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Complex geometry of Blaschke products and associated circumscribed conics
ブラシュケ積の複素幾何と外接楕円

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Abstract

We study geometrical properties of finite Blashcke products. For a Blaschke product B of degree d and the d preimages z_k $(k=1,\cdots,d)$ of $\lambda\in\partial\mathbb{D}$ by B, let L_λ be the set of d lines tangent to $\partial\mathbb{D}$ at the d preimages $z_1,\cdots,z_d$. Here, we denote by T_B the trace of the intersection points of each two elements in L_λ as λ ranges over the unit circle. We show that the trace T_B forms an algebraic curve of degree at most $d-1$.

1 Introduction

A *Blaschke product* of degree d is a rational function defined by

$$B(z)=e^{i\theta}\prod_{k=1}^{d}\frac{z-a_k}{1-\overline{a_k}z}\qquad (a_k\in\mathbb{D},\ \theta\in\mathbb{R}).$$

In the case that $\theta=0$ and $B(0)=0$, B is called *canonical.*

In [2], Daepp, Gorkin, and Mortini treated the geometrical properties of Blaschke products inside the unit disk.

Theorem 1 (U. Daepp, P. Gorkin, and R. Mortini [2])
Let B be a canonical Blaschke product of degree 3 with zeros 0, a, and b. For $\lambda\in\partial\mathbb{D}$, let z_1,z_2, and z_3 denote the points mapped to λ under B. Then the lines joining z_j and z_k for $j\neq k$ are tangent to the ellipse E with equation

$$|z-a|+|z-b|=|1-\overline{a}b|. \tag{1}$$

Conversely, each point of E is the point of tangency of a line that passes through two distinct points ζ_1, ζ_2 on $\partial\mathbb{D}$ for which

$$B(\zeta_1)=B(\zeta_2).$$

The above ellipse E is also deeply related to the numerical range of another specific matrix having the non-zero zeros of B in Theorem 1 as the eigenvalues (for example, see [4]).

Moreover, it seems that Theorem 1 is close to the following classical result in Marden's book [8] that was proved first by Siebeck [9].

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Theorem 2 (P. Siebeck 1864, (M. Marden 1966))
The zeros z_1' and z_2' of the function

$$F(z) = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \frac{m_3}{z-z_3} \left(= \frac{n(z-z_1')(z-z_2')}{(z-z_1)(z-z_2)(z-z_3)} \right)$$

are the foci of the conic which touches the line segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in the points ζ_3, ζ_1, and ζ_2 that divide these segments in the ratios $m_1 : m_2$, $m_2 : m_3$ and $m_3 : m_1$, respectively. If $n = m_1 + m_2 + m_3 \neq 0$, the conic is an ellipse or hyperbola according as $nm_1m_2m_3 > 0$ or < 0.

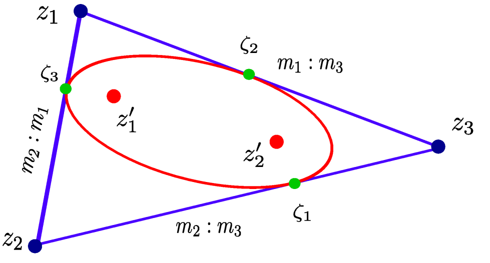

Figure 1: The foci of the inscribed ellipse are given as zero points of the function $F(z) = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \frac{m_3}{z-z_3}$.

In this report, I treat the geometrical properties of Blaschke products outside the unit disk.

2 Geometry of the Blaschke products on $\mathbb{C} \setminus \mathbb{D}$

Let B be a canonical Blaschke product of degree d. For $\lambda \in \partial\mathbb{D}$, let L_λ be the set of d lines tangent to $\partial\mathbb{D}$ at the d preimages of $\lambda \in \partial\mathbb{D}$ by B. Here, we denote by T_B the trace of the intersection points of each two elements in L_λ as λ ranges over the unit circle. Then, we will show the following.

Theorem 3
Let B be a canonical Blaschke product of degree d. Then, the trace T_B forms an algebraic curve of degree at most $d-1$.

