

Several generalizations and applications of incomplete poly-Bernoulli polynomials and incomplete poly-Cauchy polynomials

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1 Introduction

The poly-Bernoulli polynomials $B_n^{(\mu)}(x)$ ([6]) are defined by the generating function

$$\frac{\text{Li}_\mu(1 - e^{-t})}{1 - e^{-t}} e^{-tx} = \sum_{n=0}^{\infty} B_n^{(\mu)}(x) \frac{t^n}{n!}, \tag{1}$$

where

$$\text{Li}_\mu(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^\mu}$$

is the μ -th polylogarithm function. Note that e^{-tx} on the left-hand side is replaced by e^{tx} in [1, 2]. When $x = 0$, $B_n^{(\mu)}(0) = B_n^{(\mu)}$ are the poly-Bernoulli numbers ([11]). When $x = 0$ and $\mu = 1$, $B_n^{(1)}(0) = B_n$ are the classical Bernoulli numbers, defined by the generating function

$$\frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The generating function of poly-Cauchy polynomials $c_n^{(\mu)}(x)$ ([12, Theorem 2]) is given by

$$\frac{\text{Lif}_\mu(\log(1 + t))}{(1 + t)^x} = \sum_{n=0}^{\infty} c_n^{(\mu)}(x) \frac{t^n}{n!}, \tag{2}$$

where

$$\text{Lif}_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(n + 1)^\mu}$$

is the *polylogarithm factorial* or *polyfactorial* function. Note that $(1+t)^x$ on the left-hand side is replaced by $(1+t)^{-x}$ in [10]. When $x = 0$, $c_n^{(\mu)}(0) = c_n^{(\mu)}$ are the poly-Cauchy numbers $c_n^{(\mu)}$ ([12, Theorem 2]), given by

$$\text{Lif}_\mu(\log(1+t)) = \sum_{n=0}^{\infty} c_n^{(\mu)} \frac{t^n}{n!}. \quad (3)$$

When $x = 0$ and $\mu = 1$, $c_n^{(1)}(0) = c_n$ are the classical Cauchy numbers, defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}. \quad (4)$$

In this paper, by using the restricted and associated Stirling numbers of the first kind, we define the restricted and associated poly-Cauchy polynomials. By using the restricted and associated Stirling numbers of the second kind, we define the restricted and associated poly-Bernoulli polynomials. These polynomials are generalizations of original poly-Cauchy polynomials and original poly-Bernoulli polynomials, respectively. We also study their characteristic and combinatorial properties.

2 Restricted and associated Stirling numbers of the second kind

In place of the classical Stirling numbers $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ we substitute the restricted Stirling numbers and the associated Stirling numbers, denoted by

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m} \quad \text{and} \quad \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\geq m}.$$

Some combinatorial and modular properties of these numbers can be found in [21], and other properties can be found in the papers from the list of references of [21]. The generating functions of these numbers are given by

$$\sum_{n=k}^{mk} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{\leq m} \frac{x^n}{n!} = \frac{1}{k!} (E_m(x) - 1)^k \quad (5)$$

and

$$\sum_{n=mk}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\geq m} \frac{x^n}{n!} = \frac{1}{k!} (e^x - E_{m-1}(x))^k \quad (6)$$

respectively, where

$$E_m(t) = \sum_{k=0}^m \frac{t^k}{k!}$$

is the m th partial sum of the exponential function. These give the number of the k -partitions of an n -element set, such that each block contains at most or at least m elements, respectively. Notice that as particular cases, these numbers when $m = 2$ have been considered by several authors (e.g., [5, 8, 22, 23]). Since the generating function of $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is given by

$$\sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!} \quad (7)$$

(see e.g., [9]), by $E_{\infty}(x) = e^x$ and $E_0(x) = 1$, we have

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\leq \infty} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{\geq 1} = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$$

3 Restricted and associated Stirling numbers of the first kind

Denote by $\left[\begin{matrix} n \\ k \end{matrix} \right]$ the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{i=0}^n \left[\begin{matrix} n \\ i \end{matrix} \right] x^i.$$

the generating function of $\left[\begin{matrix} n \\ k \end{matrix} \right]$ is given by

$$\sum_{n=k}^{\infty} \left[\begin{matrix} n \\ k \end{matrix} \right] \frac{x^n}{n!} = \frac{(-\log(1-x))^k}{k!}.$$

