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The mean values of the Dirichlet L-functions

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1 Introduction

First, we recall the definition of the Dirichlet L-function.

Let χ be a Dirichlet character modulo $k\geq 2$ and

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{s}}$$

be the Dirichlet L-function for $\Re s>1$.

In the present paper, we consider the following mean values of the Dirichlet L-functions for a positive integer r:

$$S_{r}(m_{1},\ldots,m_{r+1})$$

$$=\left(\frac{2}{\phi(k)}\right)^{r}\sum_{\chi_{1}(-1)=(-1)^{m_{1}}}\cdots\sum_{\chi_{r}(-1)=(-1)^{m_{r}}}\left(\prod_{i=1}^{r}L(m_{i},\chi_{i})\right)L(m_{r+1},\overline{\chi_{1}\cdots\chi_{r}}),$$

where $m_{1},\ldots,m_{r+1}\in\mathbb{N}$ with $m_{1}+\cdots+m_{r+1}\equiv 0 \pmod 2$ and $\chi_{1},\ldots,\chi_{r}$ are the Dirichlet character modulo k. The aim of such studies is to express $S_{r}(m_{1},\ldots m_{r+1})$ in terms of the values of the Riemann zeta-function $\zeta(\ell)$ and Jordan's totient functions

$$J_{\ell}(k)=k^{\ell}\prod_{p|k}(1-p^{-\ell}),$$

where ℓ is a positive integer.

In the case $r = 1$, $m_1 = m_2 = 1$ and $k = p$ is a prime, Walum [8] studied $S_1(1,1)$, and Alkan [1], Qi [7], Louboutin [4] and Zhang [9] independently gave the following explicit formula

$$S_1(1,1) = \frac{\pi^2}{6k^2}\Big(J_2(k) - 3\phi(k)\Big),$$

in the case $k \geq 2$. Also Louboutin [5] studied $S_1(m,m)$ for a positive integer m and $k > 2$, and Liu and Zhang [3] studied $S_1(m,n)$ for positive integers $m, n \geq 1$ with the same parity (see [3] Theorem 1.1).

Moreover Alkan [2] considered the case $r = 2$, and he gave the explicit formula

$$S_2(1,1,2) = \pi^4\Big(\frac{J_4(k)}{90k^4} - \frac{J_2(k)}{18k^4}\Big)$$

for $k > 2$.

2 Main theorem

Now we give the main theorem, but first we have to prepare the following some notations which use in the main theorem:

Let $(b_\alpha)_{\alpha=1}^m = (b_1, b_2, \ldots, b_m)$ be an m-th row vector and $(c_{ij})_{1\leq i,j\leq m}$ be an m-th square matrix. Then, for an even positive integer$n$, we put the $\frac{n}{2}$-th column vector

$$A_n = \left(\frac{(-1)^i i a_{i,j}}{2^{2i-1}B_{2i}}\right)^{-1}_{1\leq i,j\leq \frac{n}{2}} \begin{pmatrix} J_2(k) \\ J_4(k) \\ \vdots \\ J_n(k) \end{pmatrix},$$

where B_m is the m-th Bernoulli number and

$$a_{i,j} = \begin{cases} (-1)^i 4^{i-1} & \text{if } j = 1, \\ 2(j-1)(2j-1)a_{i-1,j-1} - 4j^2 a_{i-1,j} & \text{if } 1 < j < i, \\ -(2i-1)! & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases}$$

Theorem 1. *Let $k > 2$ be a positive integer and $m_1, \ldots, m_{r+1} \in \mathbb{N}$ with $m_1 + \cdots + m_{r+1} \equiv 0 \pmod 2$.*

In the case r is odd and $m_1 = m_2 = \cdots = m_{r+1} = 1$, we have

$$S_r(1,\ldots,1) = \left(\frac{\pi}{2k}\right)^{r+1}\left\{\left(\binom{\frac{r+1}{2}}{\frac{r+1}{2}-\alpha}(-1)^{\frac{r+1}{2}-\alpha}\right)_{\alpha=1}^{\frac{r+1}{2}} A_{r+1} + (-1)^{\frac{r+1}{2}}\phi(k)\right\}$$

and the otherwise,

$$S_r(m_1,\ldots,m_{r+1}) = \frac{(-1)^{r+1}\pi^{m_1+\cdots+m_{r+1}}}{2^{r+1}k^{m_1+\cdots+m_{r+1}}\{\prod_{i=1}^{r+1}(m_i-1)!\}}$$