Proof Let l_k be a line tangent to the unit circle at a point z_k $(k = 1, \cdots, d)$, i.e., $l_k \;:\; \overline{z_k} z + z_k \overline{z} - 2 = 0$. Let

$$B(z) = \frac{z^d + \alpha_{d-1} z^{d-1} + \alpha_{d-2} z^{d-2} + \cdots + \alpha_1 z}{1 + \overline{\alpha_{d-1}} z + \cdots + \overline{\alpha_1} z^{d-1}}.$$

Eliminating λ from $B(z_1) = B(z_2) = \lambda$, we have

$$\sum_{N=1}^{d} \sum_{K=0}^{N-1} A_{N,K} (z_1 z_2)^K \Big((z_1+z_2)^{N-K-1} - \gamma_1 z_1 z_2 (z_1+z_2)^{N-K-3} + \cdots + \gamma_M (z_1 z_2)^M (z_1+z_2)^R \Big) = 0,$$

where R is the remainder after division of $N-K-1$ by 2, $M = \dfrac{N-K-1-R}{2}$, $\gamma_1 = N-K-2$, and γ_M is a non-zero coefficient. The intersection point z of two lines l_1 and l_2 satisfies

$$z_1 z_2 = \frac{z}{\overline{z}} \quad \text{and} \quad z_1 + z_2 = \frac{2}{\overline{z}}, \tag{2}$$

since each l_k $(k = 1, 2)$ is a line tangent to the unit circle at a point z_k. Note that the intersection point is the point at infinity if and only if $z_1 + z_2 = 0$. Hence, we have

$$\sum_{N=1}^{d} \sum_{K=0}^{N-1} A_{N,K} z^K \overline{z}^{d-N} \left(2^{N-K-1} - 2^{N-K-3}\gamma_1 z\overline{z} + \cdots + 2^R \gamma_M z^M \overline{z}^M\right) = 0. \tag{3}$$

This equality gives a defining equation of T_B with degree at most $d-1$. ∎

Remark 1

The degree of the trace T_B is not always $d-1$.

For every $\lambda \in \partial\mathbb{D}$, if B has a pair of parallel tangent lines, the degree is less than $d-1$. For example, if $\deg B = d = 2k$ and B has k pairs of zero points $\{(a_j, b_j)\,;\, j = 1, \cdots, k\}$ such that $a_j + b_j = 0$ $(j = 1, \cdots, k)$, the degree is less than $d-1$. We call such a Blaschke product parallel.

When the degree is low, we can check the following.

- *For every Blaschke product B of degree 3, T_B is a non-degenerate conic. (Cf. Corollary 5, below.)*
- *For a Blaschke product B of degree 4, T_B is a cubic algebraic curve if and only if B is not a parallel one. (Cf. Theorem 7, below.)*
- *For every Blaschke product B of degree 5, T_B is a algebraic curve of degree 4.*
- *For a Blaschke product B of degree 6, T_B is a algebraic curve of degree 5 if and only if B is not a parallel one.*

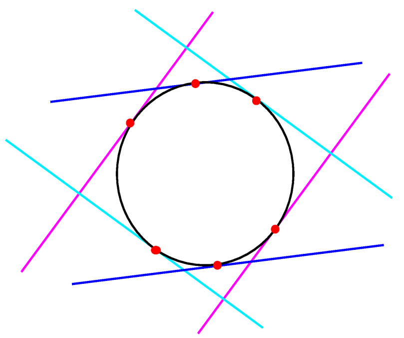

Figure 2: A parallel Blaschke product for $\deg B = 6$.

Conjecture

For a Blaschke product of degree d, the degree of the trace T_B is less than $d-1$ if and only if the degree of B is even and B is a parallel one.

3 Circumscribed conics

When the degree is low, we can describe some geometrical properties more concretely.

For a Blaschke product $B(z) = z\frac{z-a}{1-\overline{a}z}$ $(a \in \mathbb{D})$ of degree 2, let z_1 and z_2 denote the two points such that $B(z_1) = B(z_2) = \lambda \in \partial\mathbb{D}$. Then, Daepp, Gorkin, and Mortini [2] proved the property that the line joining z_1 and z_2 passes through the non-zero zero point a of B. While, if we consider the two lines tangent to the unit circle at a point z_1 and z_2, we have

Corollary 4 (Corollary of Theorem 3)
Let B be a canonical Blaschke product of degree 2 with zeros 0 and a $(\neq 0)$. Then, the trace T_B forms a line $\overline{a}z + a\overline{z} - 2 = 0$.

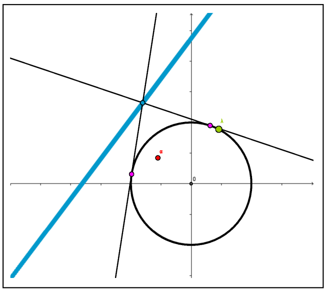

Figure 3: In the case of $\deg B = 2$, the trace T_B is a line.