In place of the classical (unsigned) Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$, we substitute the (unsigned) restricted Stirling numbers of the first kind and the (unsigned) associated Stirling numbers of the first kind, denoted by

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\leq m} \quad \text{and} \quad \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\geq m},$$

respectively. The associated Stirling number of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\geq m}$ equals the number of permutations of a set N ($|N| = n$) with k orbits such that each block contains at least m elements ([5, p.256–257], [23]). The restricted Stirling number of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\leq m}$ equals the number of permutations of n with k orbits such that each block contains at most m elements.

For $k \geq 1$ and $m \geq 1$, the generating functions of the restricted Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\leq m}$ and the associated Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\geq m}$ are given by the following ([4, p. 467, (12.8)]). Denote the m th partial sum of the logarithm function by

$$F_m(t) = \sum_{k=1}^m (-1)^{k+1} \frac{t^k}{k}. \quad (8)$$

$$\sum_{n=0}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\leq m} \frac{x^n}{n!} = \frac{1}{k!} (-F_m(-x))^k, \quad (9)$$

$$\sum_{n=0}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\geq m} \frac{x^n}{n!} = \frac{1}{k!} (-\log(1-x) + F_{m-1}(-x))^k. \quad (10)$$

It is trivial to see that

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\leq \infty} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\geq 1} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right].$$

4 Incomplete poly-Cauchy polynomials

In [10], the concept of the poly-Cauchy polynomials is introduced by replacing x by $-x$ in the definition. For integers n and μ with $n \geq 0$ and $\mu \geq 1$, define

the poly-Cauchy polynomials of the first kind $c_n^{(\mu)}(x)$ ($\mu \geq 1$) as

$$c_n^{(\mu)}(x) = \underbrace{\int_0^1 \cdots \int_0^1}_{\mu} (t_1 t_2 \cdots t_\mu - x)(t_1 t_2 \cdots t_\mu - x - 1) \cdots (t_1 t_2 \cdots t_\mu - x - n + 1) dt_1 dt_2 \cdots dt_\mu.$$

Then, $c_n^{(\mu)}(x)$ can be expressed in terms of the Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$:

$$c_n^{(\mu)}(x) = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] (-1)^{n-k} \sum_{i=0}^k \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}$$

([10, Theorem 1]).

Hence, it is natural to define two types of *incomplete poly-Cauchy polynomials of the first kind* $c_{n, \leq m}^{(\mu)}(x)$ and $c_{n, \geq m}^{(\mu)}(x)$ by

$$c_{n, \leq m}^{(\mu)}(x) = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\leq m} (-1)^{n-k} \sum_{i=0}^k \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \quad (11)$$

$$c_{n, \geq m}^{(\mu)}(x) = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_{\geq m} (-1)^{n-k} \sum_{i=0}^k \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \quad (12)$$

respectively.

The generating function of poly-Cauchy polynomials is given by (3). The generating functions of two types of incomplete poly-Cauchy polynomials of the first kind can be also given by using the polylogarithm factorial functions in terms of the m th partial sum of the logarithm function $F_m(t)$.

Theorem 1. For integers $n \geq 1$, $m \geq 1$ and $\mu \geq 1$ we have

$$\frac{\text{Lif}_\mu(F_m(t))}{e^{xF_m(t)}} = \sum_{n=0}^{\infty} c_{n, \leq m}^{(\mu)}(x) \frac{t^n}{n!}, \quad (13)$$

$$\frac{\text{Lif}_\mu(\log(1+t) - F_{m-1}(t))}{e^{x(\log(1+t) - F_{m-1}(t))}} = \sum_{n=0}^{\infty} c_{n, \geq m}^{(\mu)}(x) \frac{t^n}{n!}. \quad (14)$$

Remark. Since $F_\infty(t) = \log(1+t)$ and $F_0(t) = 0$, if we take $m \rightarrow \infty$ in (13) and $m = 1$ in (14), both of (13) and (14) are reduced to the generating function in ([17, Theorem 3]).

