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On the theory of Laplace hyperfunctions in several variables

By

NAOFUMI HONDA* and KOHEI UMETA**

Abstract

We survey the theory of Laplace hyperfunctions in several variables in [1, 2, 9]. A Laplace hyperfunction in one variable was first introduced by H. Komatsu ([3]-[8]) to consider the Laplace transform for a hyperfunction. We here construct Laplace hyperfunctions in several variables and their Laplace transform.

§ 1. A vanishing theorem of cohomology groups for the sheaf of holomorphic functions of exponential type

We briefly recall the vanishing theorem of cohomology groups on a Stein open subset with coefficients in holomorphic functions of exponential type and the edge of the wedge theorem for them.

Let $n$ be a natural number, and let $M$ be an $n$-dimensional $\mathbb{R}$-vector space. Let $E$ be the complexification of $M$. We denote by $\mathbb{D}_E$ the radial compactification of $E$ which is defined by

$$\mathbb{D}_E := E \cup ((E \setminus \{0\})/\mathbb{R}_+) \infty.$$

Let $U$ be an open subset in $\mathbb{D}_E$. A holomorphic function $f(z)$ in $U \cap E$ is said to be of exponential type if, for any compact subset $K$ in $U$, there exist positive constants $C_K$ and $H_K$ such that

$$(1.1) \quad |f(z)| \leq C_K e^{H_K |z|} \quad (z \in K \cap E).$$

We denote by $\mathcal{O}_{\mathbb{D}_E}^{\exp}$ the sheaf of holomorphic functions of exponential type on $\mathbb{D}_E$.  

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To recall the vanishing theorem of cohomology groups on a Stein open subset for $\mathcal{O}_{\mathbb{D}_{E}}^\exp$, we give the definition of the regularity condition at $\infty$ for an open subset in $\mathbb{D}_{E}$. We denote by $E_\infty$ the set $\mathbb{D}_{E} \setminus E$. For a subset $V$ in $\mathbb{D}_{E}$, we define the set $\text{clos}_\infty^1(V) \subset E_\infty$ as follows. A point $z_\infty \in E_\infty$ belongs to $\text{clos}_\infty^1(V)$ if and only if there exist points $\{z_k\}_{k \in \mathbb{N}}$ in $V \cap E$ which satisfy $z_k \to z_\infty$ in $\mathbb{D}_{E}$ and $|z_{k+1}|/|z_k| \to 1 \ (k \to \infty)$. Set

$$N_\infty^1(V) := E_\infty \setminus \text{clos}_\infty^1(E \setminus V).$$

**Definition 1.1.** An open subset $U$ in $\mathbb{D}_{E}$ is said to be regular at $\infty$ if $N_\infty^1(U) = U \cap E_\infty$ is satisfied.

Note that this condition is equivalent to saying $E_\infty \setminus U = \text{clos}_\infty^1(E \setminus U)$.

Now we state our vanishing theorem of cohomology groups for $\mathcal{O}_{\mathbb{D}_{E}}^\exp$.

**Theorem 1.2** ([2], Theorem 3.7). Let $U$ be an open subset in $\mathbb{D}_{E}$. Assume that $U \cap E$ is pseudo-convex in $E$ and $U$ is regular at $\infty$, then we have

$$H^k(U, \mathcal{O}_{\mathbb{D}_{E}}^\exp) = 0 \ (k \neq 0).$$

The regularity condition of $U$ at $\infty$ plays an essential role in our vanishing theorem of cohomology groups for $\mathcal{O}_{\mathbb{D}_{E}}^\exp$ as the following shows.

**Example 1.3** ([2], Example 3.17). We consider the radial compactification $\mathbb{D}_{\mathbb{C}^2}$ of $\mathbb{C}^2$. Let $(1,0)\infty \in \mathbb{D}_{\mathbb{C}^2} \setminus \mathbb{C}^2$. Set

$$V := \left\{(z_1, z_2) \in \mathbb{C}^2; |\arg(z_1)| < \frac{\pi}{4}, |z_2| < |z_1|\right\},$$

$$U := (\overline{V})^\circ \setminus \{(1,0)\infty\} \subset \mathbb{D}_{\mathbb{C}^2}.$$

It is easy to check that $U \cap E = V$ is pseudo-convex in $\mathbb{C}^2$ and $U$ is not regular at $\infty$. In this case, we have $H^1(U, \mathcal{O}_{\mathbb{D}_{E}}^\exp) \neq 0$.