Here, we consider a canonical Blaschke product

$$B(z) = z\frac{z-a}{1-\overline{a}z}\frac{z-b}{1-\overline{b}z} \qquad (a, b \in \mathbb{D})$$

of degree 3.

The following Corollary 5 corresponds to the result Lemma 9 in [7]. In [7], I gave a computational proof by using a symbolic computation system Risa/Asir. But here, the same result is obtained as a corollary of Theorem 3.

Corollary 5 (Corollary of Theorem 3)
Let B be a canonical Blaschke product of degree 3. Then, the trace T_B is a non-degenerate conic and the defining equation is given by

$$\overline{a}\overline{b}z^2 + (-|ab|^2 + |a+b|^2 - 1)z\overline{z} + ab\overline{z}^2 - 2(\overline{a}+\overline{b})z - 2(a+b)\overline{z} + 4 = 0. \qquad (4)$$

The foci of the conic (4) is obtained as follows by the date of non-zero zero points of B.

Proposition 6
The conic in Corollary 5 is given as follows.
In the case of $(|a+b|-1)^2 \neq |ab|^2$;

- *if $(|a+b|-1)^2 < |ab|^2$, the equation (4) is written as*

$$|z - f_1| - |z - f_2| = \pm r \qquad \text{(a hyperbola)},$$

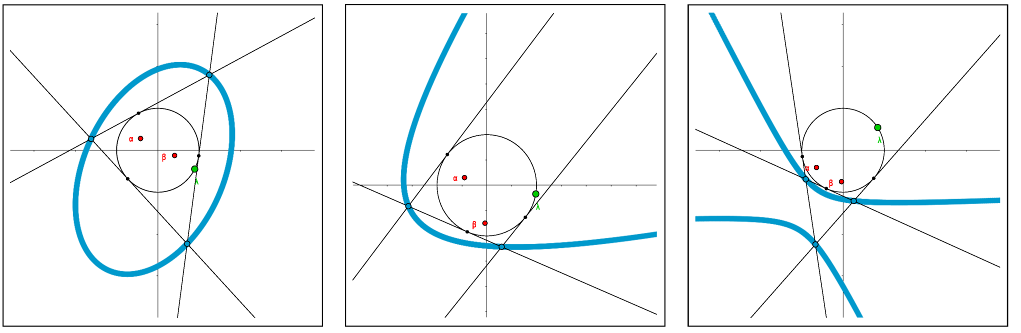

Figure 4: In the case of deg $B = 3$, the trace T_B is a non-degenerate conic.

- *if* $(|a+b|-1)^2 > |ab|^2$, *the equation* (4) *is written as*

$$|z - f_1| + |z - f_2| = r \qquad \text{(an ellipse)},$$

where f_1, f_2 *are the two solutions of*

$$\begin{aligned} F_{a,b}(t) = {} & \big((|ab|^2 + 1 - |a+b|^2)^2 - 4|ab|^2\big)t^2 \\ & + 4\big((|ab|^2+1)(a+b) - (\overline{a}+\overline{b})(a^2+b^2)\big)t + 4(a-b)^2 = 0, \end{aligned}$$

and r *is given by*

$$r = \frac{\sqrt{16(|a|^2-1)(|b|^2-1)(\overline{a}b-1)(a\overline{b}-1)(|ab|^2-|a+b|^2+2|ab|+1)}}{\big|(|ab|^2+1-|a+b|^2)^2-4|ab|\big|}.$$

Moreover, if $a = 0$ *(resp.* $b = 0$*), the equation* $F_{a,b} = 0$ *has a unique double root, and the equation* (4) *is written as*

$$\left|z + \frac{2b}{1-|b|^2}\right| = \frac{4}{1-|b|^2}, \quad \left(\text{resp. } \left|z + \frac{2a}{1-|a|^2}\right| = \frac{4}{1-|a|^2}\right), \qquad \text{(a circle)}.$$

In the case of $(|a+b|-1)^2 = |ab|^2$*; The equation* (4) *is written as*

$$|\overline{t}z + t\overline{z} + 1| = 2|t(z-s)| \qquad \text{(parabola)},$$

where s *and* t *are given by*

$$s = \frac{(a-b)^2}{(\overline{a}+\overline{b})(a^2+b^2) - (|ab|^2+1)(a+b)}, \qquad t = \frac{(\overline{a}+\overline{b})(a^2+b^2) - (|ab|^2+1)(a+b)}{2\big(2(|ab|^2+1) - |a+b|^2\big)}.$$

Here, we consider a canonical Blaschke product

$$B(z) = z\frac{z-a}{1-\overline{a}z}\frac{z-b}{1-\overline{b}z}\frac{z-c}{1-\overline{c}z} \qquad (a, b, c \in \mathbb{D})$$

of degree 4.