The generating function of two types of the incomplete poly-Cauchy polynomials of the first kind can be written in the form of iterated integrals.

Corollary 1. For $\mu \geq 1$ we have

$$\frac{1}{e^{xF_m(t)} F_m(t)} \underbrace{\int_0^t \frac{1 - (-t)^m}{(1+t)F_m(t)} \cdots \int_0^t \frac{1 - (-t)^m}{(1+t)F_m(t)} (e^{F_m(t)} - 1)}_{\mu-1} \underbrace{dt \cdots dt}_{\mu-1} = \sum_{n=0}^{\infty} c_{n, \leq m}^{(\mu)}(x) \frac{t^n}{n!}, \quad (15)$$

$$\frac{1}{e^{x(\log(1+t) - F_{m-1}(t))} (\log(1+t) - F_{m-1}(t))} \times \underbrace{\int_0^t \frac{(-t)^{m-1}}{(1+t)(\log(1+t) - F_{m-1}(t))} \cdots \int_0^t \frac{(-t)^{m-1}}{(1+t)(\log(1+t) - F_{m-1}(t))}}_{\mu-1} \times \underbrace{(e^{(\log(1+t) - F_{m-1}(t))} - 1)}_{\mu-1} dt \cdots dt = \sum_{n=0}^{\infty} c_{n, \geq m}^{(\mu)}(x) \frac{t^n}{n!}. \quad (16)$$

Remark. If we take $m \rightarrow \infty$ in (15) and $m = 1$ in (16), both of (15) and (16) are reduced to the generating function in [10, Corollary 1]. When $x = 0$, Corollary (1) is reduced to [15, Corollary 1].

For integers n and μ with $n \geq 0$ and $\mu \geq 1$, define the *poly-Cauchy polynomials of the second kind* $\widehat{c}_n^{(\mu)}(x)$ ($\mu \geq 1$) as

$$\widehat{c}_n^{(\mu)}(x) = \underbrace{\int_0^1 \cdots \int_0^1}_{\mu} (-t_1 t_2 \cdots t_\mu + x) (-t_1 t_2 \cdots t_\mu + x - 1) \cdots (-t_1 t_2 \cdots t_\mu + x - n + 1) dt_1 dt_2 \cdots dt_\mu.$$

Then, $\widehat{c}_n^{(\mu)}(x)$ can be expressed in terms of the Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$:

$$\widehat{c}_n^{(\mu)}(x) = \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] (-1)^n \sum_{i=0}^k \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}$$

([10, Theorem 4]).

Hence, it is natural to define two types of *incomplete poly-Cauchy polynomials of the second kind* $\widehat{c}_{n,\leq m}^{(\mu)}(x)$ and $\widehat{c}_{n,\geq m}^{(\mu)}(x)$ by

$$\widehat{c}_{n,\leq m}^{(\mu)}(x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{\leq m} (-1)^n \sum_{i=0}^k \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \quad (17)$$

$$\widehat{c}_{n,\geq m}^{(\mu)}(x) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_{\geq m} (-1)^n \sum_{i=0}^k \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \quad (18)$$

respectively.

The generating function of $\widehat{c}_n^{(\mu)}(x)$ is given by

$$(1+x)^z \text{Lif}_\mu(-\ln(1+t)) = \sum_{n=0}^{\infty} \widehat{c}_n^{(\mu)}(x) \frac{t^n}{n!} \quad (19)$$

([10, Theorem 5]. Note that z is replaced by $-z$).

The generating functions of two types of incomplete poly-Cauchy polynomials of the second kind can be also given by using the polylogarithm factorial functions $\text{Lif}_\mu(z)$ in terms of $F_m(t)$. Since $F_\infty(t) = \log(1+t)$ and $F_0(t) = 0$, if we take $m \rightarrow \infty$ in (20) and $m = 1$ in (21), both of (20) and (21) are reduced to the generating function in [10, Theorem 5]. On the other hand, if $x = 0$, then Theorem 2 is reduced to [15, Theorem 2].