Furthermore, by showing a Martineau type theorem for $\mathcal{O}_{\mathbb{D}_{E}}^\exp$, we have the following theorem, which is a kind of the edge of the wedge type theorem for $\mathcal{O}_{\mathbb{D}_{E}}^\exp$. Let $\overline{M}$ be the closure of $M$ in $\mathbb{D}_{E}$.

**Theorem 1.4** ([1], Corollary 3.16). The closed subset $\overline{M} \subset \mathbb{D}_{E}$ is purely $n$-codimensional relative to the sheaf $\mathcal{O}_{\mathbb{D}_{E}}^\exp$, i.e.,

$$\mathcal{H}^k_M(\mathcal{O}_{\mathbb{D}_{E}}^\exp) = 0 \ (k \neq n).$$

§ 2. Laplace hyperfunctions and their Laplace transform

In this section we construct Laplace transform for Laplace hyperfunctions with support in an $\mathbb{R}_+\text{-conic}$ closed convex cone in $\overline{M}$ and their inverse Laplace transforms. We first recall the definition of Laplace hyperfunctions:
Definition 2.1. The sheaf of Laplace hyperfunctions on $\overline{M}$ is defined by

$$\mathcal{B}_{\frac{\text{ex}}{M}} := \mathcal{H}^m_{\overline{M}}(\mathcal{O}^\text{exp}_{D_E}) \otimes \omega_{\overline{M}}.$$ (2.1)

Here $\omega_{\overline{M}}$ is the orientation sheaf $\mathcal{H}^m_{\overline{M}}(Z_{D_E})$ and $Z_{D_E}$ is the constant sheaf on $D_E$ having stalk $\mathbb{Z}$.

Let $a \in M$ and $K$ be an $\mathbb{R}_+$-conic closed convex cone in $M$. Let us denote by $K_a$ the set $\{z + a; z \in K\}$ and denote by $\overline{K}_a$ the closure of $K_a$ in $\overline{M}$. We first get the representation of $\Gamma_{\overline{K}_a}(\overline{M}, \mathcal{B}_{\frac{\text{ex}}{M}})$ by the relative Čech cohomology groups with coefficients in $\mathcal{O}^\text{exp}_{D_E}$.

Let us prepare some notation and the proposition below. For a subset $Z \subset D_E$, set

$$N_\infty(Z) := E_\infty \setminus (E \setminus Z).$$ (2.2)

For an open subset $U \subset E$, define

$$\widehat{U} := U \cup N_\infty(U).$$ (2.3)

Definition 2.2. Let $\Omega$ be an open subset in $\overline{M}$ and $\Gamma$ an $\mathbb{R}^+$-conic open cone in $M$. Let $U$ be an open subset in $D_E$. We call $U$ a wedge of the type $\Omega \times \sqrt{-1} \Gamma$ if $U$ satisfies the following conditions.

1. $U \subset (\Omega \times \sqrt{-1} \Gamma)$,
2. For any open proper subcone $\Gamma'$ of $\Gamma$, there exists an open neighborhood $V$ of $\Omega$ in $D_E$ such that

$$\left(M \times \sqrt{-1} \Gamma'\right) \cap V \subset U.$$ (2.4)

We have the following proposition.

Proposition 2.3. Let $K$ be an $\mathbb{R}_+$-conic closed cone in $M$ and $\Gamma$ a proper open cone in $M$. Assume that $\Gamma$ is given by the intersection of finite number of half-spaces in $M$. Then there exist an open neighborhood $\Omega$ of $\overline{K}$ in $\overline{M}$ and an open subset $U$ in $D_E$ such that the following conditions are satisfied.

1. $U$ is a wedge of the type $\Omega \times \sqrt{-1} \Gamma$.
2. $U$ is Stein and regular at $\infty$.
3. $U$ is an open neighborhood of $\Omega \setminus \overline{K}$ in $D_E$.

