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Connection problems for Fuchsian ordinary differential equations and regular holonomic systems

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1 Connection problem for Fuchsian ordinary differential equations

In our previous paper [4], we formulated the connection problem for regular holonomic systems. In the present note, we shall explain our idea in more detail by several examples of Fuchsian ordinary differential equations.

1.1 Gauss hypergeometric differential equation

The Gauss hypergeometric differential equation

$$x(1-x)\frac{d^2u}{dx^2}+(\gamma-(\alpha+\beta+1))x\frac{du}{dx}-\alpha\beta u=0$$

with parameters $\alpha,\beta,\gamma\in\mathbb{C}$ has the Riemann scheme

$$\left\{\begin{matrix} x=0 & x=1 & x=\infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{matrix}\right\}.$$

This scheme is a table which notes the characteristic exponents at each singular point. For example, at $x=0$, if $1-\gamma\notin\mathbb{Z}$, we see that there are local solutions

$$u_{01}(x)=\varphi_1(x),\ u_{02}(x)=x^{1-\gamma}\varphi_2(x)$$

with convergent Taylor series $\varphi_1(x),\varphi_2(x)$ at $x=0$ satisfying $\varphi_1(0)=\varphi_2(0)=1$. Similarly, if $\gamma-\alpha-\beta\notin\mathbb{Z}$, there are local solutions

$$u_{11}(x)=\psi_1(x),\ u_{12}(x)=(1-x)^{\gamma-\alpha-\beta}\psi_2(x)$$

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at $x = 1$ with convergent Taylor series $\psi_1(x), \psi_2(x)$ at $x = 1$ satisfying $\psi_1(1) = \psi_2(1) = 1$. Since the radii of convergence of these four Taylor series are at least 1, the domain $\{|x| < 1\} \cap \{|x - 1| < 1\}$ is a common domain of definition for the above four local solutions. Then there exists a linear relation among two sets $(u_{01}(x), u_{02}(x)), (u_{11}(x), u_{12}(x))$ of fundamental system of solutions. The relation can be written in the form

$$(u_{01}(x), u_{02}(x)) = (u_{11}(x), u_{12}(x))C$$

with a constant 2×2-matrix C. We call this relation a connection relation, and C a connection matrix. The entries of C are called connection coefficients. When the parameter α, β, γ are generic, we have the explicit form of C:

$$C = \begin{pmatrix} \dfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} & \dfrac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \\ \dfrac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} & \dfrac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)} \end{pmatrix}.$$

This explicit form can be obtained in several ways. We can use Gauss-Kummer identity, or an integral representation of Euler type of solutions, and so on. By a connection problem we mean a problem of obtaining connection coefficients explicitly. Thus we have a complete answer to the connection problem for the Gauss hypergeometric differential equation in generic case.

1.2 Legendre differential equation

The Legendre differential equation

$$(1-t^2)\frac{d^2u}{dt^2} - 2t\frac{du}{dt} + \lambda u = 0$$

can be obtained from the Gauss hypergeometric differential equation by the change of variables

$$x = \frac{1-t}{2}$$

and the specialization of the parameters

$$\alpha + \beta = 1, \ \gamma = 1.$$

The parameter λ is given by $\lambda = -\alpha\beta$. Note that the Legendre differential equation corresponds to a non-generic Gauss equation because $1 - \gamma = 0 \in \mathbb{Z}$.

The Legendre differential equation appears in the process of solving the Laplace equation in $\mathbb{R}^3$ by separation of variables. For example, in determining the Coulomb potential, which satisfies the Laplace equation, we come to the Legendre differential equation which possesses a solution holomorphic at both $t = 1$ and $t = -1$. The last condition determines special values of the parameter λ. We shall see how the connection problem is used to determine special values of λ.

The Riemann scheme for the Legendre equation is

$$\begin{Bmatrix} t=1 & t=-1 & t=\infty \\ 0 & 0 & \rho_1 \\ 0 & 0 & \rho_2 \end{Bmatrix},$$

where ρ_1, ρ_2 are the roots of $\rho^2 - \rho - \lambda = 0$. Since the exponents are $0,0$ at $t=1$, there are local solutions

$$u_{+1}(t) = \varphi_1(t), \ u_{+2}(t) = \varphi_2(t) + u_{+1}(t)\log(t-1)$$

with convergent Taylor series $\varphi_1(t), \varphi_2(t)$ at $t=1$ satisfying $\varphi_1(1) = 1$. Similarly, at $t=-1$ we have local solutions

$$u_{-1}(t) = \psi_1(t), \ u_{-2}(t) = \psi_2(t) + u_{-1}(t)\log(t+1)$$

with convergent Taylor series $\psi_1(t), \psi_2(t)$ at $t=-1$ satisfying $\psi_1(-1) = 1$.

