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On the theory of Laplace hyperfunctions in several variables

By

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Abstract

We survey the theory of Laplace hyperfunctions in several variables in [1, 2, 9]. A Laplace hyperfunction in one variable was first introduced by H. Komatsu ([3]-[8]) to consider the Laplace transform for a hyperfunction. We here construct Laplace hyperfunctions in several variables and their Laplace transform.

§1. A vanishing theorem of cohomology groups for the sheaf of holomorphic functions of exponential type

We briefly recall the vanishing theorem of cohomology groups on a Stein open subset with coefficients in holomorphic functions of exponential type and the edge of the wedge theorem for them.

Let \( n \) be a natural number, and let \( M \) be an \( n \)-dimensional \( \mathbb{R} \)-vector space. Let \( E \) be the complexification of \( M \). We denote by \( \mathbb{D}_E \) the radial compactification of \( E \) which is defined by

\[
\mathbb{D}_E := E \cup (E \setminus \{0\})/\mathbb{R}_+.\infty.
\]

Let \( U \) be an open subset in \( \mathbb{D}_E \). A holomorphic function \( f(z) \) in \( U \cap E \) is said to be of exponential type if, for any compact subset \( K \) in \( U \), there exist positive constants \( C_K \) and \( H_K \) such that

\[
|f(z)| \leq C_K e^{H_K|z|} \quad (z \in K \cap E).
\]

We denote by \( \mathcal{O}_{\mathbb{D}_E}^{\exp} \) the sheaf of holomorphic functions of exponential type on \( \mathbb{D}_E \).

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To recall the vanishing theorem of cohomology groups on a Stein open subset for $\mathcal{O}_{\mathbb{C}^2}^{\text{exp}}$, we give the definition of the regularity condition at $\infty$ for an open subset in $\mathbb{D}_{\mathbb{C}^2}$. We denote by $E_{\infty}$ the set $\mathbb{D}_{\mathbb{C}^2} \setminus E$. For a subset $V$ in $\mathbb{D}_{\mathbb{C}^2}$, we define the set $\text{clos}^1_{\infty}(V) \subset E_{\infty}$ as follows. A point $z_{\infty} \in E_{\infty}$ belongs to $\text{clos}^1_{\infty}(V)$ if and only if there exist points $\{z_k\}_{k \in \mathbb{N}}$ in $V \cap E$ which satisfy $z_k \to z_{\infty}$ in $\mathbb{D}_{\mathbb{C}^2}$ and $|z_{k+1}|/|z_k| \to 1$ ($k \to \infty$). Set

$(1.2) \quad N^1_{\infty}(V) := E_{\infty} \setminus \text{clos}^1_{\infty}(E \setminus V)$.

**Definition 1.1.** An open subset $U$ in $\mathbb{D}_{\mathbb{C}^2}$ is said to be regular at $\infty$ if $N^1_{\infty}(U) = U \cap E_{\infty}$ is satisfied.

Note that this condition is equivalent to saying $E_{\infty} \setminus U = \text{clos}^1_{\infty}(E \setminus U)$.

**Theorem 1.2** ([2], Theorem 3.7). Let $U$ be an open subset in $\mathbb{D}_{\mathbb{C}^2}$. Assume that $U \cap E$ is pseudo-convex in $E$ and $U$ is regular at $\infty$, then we have

$(1.3) \quad H^k(U, \mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}) = 0 \quad (k \neq 0)$.

The regularity condition of $U$ at $\infty$ plays an essential role in our vanishing theorem of cohomology groups for $\mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}$ as the following shows.

**Example 1.3** ([2], Example 3.17). We consider the radial compactification $\mathbb{D}_{\mathbb{R}}$ of $\mathbb{C}^2$. Let $(1,0)_{\infty} \in \mathbb{D}_{\mathbb{C}^2} \setminus \mathbb{C}^2$. Set

$V := \left\{ (z_1, z_2) \in \mathbb{C}^2; |\arg(z_1)| < \frac{\pi}{4}, |z_2| < |z_1| \right\}$,

$U := \left( \overline{V} \right)^\circ \setminus \{(1,0)_{\infty}\} \subset \mathbb{D}_{\mathbb{C}^2}$.