Theorem 7

Let B be a canonical Blaschke product of degree 4 with zeros $0, a, b$, and c. Assume, either that none of a, b, and c is zero, or that one of a, b, and c is zero but $a+b+c \neq 0$. Then, the trace T_B forms a cubic algebraic curve defined by

$$\begin{aligned}
&\overline{a}\overline{b}\overline{c}z^3 + \big(-\overline{a}\overline{b}\overline{c}(ab+bc+ca) + (\overline{ab}+\overline{bc}+\overline{ca})(a+b+c) - (\overline{a}+\overline{b}+\overline{c})\big)z^2\overline{z} - 2(\overline{ab}+\overline{bc}+\overline{ca})z^2 \\
&+\big(-abc(\overline{ab}+\overline{bc}+\overline{ca}) + (ab+bc+ca)(\overline{a}+\overline{b}+\overline{c}) - (a+b+c)\big)z\overline{z}^2 \\
&+2\big(|abc|^2 - (a+b+c)(\overline{a}+\overline{b}+\overline{c}) + 2\big)z\overline{z} + 4(\overline{a}+\overline{b}+\overline{c})z + abc\overline{z}^3 - 2(ab+bc+ca)\overline{z}^2 \\
&+4(a+b+c)\overline{z} - 8 = 0.
\end{aligned} \tag{5}$$

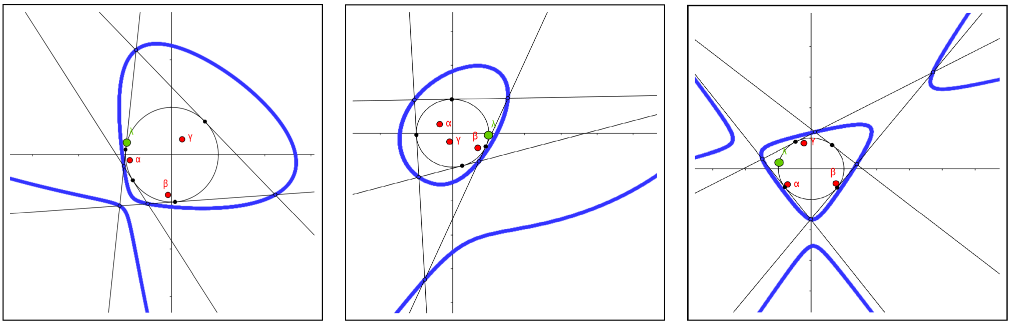

Figure 5: In the case of $\deg B = 4$, the trace T_B need not include a conic.

The trace T_B in Theorem 7 need not include a conic. But, we can give a necessary and sufficient condition that T_B includes a conic, which is similar to Lemma 5 in [5].

Theorem 8

For a canonical Blaschke product B of degree 4, the trace T_B includes a non-degenerate conic if and only if B is a composition of two Blaschke products of degree 2.

There is the following relation between the zeros of Blaschke products $B(z) = B_1 \circ B_2(z)$ and the foci of conic in Theorem 8 where $B_1(z) = z\dfrac{z-a}{1-\overline{a}z}$ and $B_2(z) = z\dfrac{z-b}{1-\overline{b}z}$.

Proposition 9

The conic in Theorem 8 is given as follows.
In the case of $\mathrm{Re}(a\overline{b}^2) + 1 \neq |a| + |b^2|$;

- *if* $\mathrm{Re}(a\overline{b}^2) + 1 < |a| + |b^2|$, *the equation of conic in Theorem 8 is written as*

$$|z - f_1| - |z - f_2| = \pm r \qquad \text{(a hyperbola)},$$

- *if* $\mathrm{Re}(a\overline{b}^2) + 1 > |a| + |b^2|$, *the equation is written as*

$$|z - f_1| + |z - f_2| = r \qquad \text{(an ellipse)},$$

where f_1, f_2 are the two solution of

$$F_{a,b}(t) = \left((a\overline{b}^2 + \overline{a}b^2 + 2 - 2|b|^2)^2 - 4|a|^2\right)t^2$$
$$- 4(a^2\overline{b}^3 + |ab|^2 b - 3a\overline{b}|b|^2 - 2b(|a|^2 - |b|^2) - \overline{a}b^3 + 4a\overline{b} - 2b)t$$
$$+ 4(a^2\overline{b}^2 + (-2|b|^2 + 4)a + b^2) = 0,$$