By solving a connection problem ([3]), we obtain the relation

$$u_{+1}(t) = e^{\pi i(1-\alpha)}\left(u_{-1}(t) + \frac{1-e^{2\pi i\alpha}}{2\pi i}\, u_{-2}(t)\right)$$

if $\alpha \notin \mathbb{Z}$. It is readily seen that the solution $u_{+1}(t)$ holomorphic at $t=1$ cannot be holomorphic at $t=-1$. Then a solution holomorphic at both $t=1$ and $t=-1$ can exist only when $\alpha \in \mathbb{Z}$. It is shown that, when $\alpha \in \mathbb{Z}$, the Legendre differential equation has a polynomial solution, which is entirely holomorphic. In this way, we can determine the special values of λ as

$$\lambda = n(n-1) \quad (n \in \mathbb{Z})$$

by using the connection problem.

1.3 Generalized hypergeometric differential equation

The generalized hypergeometric series ${}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2\end{matrix}; x\right)$ satisfies a third order Fuchsian differential equation (${}_3\mathrm{E}_2$), whose Riemann scheme is given by

$$\begin{Bmatrix} x=0 & x=1 & x=\infty \\ 0 & 0 & \alpha_1 \\ 1-\beta_1 & 1 & \alpha_2 \\ 1-\beta_2 & -\beta_3 & \alpha_3 \end{Bmatrix},$$

where β_3 is determined by $\sum_{j=1}^3 \alpha_j = \sum_{j=1}^3 \beta_j$. The exponents $0,1$ at $x=1$ implies that the dimension of the space of solutions holomorphic at $x=1$ is 2. Note that there is no canonical choice of basis of this 2 dimensional space.

We can study the connection problem by using an integral representation of Euler type of solutions

$$u_\Delta(x) = \int_\Delta s^{\alpha_2-\beta_1}(1-s)^{\beta_1-\alpha_1-1} t^{\alpha_3-\beta_2}(t-x)^{-\alpha_3}(s-t)^{\beta_2-\alpha_2-1}\, ds\, dt.$$

We consider the domains of integration

$$\begin{aligned}
\Delta_1 &= \{(s,t) \mid s < t < 0\},\\
\Delta_2 &= \{(s,t) \mid t < s < 0\},\\
\Delta_3 &= \{(s,t) \mid 0 < s < 1, t < 0\},\\
\Delta_4 &= \{(s,t) \mid s > 1, t < 0\},\\
\Delta_5 &= \{(s,t) \mid 0 < s < t < x\},\\
\Delta_6 &= \{(s,t) \mid x < t < s < 1\}.
\end{aligned}$$

With each domain we attach a branch by the standard loading (cf. [7]), and regard the domains as twisted cycles. By using the method given by Aomoto [1], we get linear relations among twisted cycles. For example, we have the relations

$$\begin{cases} \Delta_1 + e_5\Delta_2 + e_1e_5\Delta_3 + e_1e_2e_5\Delta_4 = 0,\\ \Delta_1 + e_5^{-1}\Delta_2 + (e_1e_5)^{-1}\Delta_3 + (e_1e_2e_5)^{-1}\Delta_4 = 0, \end{cases}$$

where $e_1 = e^{\pi i(\alpha_2-\beta_1)}, e_2 = e^{\pi i(\beta_1-\alpha_1-1)}, e_3 = e^{\pi i(\alpha_3-\beta_2)}, e_4 = e^{\pi i(-\alpha_3)}, e_5 = e^{\pi i(\beta_2-\alpha_2-1)}$. On the other hand, we have asymptotic behaviors

$$\begin{aligned}
u_{\Delta_5}(x) &\sim C_5 x^{1-\beta_1} \quad (x \to 0),\\
u_{\Delta_6}(x) &\sim C_6 (1-x)^{-\beta_3} \quad (x \to 1)
\end{aligned}$$

for some non-zero constants C_5, C_6. Also we see that $u_{\Delta_1}(x), u_{\Delta_2}(x), u_{\Delta_3}(x), u_{\Delta_4}(x)$ are holomorphic at $x = 1$. Then we can choose a basis of the space of holomorphic solutions at $x = 1$ from among these four solutions. Let $u_{\Delta_j}(x)$, $u_{\Delta_k}(x)$ be a chosen basis. Then we have a connection relation

$$u_{\Delta_5}(x) = c_{56}u_{\Delta_6}(x) + c_{5j}u_{\Delta_j}(x) + c_{5k}u_{\Delta_k}(x).$$

