数理解析研究所講究録
第2011巻 2016年 178-184

順序集合における不動点定理の非整数階微分方程式境界値問題への適用例

Applied results of a fixed point theorem in partially ordered sets to fractional order boundary value problems

豊田昌史 ‡　渡辺俊一 *

Masashi Toyoda　Toshikazu Watanabe

‡194–8610 東京都町田市玉川学園 6–1–1 玉川大学工学部
College of Engineering, Tamagawa University, 6–1–1 Tamagawa-gakuen, Machida-shi, Tokyo 194–8610
*101-8308 東京都千代田区神田駿河台 1-8-14 日本大学理工学部
College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo, 101-8308

1 Introduction

Let X be a partially ordered set with a metric d and let T be a mapping from X into itself. We say that T is monotone nondecreasing if for any $x, y \in X$, $x \leq y$ implies $Tx \leq Ty$. Nieto and López [3] consider the following fixed point theorem in partially ordered sets.

Theorem 1. *Let X be a partially ordered set with a metric d such that (X, d) is a complete metric space. If a nondecreasing sequence $\{x_n\}$ converges to x, then we have $x_n \leq x$ for any n. Let T be a monotone nonincreasing mapping from X into itself such that there exists $k \in [0,1)$ such that for any $x, y \in X$,*

$$x \geq y \text{ implies } d(Tx, Ty) \leq kd(x, y).$$

Assume that there exists $x_0 \in X$ with $x_0 \leq Tx_0$. Then there exists a fixed point of T. Moreover, if for any $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.

In this paper, using Theorem 1, we show the existence and uniqueness of solutions of fractional order boundary value problems.

2 Riemann-Liouville fractional derivative and integral

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function u of $(0, \infty)$ into $\mathbb{R}$ is given by

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}ds$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α and $\Gamma(\alpha)$ denotes the gamma function. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function u of $(0, \infty)$ into $\mathbb{R}$ is defined by

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds.$$

For the proof of Lemmas 2 and 3, we use the following: For $p, q > 0$ and $a \in \mathbb{R}$,

$$\int_a^t (t-s)^{p-1}(s-a)^{q-1}ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}(t-a)^{p+q-1}. \tag{1}$$

In fact, we have (1) since

$$\begin{aligned}
\int_a^t (t-s)^{p-1}(s-a)^{q-1}ds &= \int_0^{t-a} (t-a-\tau)^{p-1}\tau^{q-1}d\tau \\
&= \int_0^1 (t-a-(t-a)u)^{p-1}(t-a)^{q-1}u^{q-1}(t-a)du \\
&= (t-a)^{p+q-1}\int_0^1 (1-u)^{p-1}u^{q-1}du \\
&= (t-a)^{p+q-1}B(p,q) \\
&= (t-a)^{p+q-1}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.
\end{aligned}$$

For $\alpha, \beta > 0$, we have

$$I_{0+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \tag{2}$$

In fact, by (1), we have

$$I_{0+}^{\alpha}t^{\beta} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\beta}ds = \frac{1}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.$$

Moreover for $\beta > \alpha > 0$, we have

$$D_{0+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}. \tag{3}$$

In fact, since $D^n_{0+}t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-n)}t^{\beta-n}$ for $n = 1, 2, 3, \ldots, [\beta]$, by (1), we have

$$\begin{aligned}
D^\alpha_{0+}t^\beta &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} s^\beta ds \\
&= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \left(\frac{\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} t^{n-\alpha+\beta} \right) \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \cdot \frac{d^n}{dt^n} t^{n-\alpha+\beta} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \cdot \frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}.
\end{aligned}$$

Lemma 2. *Let $\alpha > 0$. If $u(t) = t^{\alpha-n}$ $(n = 1, 2, \ldots, [\alpha]+1)$, then $D^\alpha_{0+}u = 0$. Conversely, if $D^\alpha_{0+}u(t) = 0$, then there exists $C_1, C_2, \ldots, C_n \in \mathbb{R}$ such that*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

where $n = [\alpha] + 1$.

