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Some examples of coupling equations for differential equations of normal form

By

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Abstract

The notion of coupling equations was introduced by H. Tahara [2], for a theory of a class of transformations between some nonlinear partial differential equations in complex domains. In his original coupling theory, solutions to a coupling equation were treated as formal power series of a special form in infinitely many variables. Recently, in collaboration with R. Schäfke and H. Tahara [1], the author studied a functional analytic treatment of coupling equations and showed the unique solvability of their initial value problems.

At a glance, in both cases, solutions to coupling equations seem to correspond with linear or nonlinear differential operators of infinite order. But, actually, corresponding transformations are in general not local operators outside the initial surface.

In this report, we give some elementary and "solvable" examples of coupling equations for differential equations of normal form, in order to illustrate such a non-local nature of couplings.

§1. Introduction

The notion of coupling equations was introduced by H. Tahara in [2] for partial differential equations of normal form, and in [3] and [4] for partial differential equations of Briot-Bouquet type. where he studied a class of transformations between nonlinear partial differential equations in complex domains.

Let us recall the notion of coupling equations for partial differential equations of normal form. (Some terminologies and notations are slightly changed.)

Consider two partial differential equations

$$\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}) \quad (\mathsf{F}), \qquad \frac{\partial v}{\partial t} = G(t, x, v, \frac{\partial v}{\partial x}) \quad (\mathsf{G}),$$

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with holomorphic functions F and G in a neighborhood of the origin of $\mathbb{C}^4_{(t,x,z_0,z_1)}$, and correspondences between unknown functions $u(t,x)$ and $v(t,x)$, of form

$$\Phi : u \mapsto v, \quad v(t,x) = \Phi[u](t,x) = \phi(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x),\frac{\partial^2 u}{\partial x^2}(t,x),\dots),$$
$$\Psi : v \mapsto u, \quad u(t,x) = \Psi[v](t,x) = \psi(t,x,v(t,x),\frac{\partial v}{\partial x}(t,x),\frac{\partial^2 v}{\partial x^2}(t,x),\dots),$$

given in terms of "holomorphic functions of infinitely many variables" $\phi(t,x,z_0,z_1,\dots)$ and $\psi(t,x,z_0,z_1,\dots)$. In what follows, we shall instead write $\phi(t,x,z)$ and so on, with the notation $z = (z_i)_{i\in\mathbb{N}}$. For such correspondences to become transformations between solution spaces of the equations (F) and (G), ϕ and ψ should formally satisfy the relation

$$\text{(F}\to\text{G)} \qquad \frac{\partial \phi}{\partial t} + \sum_{m\in\mathbb{N}} D^m[F](t,x,z_0,\dots,z_{m+1})\frac{\partial \phi}{\partial z_m} = G(t,x,\phi,D[\phi]),$$

$$\text{(G}\to\text{F)} \qquad \frac{\partial \psi}{\partial t} + \sum_{m\in\mathbb{N}} D^m[G](t,x,z_0,\dots,z_{m+1})\frac{\partial \psi}{\partial z_m} = F(t,x,\psi,D[\psi]).$$

Here D denotes the formal vector field of infinitely many variables defined by

$$D := \frac{\partial}{\partial x} + \sum_{m\in\mathbb{N}} z_{m+1}\frac{\partial}{\partial z_m}.$$

The equation (F $\to$ G) is called the *coupling equation* from (F) to (G), and (G $\to$ F) is called that from (G) to (F). Sometimes (G $\to$ F) is called the *reversed equation* of (F $\to$ G). We want to solve these coupling equations under the additional initial conditions: $\phi|_{t=0} = z_0$ and $\psi|_{t=0} = z_0$ respectively. Such initial value problems are referred as $(\mathsf{F}\to\mathsf{G})_{z_0}$ and $(\mathsf{G}\to\mathsf{F})_{z_0}$. For example, the former is written as

$$(\mathsf{F}\to\mathsf{G})_{z_0} \qquad \begin{cases} \frac{\partial \phi}{\partial t} + \sum_{m\in\mathbb{N}} D^m[F](t,x,z_0,\dots,z_{m+1})\frac{\partial \phi}{\partial z_m} = G(t,x,\phi,D[\phi]), \\ \phi(0,x,z) = z_0. \end{cases}$$

In [2], $\phi(t,x,z)$ was treated as a formal power series in $(t,z_0,z_1,\dots)$ of form

$$\text{(FPS)} \qquad \phi = z_0 + \sum_{k\ge 1} \phi_k(x,z_0,\dots,z_k)t^k \in \sum_{k\ge 0} \mathcal{O}_{\mathbb{C}}(\{|x|\le R\})[[z_0,\dots,z_k]]t^k,$$

where the first term (i.e., the term corresponding to $k=0$) is fixed to be z_0, according to the initial condition $\phi|_{t=0} = z_0$. There, in the case $G \equiv 0$, that is, for the coupling

$$\frac{\partial u}{\partial t} = F(t,x,u,\frac{\partial u}{\partial x}) \quad \text{(F)} \qquad \longleftrightarrow \qquad \frac{\partial v}{\partial t} = 0 \quad \text{(0)},$$

the following problems were studied:

- the unique existence of a formal power series solution ϕ to $(\mathsf{F}\to 0)$.