The connection coefficients c_{56}, c_{5j}, c_{5k} are calculated by using the linear relations among the twisted cycles. If we choose $u_{\Delta_1}(x), u_{\Delta_4}(x)$ as a basis, we get the relation

$$u_{\Delta_5}(x) = c_{56}u_{\Delta_6}(x) + c_{51}u_{\Delta_1}(x) + c_{54}u_{\Delta_4}(x)$$

with

$$\begin{aligned}
c_{56} &= \frac{e_4e_5(e_2^2-1)}{e_{245}^2-1} = \frac{\sin\pi(\beta_1-\alpha_1)}{\sin\pi\beta_3},\\
c_{51} &= \frac{A}{e_1e_3e_5^2(e_1^2-1)(e_4^2-1)(e_{245}^2-1)},\\
c_{54} &= \frac{(e_2^2-1)(e_{1245}^2-1)(e_{345}^2-1)}{e_2e_3e_5(e_1^2-1)(e_4^2-1)(e_{245}^2-1)}\\
&= \frac{\sin\pi(\beta_1-\alpha_1)\sin\pi(\beta_2-\alpha_1-\alpha_3)\sin\pi\alpha_2}{\sin\pi(\beta_1-\alpha_2)\sin\pi\alpha_3\sin\pi\beta_3},
\end{aligned}$$

where

$$\begin{aligned}
A = 1 &- e_{15}^2 - e_{45}^2 + e_{145}^2 - e_{1245}^2 - e_{1345}^2 + e_{124}^2e_5^4\\
&+ e_{134}^2e_5^4 - e_{1234}^2e_5^4 + e_{15}^4e_{234}^2 + e_{123}^2e_{45}^4 - e_{14}^4e_{23}^2e_5^6.
\end{aligned}$$

We used the notation $e_{jk\cdots} = e_j e_k \cdots$.

In many cases as in the case of the Coulomb potential, we are interested in the vanishing or non-vanishing of connection coefficients. It is easy to see when the connection coefficient c_{56} or c_{54} vanishes, while it hard for c_{51}. However, we can easily get the condition for the vanishing of c_{51} under the condition $c_{54} = 0$. For example, if $e_2^2 = 1$, we have

$$A = -(e_{15}^2 - 1)(e_{45}^2 - 1)(e_{1345}^2 - 1).$$

Note that the space of solutions V_1 at $x = 1$ is decomposed into a direct sum

$$V_1 = V_1^1 \oplus V_1^{e^{2\pi i(-\beta_3)}},$$

where V_1^1 denotes the subspace of the holomorphic solutions at $x = 1$ and $V_1^{e^{2\pi i(-\beta_3)}} = \langle u_{\Delta_6} \rangle$. Then we can see when the component in V_1^1 of the solution u_{Δ_5} vanishes.

We note that another choice of the basis of V_1^1 works better. Namely, if we take $u_{\Delta_1}, u_{\Delta_3}$ as a basis, we have

$$u_{\Delta_5} = c'_{56} u_{\Delta_6} + c'_{51} u_{\Delta_1} + c'_{53} u_{\Delta_3}$$

with

$$\begin{aligned}
c'_{56} &= \frac{\sin \pi(\beta_1 - \alpha_1)}{\sin \pi\beta_3}, \\
c'_{51} &= \frac{\sin \pi(\beta_2 - \alpha_1) \sin \pi(\beta_2 - \alpha_2 - \alpha_3) \sin \pi\alpha_1}{\sin \pi(\alpha_1 - \alpha_2) \sin \pi\alpha_3 \sin \pi\beta_3}, \\
c'_{53} &= \frac{\sin \pi(\beta_1 - \alpha_1) \sin \pi(\beta_2 - \alpha_1 - \alpha_3) \sin \pi\alpha_2}{\sin \pi(\alpha_2 - \alpha_1) \sin \pi\alpha_3 \sin \pi\beta_3}.
\end{aligned}$$

1.4 Formulation of the connection problem

Looking at the above examples, we realize that the direct sum decomposition of the space of local solutions at a singular point is substantial for the connection problem. In order to get the direct sum decomposition, we use the local monodromy action.