Theorem 2. For integers $n \geq 1$, $m \geq 1$ and $\mu \geq 1$ we have

$$e^{xF_m(t)} \text{Lif}_\mu(-F_m(t)) = \sum_{n=0}^{\infty} \widehat{c}_{n,\leq m}^{(\mu)}(x) \frac{t^n}{n!}, \quad (20)$$

$$e^{x(\log(1+t)-F_{m-1}(t))} \text{Lif}_\mu(-\log(1+qt) + F_{m-1}(t)) = \sum_{n=0}^{\infty} \widehat{c}_{n,\geq m}^{(\mu)}(x) \frac{t^n}{n!}. \quad (21)$$

The generating function of two types of the incomplete Cauchy polynomials of the second kind can be also written in the form of iterated integrals.

Corollary 2. For $\mu \geq 1$ we have

$$\frac{e^{xF_m(t)}}{F_m(t)} \underbrace{\int_0^t \frac{1 - (-t)^m}{(1+t)F_m(t)} \cdots \int_0^t \frac{1 - (-t)^m}{(1+t)F_m(t)} (1 - e^{-F_m(t)})}_{\mu-1} \underbrace{dt \cdots dt}_{\mu-1}$$

$$= \sum_{n=0}^{\infty} \hat{c}_{n, \leq m}^{(\mu)}(x) \frac{t^n}{n!}, \quad (22)$$

$$\frac{e^{x(\log(1+t) - F_{m-1}(t))}}{\log(1+t) - F_{m-1}(t)}$$

$$\times \underbrace{\int_0^t \frac{(-t)^{m-1}}{(1+t)(\log(1+t) - F_{m-1}(t))} \cdots \int_0^t \frac{(-t)^{m-1}}{(1+t)(\log(1+t) - F_{m-1}(t))}}_{\mu-1}$$

$$\times (1 - e^{-\log(1+t) + F_{m-1}(t)}) \underbrace{dt \cdots dt}_{\mu-1} = \sum_{n=0}^{\infty} \hat{c}_{n, \geq m}^{(\mu)}(x) \frac{t^n}{n!}. \quad (23)$$

Remark. If we take $m \rightarrow \infty$ in (22) and $m = 1$ in (23), both of (22) and (23) are reduced to the generating function in [10, Corollary 2]. If $x = 0$, then Corollary 2 are reduced to [15, Corollary 2].

5 Some properties of incomplete poly-Cauchy polynomials

It is known that poly-Bernoulli numbers satisfy the duality theorem $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \geq 0$ [11, Theorem 2] because of the symmetric formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n y^k}{n! k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}. \quad (24)$$

However, incomplete poly-Cauchy polynomials do not satisfy the duality theorem for any integer $m \geq 1$, by the following results.

Theorem 3. For nonnegative integers n, k , and a positive integer m , we

have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,\leq m}^{(-k)}(z) \frac{x^n y^k}{n! k!} = \exp((e^y - z)F_m(x) + y), \quad (25)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,\geq m}^{(-k)}(z) \frac{x^n y^k}{n! k!} \\ = \exp((e^y - z)(\log(1+x) - F_{m-1}(x)) + y). \end{aligned} \quad (26)$$

Remark. If $m \rightarrow \infty$ in (25) or $m = 1$ in (26), both identities are reduced to the first identity in [10, Proposition 1]. If $z = 0$, then Theorem 3 is reduced to [15, Theorem 4], where z is replaced by $-z$.

Similarly, incomplete poly-Cauchy polynomials of the second kind have the following properties. If $m \rightarrow \infty$ in (27) or $m = 1$ in (28), both identities are reduced to the second identity in [10, Proposition 1]. If $z = 0$, then Theorem 3 is reduced to [15, Theorem 5], where z is replaced by $-z$.

Theorem 4. For nonnegative integers n , k , and a positive integer m , we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_{n,\leq m}^{(-k)}(z) \frac{x^n y^k}{n! k!} = \exp((z - e^y)F_m(x) + y), \quad (27)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \hat{c}_{n,\geq m}^{(-k)}(z) \frac{x^n y^k}{n! k!} \\ = \exp((z - e^y)(\log(1+x) - F_{m-1}(x)) + y). \end{aligned} \quad (28)$$

By using Theorem 3, we have explicit expressions of incomplete poly-Cauchy polynomials of the first kind with negative indices.