It is easy to check that $U \cap E = V$ is pseudo-convex in $\mathbb{C}^2$ and $U$ is not regular at $\infty$. In this case, we have $H^1(U, \mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}) \neq 0$.

Furthermore, by showing a Martineau type theorem for $\mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}$, we have the following theorem, which is a kind of the edge of the wedge type theorem for $\mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}$. Let $\overline{M}$ be the closure of $M$ in $\mathbb{D}_{\mathbb{C}^2}$.

**Theorem 1.4** ([1], Corollary 3.16). The closed subset $\overline{M} \subset \mathbb{D}_{\mathbb{C}^2}$ is purely $n$-codimensional relative to the sheaf $\mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}$, i.e.,

$(1.4) \quad \mathcal{H}^k_M(\mathcal{O}_{\mathbb{D}_{\mathbb{C}^2}}^{\text{exp}}) = 0 \quad (k \neq n)$.

§ 2. Laplace hyperfunctions and their Laplace transform

In this section we construct Laplace transform for Laplace hyperfunctions with support in an $\mathbb{R}_+\text{-conic}$ closed convex cone in $\overline{M}$ and their inverse Laplace transforms. We first recall the definition of Laplace hyperfunctions:
Definition 2.1. The sheaf of Laplace hyperfunctions on $\overline{M}$ is defined by

\[(2.1) \quad \mathcal{B}^\text{exp}_M := \mathcal{H}^n_M(O_D^\text{exp}) \otimes_{Z^n_M} \omega_M.\]

Here $\omega_M$ is the orientation sheaf $\mathcal{H}^n_M(Z_D^\text{exp})$ and $Z_D^\text{exp}$ is the constant sheaf on $D_E$ having stalk $\mathbb{Z}$.

Let $a \in M$ and $K$ be an $\mathbb{R}^+$-conic closed convex cone in $M$. Let us denote by $K_a$ the set $\{z + a; z \in K\}$ and denote by $\overline{K_a}$ the closure of $K_a$ in $\overline{M}$. We first get the representation of $\Gamma_{\overline{K_a}}(\overline{M}, B^\text{exp}_M)$ by the relative Čech cohomology groups with coefficients in $O_D^\text{exp}$.

Let us prepare some notation and the proposition below. For a subset $Z \subset D_E$, set

\[(2.2) \quad N_\infty(Z) := E_\infty \setminus (E \setminus Z).\]

For an open subset $U \subset E$, define

\[(2.3) \quad \hat{U} := U \cup N_\infty(U).\]

Definition 2.2. Let $\Omega$ be an open subset in $\overline{M}$ and $\Gamma$ an $\mathbb{R}^+$-conic open cone in $M$. Let $U$ be an open subset in $D_E$. We call $U$ a wedge of the type $\Omega \times \sqrt{-1} \Gamma$ if $U$ satisfies the following conditions.

1. $U \subset (\Omega \times \sqrt{-1} \Gamma),$
2. For any open proper subcone $\Gamma'$ of $\Gamma$, there exists an open neighborhood $V$ of $\Omega$ in $D_E$ such that

\[(2.4) \quad (M \times \sqrt{-1} \Gamma') \cap V \subset U.\]

We have the following proposition.

Proposition 2.3. Let $K$ be an $\mathbb{R}^+$-conic closed cone in $M$ and $\Gamma$ a proper open cone in $M$. Assume that $\Gamma$ is given by the intersection of finite number of half-spaces in $M$. Then there exist an open neighborhood $\Omega$ of $\overline{K}$ in $\overline{M}$ and an open subset $U$ in $D_E$ such that the following conditions are satisfied.

1. $U$ is a wedge of the type $\Omega \times \sqrt{-1} \Gamma$.
2. $U$ is Stein and regular at $\infty$.
3. $U$ is an open neighborhood of $\Omega \setminus \overline{K}$ in $D_E$.