We consider a Fuchsian ordinary differential equation (L) on the projective line $\mathbb{P}^1$. Let $a_0, a_1, \ldots, a_p$ be the regular singular points, and set $X = \mathbb{P}^1 \setminus \{a_0, a_1, \ldots, a_p\}$. For each a_j, we take a point $b_j \in X$ near a_j so that the circle K_j with center a_j of radius $|b_j - a_j|$ does not contain the other a_k's in its inside. We attach the positive direction to K_j. Let V_j be the vector space of solutions of (L) at b_j. The analytic continuation along K_j induces a linear transformation of V_j, which we call the local monodromy action at a_j. Then we can decompose V_j into a direct sum

$$V_j = \bigoplus_{\alpha} V_j^{\alpha}$$

by this action, where α is an eigenvalue and V_j^α is the generalized eigenspace for the eigenvalue α. Each V_j^α is stable under the action. For each j, α, we denote by

$$\pi_j^\alpha : V_j \to V_j^\alpha$$

the projection onto the component V_j^α. If V_j^α is not an eigenspace, we have a filtration

$$V_j^{\alpha,0} \subset V_j^{\alpha,1} \subset \cdots \subset V_j^\alpha,$$

which is called the logarithmic filtration, where $V_j^{\alpha,k}$ consists of solutions containing $(\log(x - a_j))^l$ with $l \leq k$. Each $V_j^{\alpha,k}$ is also stable under the action.

In the previous examples, the decompositions are given as follows. In the Gauss case, we have

$$V_0 = V_0^1 \oplus V_0^{e^{2\pi i(1-\gamma)}}, \ V_1 = V_1^1 \oplus V_1^{e^{2\pi i(\gamma-\alpha-\beta)}}$$

with

$$\begin{aligned} V_0^1 &= \langle u_{01} \rangle, \ V_0^{e^{2\pi i(1-\gamma)}} = \langle u_{02} \rangle, \\ V_1^1 &= \langle u_{11} \rangle, \ V_1^{e^{2\pi i(\gamma-\alpha-\beta)}} = \langle u_{12} \rangle. \end{aligned}$$

In the Legendre case, we have

$$V_1 = V_1^1, \ V_{-1} = V_{-1}^1$$

with the filtrations

$$V_1^{1,0} \subset V_1^{1,1} = V_1^1, \ V_{-1}^{1,0} \subset V_{-1}^{1,1} = V_{-1}^1,$$

where

$$\begin{aligned} V_1^{1,0} &= \langle u_{+1} \rangle, \ V_1^{1,1} = \langle u_{+1}, u_{+2} \rangle, \\ V_{-1}^{1,0} &= \langle u_{-1} \rangle, \ V_{-1}^{1,1} = \langle u_{-1}, u_{-2} \rangle. \end{aligned}$$

In the generalized hypergeometric case, we have

$$V_0 = V_0^1 \oplus V_0^{e^{2\pi i(1-\beta_1)}} \oplus V_0^{e^{2\pi i(1-\beta_2)}}, \ V_1 = V_1^1 \oplus V_1^{e^{2\pi i(-\beta_3)}}.$$

The second one has already been given in the previous subsection. We note that $\dim V_1^1 = 2$, and that

$$V_0^{e^{2\pi i(1-\beta_1)}} = \langle u_{\Delta_5} \rangle.$$

We shall go back to the general case. For each pair (j, k) of indices, we take a path γ_{jk} in X with the starting point b_j and the end point b_k. The result of the analytic continuation of V_j^α along γ_{jk} becomes a subspace of V_k, and hence is decomposed according to the direct sum decomposition of V_k. The connection problem can be understood as a problem to obtain each component

$$\pi_k^\beta \left((\gamma_{jk})_* V_j^\alpha \right)$$

for β. If we take bases of V_j^α and of V_k^β, the problem reduces to the usual connection problem, the evaluation of the connection coefficients.

The vanishing of a connection coefficient can be generalized in basis free manner as follows. We see

$$\dim \pi_k^\beta\left((\gamma_{jk})_* V_j^\alpha\right) \le \min\{\dim V_j^\alpha, \dim V_k^\beta\}.$$

Then, the vanishing of some connection coefficients corresponds to the inequality

$$\dim \pi_k^\beta\left((\gamma_{jk})_* V_j^\alpha\right) < \min\{\dim V_j^\alpha, \dim V_k^\beta\}.$$

When we consider the logarithmic filtrations, we are also interested in, for each l, the minimum of m such that

$$\pi_j^\beta\left((\gamma_{jk})_* V_j^{\alpha,l}\right) \subset V_k^{\beta,m}$$

holds.