Theorem 5. For nonnegative integers n , k , and a positive integer m , we

have

$$c_{n, \leq m}^{(-k)}(z) = \sum_{j=0}^k \sum_{i=0}^n \sum_{l=0}^i j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \binom{i}{l} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\leq m} (-1)^{n-i-i+l} z^l, \quad (29)$$

$$c_{n, \geq m}^{(-k)}(z) = \sum_{j=0}^k \sum_{i=0}^n \sum_{l=0}^i j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \binom{i}{l} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\geq m} (-1)^{n-i-i+l} z^l. \quad (30)$$

Remark. If $m \rightarrow \infty$ in (29), or if $m = 1$ in (30), both identities in Theorem 5 are reduced to the first identity in [10, Theorem 8], where z is replaced by $-z$, and only classical Stirling numbers of both kinds are used.

If $z = 0$, then we have the following. These are different expressions seen in [15, Theorem 6].

Corollary 3. For nonnegative integers n , k , and a positive integer m , we have

$$c_{n, \leq m}^{(-k)} = \sum_{j=0}^k \sum_{i=0}^n j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\leq m} (-1)^{n-i-j}, \quad (31)$$

$$c_{n, \geq m}^{(-k)} = \sum_{j=0}^k \sum_{i=0}^n j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\geq m} (-1)^{n-i-j}. \quad (32)$$

Similarly, using Theorem 4, we have explicit expressions of incomplete poly-Cauchy polynomials of the second kind with negative indices. If $m \rightarrow \infty$ in (33), or if $m = 1$ in (34), any identity in Theorem 6 is reduced to the second identity in [10, Theorem 8], where z is replaced by $-z$, and only classical Stirling numbers of both kinds are used.

Theorem 6. For nonnegative integers n , k , and a positive integer m , we have

$$\hat{c}_{n, \leq m}^{(-k)}(z) = \sum_{j=0}^k \sum_{i=0}^n \sum_{l=0}^i j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \binom{i}{l} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\leq m} (-1)^{n-l} z^l, \quad (33)$$

$$\hat{c}_{n, \geq m}^{(-k)}(z) = \sum_{j=0}^k \sum_{i=0}^n \sum_{l=0}^i j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \binom{i}{l} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\geq m} (-1)^{n-l} z^l. \quad (34)$$

If $z = 0$, then then we have the following. These are different expressions seen in [15, Theorem 7].

Corollary 4. *For nonnegative integers n , k , and a positive integer m , we have*

$$\hat{c}_{n, \leq m}^{(-k)} = (-1)^n \sum_{j=0}^k \sum_{i=0}^n j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\leq m}, \quad (35)$$

$$\hat{c}_{n, \geq m}^{(-k)} = (-1)^n \sum_{j=0}^k \sum_{i=0}^n j! \left\{ \begin{matrix} k+1 \\ j+1 \end{matrix} \right\} \binom{i+j}{j} \left[\begin{matrix} n \\ i+j \end{matrix} \right]_{\geq m}. \quad (36)$$

6 Incomplete poly-Bernoulli polynomials

In [6], poly-Bernoulli polynomials $B_n^{(\mu)}(x)$ are defined in (1). In [1, 2], x and $-x$ are interchanged in the definition of $B_n^{(\mu)}(x)$. In [3], an extended concept named poly-Bernoulli polynomials with a q parameter is introduced. In [16], still different poly-Bernoulli polynomials $B_n^{(\mu)}(x)$ are defined. Poly-Bernoulli polynomials have an explicit expression in terms of the Stirling numbers of the second kind:

$$B_n^{(\mu)}(x) = \sum_{k=0}^n \frac{(-1)^{n-k} k!}{(k+1)^\mu} \sum_{l=0}^n \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\} x^{n-l}. \quad (37)$$

When $x = 0$, we get the expression of poly-Bernoulli numbers $B_n^{(\mu)}(0) = B_n^{(\mu)}$:

$$B_n^{(\mu)} = \sum_{k=0}^n \frac{(-1)^{n-k} k!}{(k+1)^\mu} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

([11, Theorem 1]).