Now let us consider the representation of $\Gamma_{K_a}(\overline{M}, B^\text{exp}_M)$ by the relative Čech cohomology with coefficients in $O_D^\text{exp}$. Choose vectors $\gamma_0, \ldots, \gamma_n \in S^{n-1}$. By Proposition 2.3, we can take an open neighborhood $\Omega$ of $\overline{K_a}$ in $\overline{M}$ and an open subset $U_j \subset D_E$ which is the wedge of the type $\Omega \times \sqrt{-1} \gamma_j^\alpha$, Stein and regular at $\infty$, and furthermore,
an open neighborhood of \( \Omega \setminus \overline{K_a} \). Here \( \gamma_j^o \) denotes the polar set \( \{ y \in M; y \gamma_j > 0 \} \) of \( \gamma_j \). We also take a neighborhood \( U \) of \( \overline{K_a} \) in \( D_E \) which is Stein and regular at \( \infty \). Then \( \Omega = \{ U, U_0, \ldots, U_n \} \) and \( \Omega' = \{ U_0, \ldots, U_n \} \) give a relative open covering of the pair \((U, U \setminus \overline{K_a})\). Hence we have

\[
\Gamma_{K_a}(\overline{M}, B_{M}^{\exp}) = \frac{\text{Ker}\{ \bigoplus_{j=0}^{n} O_{D_E}^{\exp}(\bigcap_{l \neq j} U_l) \} \to O_{D_E}^{\exp}(\bigcap_{l=0}^{n} U_l) \}}{\text{Im}\{ \bigoplus_{j \neq k} O_{D_E}^{\exp}(\bigcap_{l \neq j, k} U_l) \to \bigoplus_{j=0}^{n} O_{D_E}^{\exp}(\bigcap_{l \neq j} U_l) \} \}}.
\]

Let us define the Laplace transform for an element \( f = \bigoplus_{j=0}^{n} F_j \) of the above representation of \( \Gamma_{K_a}(\overline{M}, B_{M}^{\exp}) \). Set, for \( j = 0, 1, \ldots, n \),

\[
D_j := \{ x + \sqrt{-1}y \in E; x \in \Gamma, y = \varphi(x) \gamma \},
\]

where we take an appropriate closed cone \( \Gamma \subset \Omega \) which contains \( K \) and a point \( \gamma \in \bigcap_{l \neq j} \gamma_l^o \). Further, the continuous function \( \varphi: \Gamma \rightarrow \mathbb{R}_+ \cup \{0\} \) is chosen to satisfy the following conditions: (1) \( \varphi(x) = 0 \) in \( \partial \Gamma \), (2) \( \overline{D_j} \cap \overline{K_a} = \emptyset \), (3) \( \overline{D_j} \subset U_j \). Note that such \( \Gamma, \gamma \) and \( \varphi \) always exist for each \( j \).

**Definition 2.4.** Under the above situation, the Laplace transform of \( f = \bigoplus_{j=0}^{n} F_j \in \Gamma_{K_a}(\overline{M}, B_{M}^{\exp}) \) is defined by the integral

\[
\mathcal{L}(f)(\lambda) := \sum_{j=0}^{n} \sigma_j \int_{D_j} F_j(z) e^{-\lambda z} dz,
\]

where \( \sigma_j := \text{sgn} \left( \det(\omega_0, \cdots, \omega_{j-1}, \omega_{j+1}, \cdots, \omega_n) \right) \).

Note that the Laplace transform does not depend on the choice of \( \Gamma, \gamma \) and \( \varphi \).

**Definition 2.5.** Let \( \Omega \) be an open subset in \( D_E \). The set \( O_{D_E}^{a,\inf}(\Omega) \) consists of a holomorphic function \( f(z) \) on \( \Omega \cap E \) such that, for any compact subset \( K \subset \Omega \) and \( \epsilon > 0 \), \( f(z) \) satisfies

\[
|e^{az} f(z)| \leq C_{K, \epsilon} e^{\epsilon |z|}, \quad z \in K \cap E.
\]

with a positive constant \( C_{K, \epsilon} \).