Proof. Let $n = 1, 2, \ldots, [\alpha]+1$ and $u(t) = t^{\alpha-n}$. By (1), we have

$$\begin{aligned}
D^\alpha_{0+}u(t) &= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} s^{\alpha-n} ds \\
&= \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \left(\frac{\Gamma(n-\alpha)\Gamma(\alpha-n+1)}{\Gamma(1)} \right) \\
&= 0.
\end{aligned}$$

Conversely, assume that $D^\alpha_{0+}u(t) = 0$. Then we have

$$\frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = 0.$$

Since $\frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = 0$, we have

$$\frac{d^{n-1}}{dt^{n-1}} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1.$$

Moreover we have

$$\frac{d^{n-2}}{dt^{n-2}} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1 t + C_2.$$

Similarly we obtain that

$$\frac{d^{n-3}}{dt^{n-3}} \int_0^t (t-s)^{n-\alpha-1} u(s) ds = C_1 t^2 + C_2 t + C_3$$

where we change $\frac{C_1}{2}$ by C_1. Hence we obtain that

$$\int_0^t (t-s)^{n-\alpha-1} u(s)ds = C_1 t^{n-1} + C_2 t^{n-2} + \cdots + C_n$$

for some $C_1, C_2, \ldots, C_n$. By (1), we have

$$\begin{aligned} I_{0+}^{\alpha} \int_0^t (t-s)^{n-\alpha-1} u(s)ds &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left(\int_0^s (s-\tau)^{n-\alpha-1} u(\tau)d\tau \right) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t u(\tau) \left(\int_\tau^t (t-s)^{\alpha-1} (s-\tau)^{n-\alpha-1} ds \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)\Gamma(n-\alpha)}{\Gamma(n)} (t-\tau)^{n-1} \\ &= \frac{\Gamma(n-\alpha)}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} u(\tau)d\tau \\ &= \Gamma(n-\alpha) I_{0+}^n u(t). \end{aligned}$$

Since

$$I_{0+}^{\alpha}(C_1 t^{n-1} + C_2 t^{n-2} + \cdots + C_n) = C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \cdots + C_n t^{\alpha},$$

we have

$$\Gamma(n-\alpha) I_{0+}^n u(t) = C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \cdots + C_n t^{\alpha}.$$

Hence, for some $C_1, C_2, \ldots, C_n$, we obtain that

$$I_{0+}^n u(t) = C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \cdots + C_n t^{\alpha}.$$

Since $D_{0+}^n I_{0+}^n u(t) = u(t)$ and $D_{0+}^n(C_1 t^{\alpha+n-1} + C_2 t^{\alpha+n-2} + \cdots + C_n t^{\alpha}) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$, we have

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}.$$

□

The following lemma can be found in [4].

Lemma 3. *Let $\alpha > 0$ and let $u \in L(a,b)$. Then we have*

$$D_{0+}^{\alpha} I_{0+}^{\alpha} u = u.$$

Proof. Let $n = [\alpha] + 1$. By (1), we have

$$\begin{aligned} D_{0+}^{\alpha} I_{0+}^{\alpha} u &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_0^x \frac{1}{(t-s)^{\alpha-n+1}} \left(\int_\tau^t (s-\tau)^{\alpha-1} u(\tau)d\tau \right) ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_0^t \left(\int_\tau^t (t-s)^{n-\alpha-1} (s-\tau)^{\alpha-1} u(\tau)ds \right) d\tau \\ &= \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_0^t u(\tau) \left(\int_\tau^t (t-s)^{n-\alpha-1} (s-\tau)^{\alpha-1} ds \right) d\tau \\ &= \frac{1}{\Gamma(n)} \cdot \frac{d^n}{dx^n} \int_0^t (t-\tau)^{n-1} u(\tau)d\tau. \end{aligned}$$

Since $\int_0^t \left(\int_0^t \cdots \left(\int_0^t u(s)ds\right) \cdots ds\right) ds = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds$, we have $D_{0+}^\alpha I_{0+}^\alpha u = u$. □

The following lemma can be found in [1]. See also [2].

Lemma 4. *Let $\alpha > 0$. Let $u \in C(0,1) \cap L(0,1)$ satisfying $D_{0+}^\alpha u \in C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$$

for some $C_1, C_2, \ldots, C_n \in \mathbb{R}$ and $n = [\alpha] + 1$.

Proof. By Lemma 2, we have $D_{0+}^\alpha(I_{0+}^\alpha D_{0+}^\alpha u - u) = D_{0+}^\alpha I_{0+}^\alpha D_{0+}^\alpha u - D_{0+}^\alpha u = D_{0+}^\alpha u - D_{0+}^\alpha u = 0$. By Lemma 3, there exists $C_1, C_2, \ldots, C_n \in \mathbb{R}$ such that $I_{0+}^\alpha D_{0+}^\alpha u(t) - u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}$. □

3 Applied results to fractional order boundary value problems

Using Lemma 4, we obtain the following [1]. For the sake of completeness, we show the proof.