$$\times\left(\sum_{n=0}^{[\frac{t}{2}]}(-1)^{[\frac{t}{2}]-n}\binom{[\frac{t}{2}]}{[\frac{t}{2}]-n}\sum_{\substack{j_1+\cdots+j_{r+1}=\alpha-n\\ 1\le j_1\le[\frac{m_1}{2}],\cdots,1\le j_{r+1}\le[\frac{m_{r+1}}{2}]}}\cdots\sum\left\{\prod_{v=1}^{r+1}f(m_v,j_v)\right\}\right)_{\alpha=1}^{\frac{m_1+\cdots+m_{r+1}}{2}} A_{m_1+\cdots+m_{r+1}},$$

where for $m_v \geq 2$, $f(m_v,j_v) = a_{[\frac{m_v}{2}],j_v}\rho(m_v,j_v)$,

$$\rho(m_v;j_v) = \begin{cases} -2j_v & \text{if } m_v \in 2\mathbb{N}+1,\\ 1 & \text{otherwise.}\end{cases}$$

and we put

$$t =^{\#} \{m_v \mid m_v \text{ is odd for } 1 \leq v \leq r+1\}.$$

Also, in the case $m_v = 1$*, the sum and product corresponding* v *does not appear.*

We note that this result is a generalization of results of [6].

3 Sketch of proof

The proof of Theorem 1 is similar to the proof of Theorem 1.1 and 1.2 of [6]. Therefore we give only the sketch of the proof of Theorem 1 in this paper.

The key of the proof is to use the following result of Louboutin ([5] Proposition 3 (1)):

Let $n \geq 1$ and $k > 2$ be positive integers. Let $\cot^{(n)} x$ denote the n-th derivative of $\cot x$. If χ is a Dirichlet character modulo k and if $\chi(-1) = (-1)^n$ then we have

$$L(n,\chi) = \frac{(-1)^{n-1}\pi^n}{2k^n(n-1)!}\sum_{j=1}^{k-1}\chi(j)\cot^{(n-1)}(\pi j/k). \tag{1}$$

Then, by (1) and

$$\sum_{\chi(-1)=(-1)^n} \chi(j_1)\overline{\chi}(j_2) = \begin{cases} \dfrac{\phi(k)}{2} & \text{if } j_1 \equiv j_2 \mod k,\ (j_1,k)=1, \\ (-1)^n \dfrac{\phi(k)}{2} & \text{if } j_1 \equiv -j_2 \mod k,\ (j_1,k)=1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$S_r(m_1,\ldots,m_{r+1}) = \frac{(-1)^{r+1}\pi^{m_1+\cdots+m_{r+1}}}{2k^{m_1+\cdots+m_{r+1}}(\prod_{i=1}^{r+1}(m_i-1)!)} \times \sum_{\substack{l=1\\(l,k)=1}}^{k-1} \cot^{(m_1-1)}(\pi l/k)\cot^{(m_2-1)}(\pi l/k)\cdots\cot^{(m_{r+1}-1)}(\pi l/k).$$

Now, using the following formula:
For any positive integer n, we have

$$\cot^{(2n-1)} x = \sum_{j=1}^{n} a_{n,j}\sin^{-2j} x, \tag{2}$$

where $a_{i,j}$ is defined in Section 2, we can express $S_r(m_1,\ldots m_{r+1})$ to the terms of $\sum_{j=1}^{n} a_{n,j}\sin^{-2j}(\pi l/k)$ for $1 \le j \le (m_1+\cdots+m_{r+1})2$.

On the other hand, applying (2) to (1), we have

$$L(2n,\chi_0) = \frac{-\pi^{2n}}{2k^{2n}(2n-1)!}\sum_{j=1}^{n} a_{n,j}\sum_{\substack{l=1\\(l,k)=1}}^{k-1}\sin^{-2j}\left(\frac{\pi l}{k}\right), \tag{3}$$

where χ_0 is a principal Dirichlet character modulo k. Also, by the Euler product expansion, we have

$$L(2n,\chi_0) = \zeta(2n)k^{-2n}J_{2n}(k). \tag{4}$$

Then, by (3) and (4), we can express $\sum_{j=1}^{n} a_{n,j}\sin^{-2j}(\pi l/k)$ to the terms of J_{2j} for $1 \le j \le (m_1+\cdots+m_{r+1})/2$. Therefore we can express $S_r(m_1,\ldots,m_{r+1})$ to the terms of J_{2j} for $1 \le j \le (m_1+\cdots+m_{r+1})/2$ and we can show Theorem 1.

4 Examples

Lastly we give the following several evaluation formulas for $S_r(m_1, \dots m_{r+1})$ in the case $r = 2$:

$$S_2(1,1,4) = \frac{\pi^6}{12k^6}\left(\frac{4}{315}J_6(k) - \frac{4}{45}J_4(k) + \frac{8}{45}J_2(k)\right),$$

$$S_2(1,1,6) = \frac{\pi^8}{240k^8}\left(\frac{8}{315}J_8(k) - \frac{32}{189}J_6(k) + \frac{8}{45}J_4(k) + \frac{32}{63}J_2(k)\right).$$

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