and r is given by

$$r = \frac{\sqrt{16(|a|^2 - 1)(|b|^2 - 1)(a\overline{b}^2 + \overline{a}b^2 - 2|b|^2 + 4)(a\overline{b}^2 + \overline{a}b^2 - 2|b|^2 + 2|a| + 2)}}{|(a\overline{b}^2 + \overline{a}b^2 + 2 - 2|b|^2)^2 - 4|a|^2|}.$$

Moreover, if $a = 0$, the equation $F_{a,b} = 0$ has a unique double root, and the equation of conic is written as

$$\left|(1 - |b|^2)z + b\right| = \sqrt{2 - |b|^2} \qquad \text{(a circle)}.$$

In the case of $\mathrm{Re}(a\overline{b}^2) + 1 = |a| + |b^2|$; *The equation of conic in Theorem 8 is written as*

$$|\overline{t}z + t\overline{z} + 1|^2 = 2|t(z - s)| \qquad \text{(a parabola)},$$

where s and t are given by

$$s = \frac{a^2\overline{b}^2 - (2|b|^2 - 4)a + b^2}{a^2\overline{b}^3 + |ab|^2 b - 3a\overline{b}|b|^2 - 2b(|a|^2 - |b|^2) - \overline{a}b^3 + 4a\overline{b} - 2b},$$

and

$$t = \frac{a^2\overline{b}^3 + |ab|^2 b - 3a\overline{b}|b|^2 - 2b(|a|^2 - |b|^2) - \overline{a}b^3 + 4a\overline{b} - 2b}{2(|ab|^2 + a\overline{b}^2 + \overline{a}b^2 - 3|b|^2 + 4)}.$$

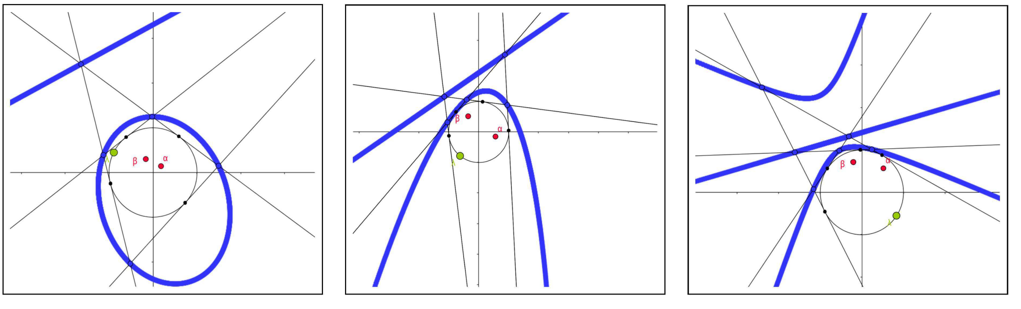

Figure 6: If B is a composition of two Blaschke products of degree 2, the trace T_B includes a conic.

Acknowledgements

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References

[1] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms, 4th edition*, Springer, (2015).

[2] U. Daepp, P. Gorkin, and R. Mortini, Ellipses and finite Blaschke products, *Amer. Math. Monthly*, **109** (2002) 785–795.

[3] H.-L. Gau and P. Y. Wu, Numerical range and Poncelet property, *Taiwanese J. Math.*, **7** (2003) 173–193.

[4] P. Gorkin and E. Skubak, Polynomials, ellipses, and matrices: Two questions, one answer, *Amer. Math. Monthly*, **118** (2011) 522–533.

[5] M. Fujimura, Inscribed ellipses and Blaschke products, *Comput. Methods Funct. Theory*, **13** (2013) 557–573.

[6] M. Fujimura, The complex geometry of Blaschke products of degree 3 and associated ellipses, accepted to *Filomat.*

[7] M. Fujimura, Circumscribed and inscribed ellipses induced by Blaschke products of degree 3, *RIMS Kôkyûroku*, **1976** (2015) 71–80.

[8] M. Marden, *Geometry of Polynomials, 2nd edition*, Amer. Math. Soc., (1966).

[9] P. Siebeck, Ueber eine neue analytische Behandlungsweise der Brennpunkte *J. Reine Angew. Math.*, **64** (1865) 175–182.

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