- the estimate of ϕ so that $v = \Phi[u] = \phi(t, x, u, \partial u/\partial x, \cdots)$ makes sense as a transformation between some classes of solutions.
- similar statements for the solution ψ to $(0 \to \mathsf{F})$.
- "reversibility" of ϕ and ψ, (i.e., Φ and Ψ are inverses each other, for some classes of solutions).

Recently in [1], R. Schäfke, H. Tahara and the author studied a functional analytic treatment of coupling equations for the purpose of further applicability. Let us briefly recall its main result.

We defined a "domain" $\Omega^{\mathbb{C}}_{d,c,+0}$ in $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}^{\mathbb{N}}_z$, a weight function $\omega_{d,c,0}(|t|, x, z)$ of Nagumo type on $\Omega^{\mathbb{C}}_{d,c,+0}$, and a Banach space $\mathscr{X}^{\mathbb{C}}_{d,c}$ consisting of "holomorphic" functions on $\Omega^{\mathbb{C}}_{d,c,+0}$ satisfying some estimates given in terms of $\omega_{d,c,0}$, which are parametrized by a function $d(x)$ and a constant $c \geq 1$. Moreover, we defined a closed subset $\mathscr{X}^{\mathbb{C}}_{d,c}(\alpha, \beta)$ of $\mathscr{X}^{\mathbb{C}}_{d,c}$ parametrized by positive constants α and β. Finally we constructed an integral operator T such that the integral equation $\phi = T[\phi]$ for $\phi \in \mathscr{X}^{\mathbb{C}}_{d,c}(\alpha, \beta)$ is expected to be equivalent to the initial value problem $(\mathsf{F} \to \mathsf{G})_{z_0}$.

Theorem 1.1 (c.f. [1, Theorem 3.3]). *Under a suitable choice of constants α, β and c, the integral operator T becomes a contraction map from $\mathscr{X}^{\mathbb{C}}_{d,c}(\alpha, \beta)$ to itself. As a conclusion, the initial value problem $(\mathsf{F} \to \mathsf{G})_{z_0}$ has a unique solution in $\mathscr{X}^{\mathbb{C}}_{d,c}(\alpha, \beta)$.*

Moreover, the aforementioned solution can be expanded into a formal power series of form given in (FPS).

In this report, we do not go into details on this functional analytic approach. But, by comparing both results on unique solvability, we can at least say that once we find a formal power series solution, it corresponds to a holomorphic solution defined on some domain.

In what follows, we use ∂_t, ∂_x and ∂_{z_i} instead of $\partial/\partial t$, $\partial/\partial x$ and $\partial/\partial z_i$. Therefore, the formal vector field D shall be written as $D = \partial_x + \sum_{i \in \mathbb{N}} z_{i+1} \partial_{z_i}$.

§ 2. Examples

In this section, we shall give 3 examples of couplings among 3 differential equations. Throughout this section, F and G denote the functions

$$F(t, x, z_0, z_1) := z_1, \quad G(t, x, z_0, z_1) := z_0^2,$$

and (F), (G) and (0) denote the differential equations with right hand side given by F, G and 0.

Example 2.1. Consider a pair of partial differential equations

$$\partial_t u = \partial_x u \quad (\mathsf{F}), \qquad \partial_t v = 0 \quad (0).$$

First, we study the initial value problem of the coupling equation $(\mathsf{F}\to\mathsf{0})_{z_0}$, given by

$$(\mathsf{F}\to\mathsf{0})_{z_0} \qquad \begin{cases} \partial_t\phi + \sum_{m\in\mathbb{N}} z_{m+1}\partial_{z_m}\phi = 0, \\ \phi(0,x,z) = z_0. \end{cases}$$

Here we used $D^m z_1 = z_{m+1}$ in the derivation of the equation. This problem seems to be an initial value problem of a partial differential equation, but it involves infinitely many variables $(t, x, z_0, z_1, \dots)$.

We will take a heuristic argument, and try to solve it as an initial value problem of a partial differential equation in (t, x, z) of 1st order. Using parameters τ, ξ and $\zeta = (\zeta_i)_{i\in\mathbb{N}}$, the calculation goes as follows. The relations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dz_i}{z_{i+1}} = \frac{d\phi}{0} = d\tau, \quad (t,x,z,\phi)|_{\tau=0} = (0,\xi,\zeta,\zeta_0)$$

imply

$$(t, x, \phi) = (\tau, \xi, \zeta_0),$$

and also

$$z_{i+j} = \partial_\tau^j z_i, \quad z_i|_{\tau=0} = \zeta_i, \quad \text{for any } i, j \in \mathbb{N}.$$

