 \in C$, $C_1 = C$, *and*

$$C_{n+1} = \{z \in C : \phi(z, Tx_n) \leq \phi(z, x_n)\} \cap C_n,$$
$$x_{n+1} \in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1}$$

for $n \in \mathbb{N}$. *Then,*

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq 2g_r^{-1}\left(\frac{1}{2}\delta_0 + \frac{1}{2}g_r^*(g_r^{-1}(\delta_0))\right).$$

Moreover, if $\delta_0 = 0$ *and* $I - T$ *is closed at zero, then* $\{x_n\}$ *converges strongly to* $P_{F(T)}u$.

As a direct result, we obtain the following convergence theorem of another type of iterative sequence.

Theorem 6. *Let* E, C, r, *and* x_0 *be the same as in Theorem 5. Let* $T : C \to E$ *be such that* $\phi(z, Tx) \leq \phi(z, x)$ *for every* $x \in C$ *and* $z \in F(T) \neq \emptyset$. *Suppose that* $I - T$ *is closed at zero. Let* $\{u_n\}$ *be a sequence in* E *converging to* x_0 *and generate an iterative sequence* $\{y_n\} \subset C$ *as follows:* $y_1 \in C$, $C_1 = C$, *and*

$$C_{n+1} = \{z \in E : \phi(z, Ty_n) \leq \phi(z, y_n)\} \cap C_n,$$
$$y_{n+1} = P_{C_{n+1}} u_{n+1}$$

for each $n \in \mathbb{N}$. *Then* $\{y_n\}$ *converge to* $P_{F(T)}x_0$.

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