Then we find that the Laplace transform gives the following morphism.

\[
\mathcal{L} : \Gamma_{K_a}(\overline{M}, B_{M}^{\exp}) \longrightarrow O_{D_E}^{a,\inf}(N_{\infty}(K^o)).
\]

Here \( K^o \) denotes the dual open cone of \( K \) in \( E \). Since the above morphism does not depend on the representation of \( \Gamma_{K_a}(\overline{M}, B_{M}^{\exp}) \), \( \mathcal{L} \) is well-defined.

**Definition 2.6.** Let \( T \) be an open subset in \( E_\infty \), and \( U \) an open subset in \( D_E \). We say that \( U \) has the opening wider than or equal to \( T \) at \( \infty \) if \( T \subset N_{\infty}(U) \) is satisfied.
We have the following lemma which plays an important role in establishing the inverse Laplace transform.

**Lemma 2.7.** The following conditions are equivalent:

1. \( f \in \mathcal{O}_{D_E}^{a, \text{inf}}(N_{\infty}(K^o)) \).
2. There exists an open subset \( U \) in \( E \) whose opening is wider than or equal to \( N_{\infty}(K^o) \) such that \( f \) is holomorphic on \( U \) and, for any compact subset \( K \) in \( \tilde{U} \), there exists an infra-linear function \( \phi_K(s) \) satisfying
   \[
   |e^{az}f(z)| \leq e^{\phi_K(|z|)}, \quad z \in K \cap E.
   \]
3. There exists an infra-linear function \( \phi(s) \) and an open subset \( U \) in \( E \) whose opening is wider than or equal to \( N_{\infty}(K^o) \) such that \( f \) is holomorphic on \( U \) with
   \[
   |e^{az}f(z)| \leq e^{\phi(|z|)}, \quad z \in U.
   \]

Let us define the inverse Laplace transform.

**Definition 2.8.** We define the morphism

\[
\mathcal{S} : \mathcal{O}_{D_E}^{a, \text{inf}}(N_{\infty}(K^o)) \rightarrow \mathcal{B}_{\frac{\mathrm{e}\mathrm{x}}{M}}^\text{p}(\overline{M})
\]

by

\[
\mathcal{S}(f) = \bigoplus_{0 \leq k \leq n} \sigma_k f_k, \quad f \in \mathcal{O}_{D_E}^{a, \text{inf}}(N_{\infty}(K^o)).
\]

Here \( f_k \) is given by the integral

\[
f_k(z) := \frac{1}{(2\pi \sqrt{-1})^n} \int_{T_k} f(\lambda)e^{\lambda z}d\lambda.
\]

The path of the integration \( T_k \) is given as follows. Set

\[
\Sigma_k := \{ \eta \in M; \eta = \sum_{j \neq k} t_j \gamma_j, t_j \geq 0 \}.
\]

Let \( \psi \) be an infra-linear function, and let \( \hat{\xi} \) be a point in the dual open cone of \( K \) in \( M \). Then we put

\[
T_k := \left\{ \lambda = \xi + \sqrt{-1}\eta \in E ; \eta \in \Sigma_k, \quad \xi = \psi(|\eta|)\hat{\xi} \right\}.
\]

Note that the integral \( f_k \) does not depend on the choice of \( \psi \) and \( \hat{\xi} \) if \( \psi \) is rapidly increasing. We can see that \( f_k \) is a holomorphic function of exponential type on \( (M \times \sqrt{-1}\cap_{j \neq k} \gamma_j^o) \) by Lemma 2.7.

Furthermore, we have:
Lemma 2.9. \( \text{supp}(\mathcal{S}(f)) \subset \overline{K_a} \) for \( f \in \mathcal{O}^a_{D_E}^{\inf}(N_{\infty}(K^o)) \).

Hence we have the inverse Laplace transform, and we can show that it satisfies the following theorem.

Theorem 2.10. \( \mathcal{S} \circ \mathcal{L} = \text{id}_{\mathcal{O}^a_{D_E}^{\inf}(N_{\infty}(K^o))} \), \( \mathcal{L} \circ \mathcal{S} = \text{id}_{\Gamma_{\overline{K_a}}(\overline{M}, B_{\text{ex}}^N)} \).

References