Now let us consider the representation of $\Gamma_{\overline{K}_a}(\overline{M}, \mathcal{B}_{\frac{\text{ex}}{M}})$ by the relative Čech cohomology with coefficients in $\mathcal{O}^\text{exp}_{D_E}$. Choose vectors $\gamma_0, \ldots, \gamma_n \in S^{n-1}$. By Proposition 2.3, we can take an open neighborhood $\Omega$ of $\overline{K}_a$ in $\overline{M}$ and an open subset $U_j \subset D_E$ which is the wedge of the type $\Omega \times \sqrt{-1} \gamma_j^\circ$, Stein and regular at $\infty$, and furthermore,
an open neighborhood of $\Omega \setminus \overline{K_a}$. Here $\gamma_j^o$ denotes the polar set $\{ y \in M; y\gamma_j > 0 \}$ of $\gamma_j$. We also take a neighborhood $U$ of $\overline{K_a}$ in $\mathbb{D}_E$ which is Stein and regular at $\infty$. Then $\mathcal{U} = \{ U, U_0, \ldots, U_n \}$ and $\mathcal{U}' = \{ U_0, \ldots, U_n \}$ give a relative open covering of the pair $(U, U \setminus \overline{K_a})$. Hence we have

$$\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}_{\frac{\text{e}m}{M}}^\exp) = \frac{\text{Ker}\{ \bigoplus_{j=0}^n \mathcal{O}_{\mathbb{D}_E}^{\exp}(\bigcap_{l \neq j} U_l) \to \mathcal{O}_{\mathbb{D}_E}^{\exp}(\bigcap_{l=0}^n U_l) \}}{\text{Im}\{ \bigoplus_{j \neq k} \mathcal{O}_{\mathbb{D}_E}^{\exp}(\bigcap_{l \neq j, k} U_l) \to \bigoplus_{j=0}^n \mathcal{O}_{\mathbb{D}_E}^{\exp}(\bigcap_{l \neq j} U_l) \}}.$$

Let us define the Laplace transform for an element $f = \bigoplus_{j=0}^n F_j$ of the above representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}_{\frac{\text{e}m}{M}}^\exp)$. Set, for $j = 0, 1, \ldots, n$,

$$D_j := \{ x + \sqrt{-1}y \in E; x \in \Gamma, y = \varphi(x) \gamma \},$$

where we take an appropriate closed cone $\Gamma \subset \Omega$ which contains $K$ and a point $\gamma \in \bigcap_{l \neq j} \gamma_l^o$. Further, the continuous function $\varphi : \Gamma \to \mathbb{R}_+ \cup \{0\}$ is chosen to satisfy the following conditions: (1) $\varphi(x) = 0$ in $\partial \Gamma$, (2) $\overline{D}_j \cap \overline{K_a} = \emptyset$, (3) $\overline{D}_j \subset U_j$. Note that such $\Gamma$, $\gamma$ and $\varphi$ always exist for each $j$.

**Definition 2.4.** Under the above situation, the Laplace transform of $f = \bigoplus_{j=0}^n F_j \in \Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}_{\frac{\text{e}m}{M}}^\exp)$ is defined by the integral

$$\mathscr{L}(f)(\lambda) := \sum_{j=0}^n \sigma_j \int_{D_j} F_j(z) e^{-\lambda z} dz,$$

where $\sigma_j := \text{sgn} \left( \det(\omega_0, \cdots, \omega_{j-1}, \omega_{j+1}, \cdots, \omega_n) \right)$.

Note that the Laplace transform does not depend on the choice of $\Gamma$, $\gamma$ and $\varphi$.

**Definition 2.5.** Let $\Omega$ be an open subset in $\mathbb{D}_E$. The set $\mathcal{O}_{\mathbb{D}_E}^{a, \inf}(\Omega)$ consists of a holomorphic function $f(z)$ on $\Omega \cap E$ such that, for any compact subset $K \subset \Omega$ and $\epsilon > 0$, $f(z)$ satisfies

$$|e^{\epsilon z} f(z)| \leq C_{K, \epsilon} e^{\epsilon |z|}, \quad z \in K \cap E.$$

with a positive constant $C_{K, \epsilon}$.