By using two types of incomplete Stirling numbers, define *restricted poly-Bernoulli polynomials* $B_{n, \leq m}^{(\mu)}(x)$ and *associated poly-Bernoulli polynomials* $B_{n, \geq m}^{(\mu)}(x)$ by

$$B_{n, \leq m}^{(\mu)}(x) = \sum_{k=0}^n \frac{(-1)^{n-k} k!}{(k+1)^\mu} \sum_{l=0}^n \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_{\leq m} x^{n-l} \quad (n \geq 0) \quad (38)$$

and

$$B_{n, \geq m}^{(\mu)}(x) = \sum_{k=0}^n \frac{(-1)^{n-k} k!}{(k+1)^\mu} \sum_{l=0}^n \binom{n}{l} \left\{ \begin{matrix} l \\ k \end{matrix} \right\}_{\geq m} x^{n-l} \quad (n \geq 0), \quad (39)$$

respectively. We call these numbers as *incomplete poly-Bernoulli polynomials*. If $x = 0$, then they are reduced to restricted poly-Bernoulli numbers and associated poly-Bernoulli numbers, respectively ([18]).

One can deduce that the numbers $B_{n, \leq m}^{(\mu)}(x)$ and $B_{n, \geq m}^{(\mu)}(x)$ have the generating functions by using the polylogarithms.

Theorem 7. *We have*

$$\frac{\text{Li}_\mu(1 - E_m(-t))}{1 - E_m(-t)} e^{-tx} = \sum_{n=0}^{\infty} B_{n, \leq m}^{(\mu)}(x) \frac{t^n}{n!} \quad (40)$$

and

$$\frac{\text{Li}_\mu(E_{m-1}(-t) - e^{-t})}{E_{m-1}(-t) - e^{-t}} e^{-tx} = \sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)}(x) \frac{t^n}{n!}. \quad (41)$$

Remark. If $x = 0$, they are reduced to the generating functions in [18, Theorem 1].

If $m \rightarrow \infty$ in (40) or $m = 1$ in (41), they are reduced to the generating function in [3, Theorem 2.1] by putting $q = 1$.

For $\mu \geq 1$, the generating functions can be written in the form of iterated integrals. We set $E_{-1}(-t) = 0$ for convenience.

Theorem 8.

$$\frac{e^{-tx}}{1 - E_m(-t)} \cdot \underbrace{\int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} \cdots \int_0^t \frac{E_{m-1}(-t)}{1 - E_m(-t)} (-\log(E_m(-t)))}_{\mu-1} \underbrace{dt \dots dt}_{\mu-1}$$

$$= \sum_{n=0}^{\infty} B_{n, \leq m}^{(\mu)}(x) \frac{t^n}{n!}, \quad (42)$$

$$\frac{e^{-tx}}{E_{m-1}(-t) - e^{-t}} \cdot \underbrace{\int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}} \cdots \int_0^t \frac{e^{-t} - E_{m-2}(-t)}{E_{m-1}(-t) - e^{-t}}}_{\mu-1}$$

$$\times (-\log(1 + e^{-t} - E_{m-1}(-t))) \underbrace{dt \dots dt}_{\mu-1}$$

$$= \sum_{n=0}^{\infty} B_{n, \geq m}^{(\mu)}(x) \frac{t^n}{n!}. \quad (43)$$

Remark. If $x = 0$, then Theorem 8 is reduced to [18, Theorem 2]. In addition, if $m \rightarrow \infty$ in (42), by $E_{\infty}(-t) = e^{-t}$, and if $m = 1$ in (43), both of them are reduced to the iterated form (2) in ([11]) by setting $t = 0$.

A symmetric property does not hold for generalized poly-Bernoulli numbers with negative indices because of the following.

Theorem 9. *We have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_{n, \leq m}^{(-k)}(z) \frac{x^n y^k}{n! k!} = \frac{e^{y-xz}}{1 - e^y(1 - E_m(-x))}, \quad (44)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_{n, \geq m}^{(-k)}(z) \frac{x^n y^k}{n! k!} = \frac{e^{y-xz}}{1 - e^y(E_{m-1}(-x) - e^{-x})}. \quad (45)$$

Remark. If $m \rightarrow \infty$ in (44) or $m = 1$ in (45), both identities are reduced to (24) with $z = 0$.

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