数理解析研究所講究録
第2014巻 2017年 151-154

Asymptotics of L-functions

Gautami Bhowmik

1 Motivation

The vanishing or non-vanishing of families of L-functions are important in the theory itself and has applications in many areas of number theory like elliptic curves in the Swinnerton-Dyer conjecture (see, for instance [4]). In theory non-vanishing results could be obtained by calculating moments or their asymptotics but in general this is a difficult task.

For quadratic cases Soundararajan [8] showed a positive proportion of Dirichlet L-functions are not zero at $s=1/2$. For degree three cases bounds were given (eg.[6],[3]) and for the level 1 case we have a result of Khan [5] who calculated the first moment. Blomer [1] calculated the first as well as second moment for an arbitrary level and real primitive nebentypus and proved the non-vanishing of the symmetric square L-function at the critical value while Sun [9] extended the non-vanishing results to the critical line.

The simultaneous non-vanishing of L-functions was considered by Munshi and Sengupta [7] who showed that for large levels and trivial nebentypus the product of certain degree two and degree three L-functions do not vanish. Here we indicate similar results for level 1 functions, obtained by expressing the central values as rapidly converging series and the use of the trace formula. The details of the proof will be published later jointly with Nicole Raulf and Jyoti Sengupta.

For f a primitive cusp form of weight $k\geq 2$, level N and nebentypus χ, we denote by $\mathcal{S}_k(N,\chi)$ the finite-dimensional Hilbert space of such forms with the Petersson norm

$$||f||^2=\int_{\mathbb{H}/\Gamma_0(N)}|f(z)|^2y^{k-2}dxdy$$

with respect to the congruence subgroup $\Gamma_0(N)$ and and by $\mathcal{H}_k(N)$ the orthogonal basis of newforms. The Fourier expansion of $f\in\mathcal{H}_k(N)$ for $\Im(z)>0$ is given by

$$f(z)=\sum\lambda_f(n)n^{\frac{k-1}{2}}e(nz)$$

where the λ are Hecke eigenvalues and $\lambda_f(1)=1$.

Let the L-function associated to f be

$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}$$

while the associated symmetric square function for $\Re s > 1$ be defined as

$$L(s, \mathrm{sym}^2 f) := \zeta^{(N)}(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}$$

where $\zeta^{(N)}(2s) = \zeta(2s) \prod p|N(1-p^{-2s})$ so that up to Euler factors for prime divisors of N this becomes $\zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}$.

It is known that the symmetric square function can be continued to an entire function with a possible simple pole at s=1 and is automorphic. The spectral weight w_f is given by

$$w_f = \frac{(4\pi)^{k-1}||f||^2}{\Gamma(k-1)}.$$

2 Asymptotics

We are interested in the behaviour of

$$\sum_{f \in \mathcal{H}_k(1)} w_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right) L\left(\frac{1}{2}, f\right).$$

From the approximate functional equation we get

$$L\left(\frac{1}{2}, f\right) = 2 \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^{1/2}} W_{1/2}(n)$$

where

$$W_{1/2}(y) = \frac{1}{2\pi i} \int_{(3)} (2\pi y)^{-u} \frac{\Gamma\left(u + \frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)} \frac{du}{u}.$$

We can show that

$$W_{1/2}(m) \ll k^{1/2} \left(\frac{k}{m}\right)^A$$

for each $A > 0$. Now

$$\sum_{f \in \mathcal{H}_k(1)} w_f^{-1} L\left(\frac{1}{2}, \mathrm{sym}^2 f\right) L\left(\frac{1}{2}, f\right) = 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)\lambda_f(m)}{\sqrt{n}\sqrt{m}} V_{1/2}(n) W_{1/2}(m).$$

where the above, as in [5], is defined as

$$V_{1/2}(n) = \frac{1}{2\pi i}\int_{(2)} \frac{L^1_\infty(1/2+w)}{L^1_\infty(1/2)}\zeta(1+2w)n^{-w}\frac{dw}{w}$$

with

$$L^1_\infty(s) = \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right).$$

The bound for W and the Petersson trace formula gives, for $\alpha > 0$,

$$\begin{aligned}\sum_{f\in\mathcal{H}_k(1)} & w_f^{-1} L\left(\frac{1}{2},\mathrm{sym}^2 f\right) L\left(\frac{1}{2},f\right) \\ &= 4\sum_{n\leq k^{1+\epsilon'}} \frac{V_{1/2}(n)W_{1/2}(n^2)}{n^{3/2}} \\ &\quad + 8\pi \sum_{m,n\leq k^{1+\epsilon'}} \sum_{c=1}^{\infty} \frac{S(n^2,m,c)}{c} J_{k-1}\left(4\pi\frac{\sqrt{n^2 m}}{c}\right)\frac{V_{1/2}(n)W_{1/2}(m)}{m^{1/2}n^{1/2}} + O\left(k^{-\alpha}\right).\end{aligned}$$

with J being the Bessel function and $S(n^2,m,c)$ the twisted Kloosterman sum. We can estimate the first sum as

$$4\sum_{n\leq k^{1+\epsilon'}} \frac{V_{1/2}(n)W_{1/2}(n^2)}{n^{3/2}} = 2\log(k) + 2C + O\left(\frac{1}{k}\right).$$

To estimate the Kloosterman sums we use a result of [2] which gives an error term of $\sum_{f\in\mathcal{H}_k(1)} w_f^{-1} L\left(\frac{1}{2},\mathrm{sym}^2 f\right) L\left(\frac{1}{2},f\right)$ and as a consequence we get a non-vanishing result :

As $k\to\infty$ there exists a $f\in\mathcal{H}_k(N)$ such that

$$L\left(\frac{1}{2},\mathrm{sym}^2 f\right) L\left(\frac{1}{2},f\right) \neq 0.$$

References

[1] V. Blomer , On the central value of symmetric square L-functions, *Math.Z.* 260(4) : 755–777, (2008).

[2] H. Iwaniec, W. Luo and P. Sarnak, Low lying zeros of families of L-functions , *IHES. Publ. Math.* (91) : 55–131, (2001).

[3] H. Iwaniec and P. Michel, The second moment of the symmetric square L-functions, *Ann. Acad. Sci. Fenn. Math.* 26 (2) : 465–482, (2001).

[4] H. Iwaniec and P. Sarnak, Perspectives on the analytic theory of L-functions , *Geom. Funct. Anal.* : 705–741, (2000).

[5] R. Khan, The first moment of the symmetric square L-functions , *J. Number Theory* 124: 259–266, (2007).

[6] Y-K . Lau, Non-vanishing of symmetric square L-functions at $s = 1/2$, *Proc. Amer. Math. Soc.* 130 (2) : 3133–3139, (2002).

[7] R. Munshi and J. Sengupta, A case of simultaneous non-vanishing, arXiv:1504.07727v1.

[8] K. Soundararajan, Non-vanishing of quadratic Dirichlet L-functions , *Ann. of Math.* (2), 152 (2): 447–488, (2000).

[9] Q. Sun, The symmetric -square L-function on the critical line, *J. Number Theory* 140: 196–214, (2014).

Gautami Bhowmik
Laboratoire Paul Painlevé LABEX-CEMPI
Université Lille 1
59655 Villeneuve d'Ascq Cedex
France
E-mail address: bhowmik@math.univ-lille1.fr