Then we find that the Laplace transform gives the following morphism.

$$\mathscr{L} : \Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}_{\frac{\text{e}m}{M}}^\exp) \to \mathcal{O}_{\mathbb{D}_E}^{a, \inf}(N_{\infty}(K^o)).$$

Here $K^o$ denotes the dual open cone of $K$ in $E$. Since the above morphism does not depend on the representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}_{\frac{\text{e}m}{M}}^\exp)$, $\mathscr{L}$ is well-defined.

**Definition 2.6.** Let $T$ be an open subset in $E_\infty$, and $U$ an open subset in $\mathbb{D}_E$. We say that $U$ has the opening wider than or equal to $T$ at $\infty$ if $T \subset N_{\infty}(U)$ is satisfied.
We have the following lemma which plays an important role in establishing the inverse Laplace transform.

**Lemma 2.7.** The following conditions are equivalent:
1. \( f \in \mathcal{O}_{\mathrm{D}_{E}}^{a,\inf}(N_{\infty}(K^{o})) \).
2. There exists an open subset \( U \) in \( E \) whose opening is wider than or equal to \( N_{\infty}(K^{o}) \) such that \( f \) is holomorphic on \( U \) and, for any compact subset \( K \) in \( \hat{U} \), there exists an infra-linear function \( \phi_{K}(s) \) satisfying
   \[
   |e^{az}f(z)| \leq e^{\phi_{K}(|z|)}, \quad z \in K \cap E.
   \]
3. There exists an infra-linear function \( \phi(s) \) and an open subset \( U \) in \( E \) whose opening is wider than or equal to \( N_{\infty}(K^{o}) \) such that \( f \) is holomorphic on \( U \) with
   \[
   |e^{az}f(z)| \leq e^{\phi(|z|)}, \quad z \in U.
   \]

Let us define the inverse Laplace transform.

**Definition 2.8.** We define the morphism

\[
\mathcal{S} : \mathcal{O}_{\mathrm{D}_{E}}^{a,\inf}(N_{\infty}(K^{o})) \rightarrow \mathcal{B}_{\frac{\mathrm{e}\mathrm{x}}{M}}^{\mathrm{p}}(\overline{M})
\]

by

\[
\mathcal{S}(f) = \bigoplus_{0 \leq k \leq n} \sigma_{k}f_{k}, \quad f \in \mathcal{O}_{\mathrm{D}_{E}}^{a,\inf}(N_{\infty}(K^{o})).
\]

Here \( f_{k} \) is given by the integral

\[
f_{k}(z) := \frac{1}{(2\pi\sqrt{-1})^{n}} \int_{T_{k}} f(\lambda)e^{\lambda z}d\lambda.
\]

The path of the integration \( T_{k} \) is given as follows. Set

\[
\Sigma_{k} := \{ \eta \in M; \eta = \sum_{j \neq k} t_{j}\gamma_{j}, t_{j} \geq 0 \}.
\]

Let \( \psi \) be an infra-linear function, and let \( \hat{\xi} \) be a point in the dual open cone of \( K \) in \( M \). Then we put

\[
T_{k} := \left\{ \lambda = \xi + \sqrt{-1}\eta \in E ; \eta \in \Sigma_{k}, \quad \xi = \psi(|\eta|)\hat{\xi} \right\}.
\]

Note that the integral \( f_{k} \) does not depend on the choice of \( \psi \) and \( \hat{\xi} \) if \( \psi \) is rapidly increasing. We can see that \( f_{k} \) is a holomorphic function of exponential type on \( (M \times \sqrt{-1}\cup_{j \neq k} \gamma_{j}^{o}) \) by Lemma 2.7.

Furthermore, we have:
Lemma 2.9. $\text{supp}(\mathcal{S}(f)) \subset \overline{K_a}$ for $f \in \mathcal{O}_{\mathcal{D}_E}^{a,\inf}(N_{\infty}(K^\circ))$.

Hence we have the inverse Laplace transform, and we can show that it satisfies the following theorem.

Theorem 2.10. $\mathcal{I} \circ \mathcal{L} = \text{id}_{\Gamma_{\overline{K_a}}(\overline{M}, B_{\text{exp}}^\mathbb{M})}$, $\mathcal{L} \circ \mathcal{I} = \text{id}_{\mathcal{O}_{\mathcal{D}_E}^{a,\inf}(N_{\infty}(K^\circ))}$.

References