2 Connection problem for regular holonomic systems

Our formulation of the connection problem for Fuchsian ordinary differential equations depends on the direct sum decomposition of the space of a local solution by the local monodromy action. In order to extend the problem to regular holonomic case, we need to define the local monodromy action, and for the purpose, here we recall the definition of the local monodromy for regular holonomic case.

Let $D \subset \mathbb{C}^n$ be a hypersurface, and

$$D = \bigcup_j D_j$$

its irreducible decomposition. Set $X = \mathbb{C}^n \setminus D = \mathbb{P}^n \setminus (D \cup H_\infty)$, where H_∞ is the hyperplane at infinity, and take a base point $b \in X$. We denote by D° the set of regular points of D. Consider an irreducible component D_j. For any point $a \in D_j \cap D^\circ$, we can take a complex line Π which passes through a and is in general position with respect to D. Take a $(+1)$-loop $\tilde{\gamma}$ for a in Π. Connecting b to the starting point of the $(+1)$-loop by a path μ in X, we get a $(+1)$-loop $\mu\tilde{\gamma}\mu^{-1}$ for a in X. It can be shown that the conjugacy class of such $(+1)$-loop in $\pi_1(X, b)$ is uniquely determined by D_j. Then, if we consider a representation

$$\rho : \pi_1(X, b) \to \mathrm{GL}(V),$$

the conjugacy class $[\rho(\gamma)]$ of the image of a $(+1)$-loop γ for $a \in D_j \cap D^\circ$ is uniquely determined by D_j, which we call the local monodromy at D_j. Thus we understand that, in holonomic case, each irreducible component of the singular locus plays a similar role as a singular point of ordinary differential equations.

Now we know how to obtain a direct sum decomposition of the space of local solutions. Let (M) be a regular holonomic system with the singular locus D.

Take an irreducible component D_j of D, and a point b_j near D_j. Let V_j be the space of solutions at b_j. Then, similarly as in ODE case, we can decompose V_j by the local monodromy action into the direct sum

$$V_j = \bigoplus_{\alpha} V_j^{\alpha},$$

where α is an eigenvalue of the local monodromy action and V_j^{α} the generalized eigenspace for α. (We also have the logarithmic filtration for each generalized eigenspace.) The connection problem will be a problem to study the relation among V_j^{α} and V_k^{β}, and, in ODE case, we took a path γ_{jk} to relate V_j to V_k. However, in holonomic case, we do not need to take such path, since two irreducible components D_j and D_k may meet. Thus we take a point b_{jk} near an intersection point of D_j and D_k, and consider the space V_{jk} of solutions at b_{jk}. The space V_{jk} can be decomposed in two ways as

$$\begin{aligned} V_{jk} &= \bigoplus_{\alpha} V_j^{\alpha} \\ &= \bigoplus_{\beta} V_k^{\beta}. \end{aligned}$$

According to these decompositions, we have two sets of projections

$$\begin{aligned} \pi_j^{\alpha} &: V_{jk} \to V_j^{\alpha}, \\ \pi_k^{\beta} &: V_{jk} \to V_k^{\beta}. \end{aligned}$$

Then the connection problem is the study of the components

$$\pi_k^{\beta}(V_j^{\alpha})$$

for α and β. This is our formulation of the connection problem for regular holonomic systems.

Another distinguished nature for regular holonomic case is the existence of simultaneous basis for several direct sum decompositions. Let a be an intersection point of several irreducible components $D_{j_1}, D_{j_2}, \ldots, D_{j_m}$. Take a point $b \in X$ near a, and let V be the space of local solutions at b. If these irreducible components are normally crossing at a, thanks to the results by Gérard [2] and Yoshida-Takano [12], we have a basis of V such that each member of the basis belongs to some direct sum component simultaneously for every decomposition by a local monodromy action. We call the problem to find such simultaneous basis a *trivialization*, which is a solution of the connection problem at a normally crossing point. At a non-normally crossing point, we should solve a usual connection problem.

In our paper [4], we solved connection problems for Appell's hypergeometric series F_1 and F_2 along the above formulation. We do not repeat the results here, however, we find that our formulation works well for these cases.

There are not so many works on the connection problem for regular holonomic systems. We refer the readers to the works [5], [6],[8], [9], [10] and [11].

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