Lemma 5. *Let $h \in C[0,1]$ and $1 < \alpha \leq 2$. Then the unique solution of the problem*

$$\begin{cases} D_{0+}^\alpha u(t) + h(t) = 0, \\ u(0) = u(1) = 0 \end{cases}$$

is

$$u(t) = \int_0^1 G_\alpha(t,s)h(s)ds$$

where

$$G_\alpha(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\left(t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\right) & (0 \leq s \leq t \leq 1), \\ \dfrac{1}{\Gamma(\alpha)}\left(t^{\alpha-1}(1-s)^{\alpha-1}\right) & (0 \leq t \leq s \leq 1). \end{cases}$$

Proof. By Lemma 4, we have

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}$$

for some $C_1, C_2 \in \mathbb{R}$. By $u(0) = 0$, we have $C_2 = 0$. Moreover, by $u(1) = 0$, we have $C_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds$. Thus we obtain that $u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}t^{\alpha-1}h(s)ds = \int_0^1 G_\alpha(t,s)h(s)ds$. □

Using Lemma 5, we obtain the following.

Theorem 6. *Let f be a mapping of $[0,1]\times[0,\infty)$ into $[0,\infty)$ such that f is continuous and nondecreasing with respect to second argument. Let $1<\alpha,\beta\le 2$. Assume that there exists $\lambda\in[0,\Gamma(\alpha)\Gamma(\beta))$, for any $u,v\in[0,\infty)$ with $u\ge v$ and $t\in[0,1]$,*

$$0\le f(t,u)-f(t,v)\le\lambda(u-v).$$

Then the problem

$$\begin{cases} D_{0+}^{\beta}(D_{0+}^{\alpha}u(t))=f(t,u(t)), \\ u(0)=u(1)=(D_{0+}^{\alpha}u)(0)=(D_{0+}^{\alpha}u)(1)=0 \end{cases} \tag{4}$$

has a unique nonnegative solution.

Proof. We first show that the unique solution of the problem (4) is

$$u(t)=\int_0^1 G_\alpha(t,s)\left(\int_0^1 G_\beta(s,r)f(r,u(r))dr\right)ds$$

where G_α is the function in Lemma 5. In fact, let $y(t)=-D_{0+}^{\alpha}u(t)$. Then the problem

$$\begin{cases} D_{0+}^{\beta}(D_{0+}^{\alpha}u(t))=f(t,u(t)), \\ (D_{0+}^{\alpha}u)(0)=(D_{0+}^{\alpha}u)(1)=0 \end{cases}$$

is equal to the problem

$$\begin{cases} D_{0+}^{\beta}y(t)+f(t,u(t))=0, \\ y(0)=y(1)=0. \end{cases}$$

By Lemma 5, we have the unique solution

$$y(t)=\int_0^1 G_\beta(t,s)f(s,u(s))ds,$$

that is,

$$D_{0+}^{\alpha}u(t)+\int_0^1 G_\beta(t,s)f(s,u(s))ds=0.$$

Furthermore, by Lemma 5, the problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+\displaystyle\int_0^1 G_\beta(t,s)f(s,u(s))ds=0, \\ u(0)=u(1)=0 \end{cases}$$

has the unique solution

$$u(t)=\int_0^1 G_\alpha(t,s)\left(\int_0^1 G_\beta(s,r)f(r,u(r))dr\right)ds.$$

Let $X = \{u \in C[0,1] \mid u(t) \geq 0\}$. Then (X, d) is a complete metric space where d is defined by $d(u,v) = \sup_{0 \leq t \leq 1} |u(t) - v(t)|$ for $u, v \in X$. We define a mapping T of X by

$$(Tu)(t) = \int_0^1 G_\alpha(t,s) \left(\int_0^1 G_\beta(s,r) f(r, u(r)) dr \right) ds$$

for $u \in X$. Using Theorem 1, we obtain the unique fixed point of T. This is the unique solution of (4). For more details, see [5]. □

In the case that $\alpha = \beta = 2$ in Theorem 6, we have the following.

Corollary 7. *Let f be a mapping of $[0,1] \times [0,\infty)$ into $[0,\infty)$ such that f is continuous and nondecreasing with respect to second argument. Assume that there exists $\lambda \in [0,1)$, for any $u, v \in [0,\infty)$ with $u \geq v$ and $t \in [0,1]$,*

$$0 \leq f(t,u) - f(t,v) \leq \lambda(u - v).$$

Then the problem

$$\begin{cases} u'''' = f(t, u(t)), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

has a unique nonnegative solution.

References

[1] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Journal of Mathematical Analysis and Applications, 311(2005), 495–505.

[2] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, In North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.

[3] J J. Nieto and R. R. López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, 22(2005), 223–239.

[4] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers,Switzerland, 1993.

[5] M. Toyoda and T. Watanabe, *Existence and uniqueness theorem for fractional order differential equations with boundary conditions and two fractional order*, to appear in Journal of Nonlinear and Convex Analysis.