The latter relations give the Taylor expansion of z_i in τ as $z_i = \sum_{j\in\mathbb{N}} \zeta_{i+j}\tau^j/j!$, $\forall i \in \mathbb{N}$, or,

$$z = \exp(\tau J)\zeta,$$

where J denotes the shift operator $\zeta = (\zeta_i)_{i\in\mathbb{N}} \mapsto J\zeta = (\zeta_{i+1})_{i\in\mathbb{N}}$. Therefore, we have

$$(t, x, z, \phi) = (\tau, \xi, \exp(\tau J)\zeta, \zeta_0).$$

Since the relation $\exp(\tau J)\zeta = z$ can be inverted into $\zeta = \exp(-\tau J)z$, we can eliminate the parameters and get the solution

$$\phi(t,x,z) = \sum_{j\in\mathbb{N}} z_j(-t)^j/j!.$$

Note that the right hand side can be understood as a formal power series as in (FPS), and we can easily check that it really satisfies $(\mathsf{F}\to\mathsf{0})_{z_0}$. This solution ϕ corresponds to the transformation $\Phi_{\mathsf{F}\to\mathsf{0}}$ given by

$$\begin{aligned}\Phi_{\mathsf{F}\to\mathsf{0}}[u](t,x) &= \sum_{j\in\mathbb{N}} \frac{(-t)^j}{j!}\partial_x^j u(t,x) = \sum_{j\in\mathbb{N}} \frac{(-t\partial_x)^j}{j!} u(t,x) = \exp(-t\partial_x)u(t,x) \\ &= u(t, x-t),\end{aligned}$$

if $u(t,x)$ is analytic in x and if any Taylor expansion of $u(t,x)$ in x at each (t,x) admits a radius of convergence larger than $|t|$.

We can see that a solution to (F) is written as $u = f(x+t)$ with an arbitrary function $f(x)$ in x, and that $v = \Phi[u](t,x) = f(x)$ is a solution to (0) for any f.

Next, we study the reversed problem $(0 \to \mathsf{F})_{z_0}$.

$$(0 \to \mathsf{F})_{z_0} \qquad \begin{cases} \partial_t \psi = \partial_x \psi + \sum_{m\in\mathbb{N}} z_{m+1} \partial_{z_m} \psi, \\ \psi(0,x,z) = z_0. \end{cases}$$

Similarly, a heuristic argument gives the following calculation. The relations

$$\frac{dt}{1} = \frac{dx}{-1} = \frac{dz_i}{-z_{i+1}} = \frac{d\psi}{0} = d\tau, \quad (t,x,z,\psi)|_{\tau=0} = (0,\xi,\zeta,\zeta_0)$$

imply

$$(t,x,\psi) = (\tau, -\tau+\xi, \zeta_0),$$

and also

$$z_{i+j} = (-\partial_\tau)^j z_i, \quad z_i|_{\tau=0} = \zeta_i, \quad \text{for any } i,j \in \mathbb{N}.$$

The latter relations give $z_i = \sum_{j\in\mathbb{N}} \zeta_{i+j}(-\tau)^j/j!$, $i \in \mathbb{N}$, or,

$$z = \exp(-\tau J)\zeta.$$

Therefore, we have

$$(t,x,z,\psi) = (\tau, -\tau+\xi, \exp(-\tau J)\zeta, \zeta_0).$$

Using the similar inversion formula for $\exp(-\tau J)$, we can eliminate the parameters and get the solution

$$\psi(t,x,z) = \sum_{j\in\mathbb{N}} z_j t^j/j!,$$

which corresponds to the transformation $\Phi_{0\to\mathsf{F}}$ given by

$$\Phi_{0\to\mathsf{F}}[v](t,x) = \exp(t\partial_x)v(t,x) = v(t,x+t),$$

if $v(t,x)$ satisfies a similar regularity requirement.

We can similarly see the correspondence of solutions. Moreover, we can also see, in a straightforward manner, the reversibility of $\Phi_{\mathsf{F}\to 0}$ and $\Phi_{0\to\mathsf{F}}$, that is, the fact that $\Phi_{\mathsf{F}\to 0}$ and $\Phi_{0\to\mathsf{F}}$ are inverses each other.

Example 2.2. Consider a pair of ordinary differential equations

$$\partial_t u = 0 \quad (0), \qquad \partial_t v = v^2 \quad (\mathsf{G}),$$

in the variable t, while we regard them as partial differential equations in two variables (t,x).

First, we study the initial value problem $(0\to\mathsf{G})_{z_0}$.

$$(0\to\mathsf{G})_{z_0} \qquad \begin{cases} \partial_t\phi = \phi^2, \\ \phi(0,x,z) = z_0. \end{cases}$$

Using the uniqueness result of the initial value problems of coupling equations, we can immediately see that ϕ depends only on t, x and z_0. Note that ϕ is independent of x, although we will not use this fact. Therefore, it can be solved as an initial value problem of a partial differential equation in (t,x,z_0) of 1st order. In fact, we can calculate as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dz_0}{0} = \frac{d\phi}{\phi^2} = d\tau, \quad (t,x,z_0,\phi)|_{\tau=0} = (0,\xi,\zeta_0,\zeta_0)$$
$$\Longrightarrow \quad (t,x,z_0,\phi) = (\tau,\xi,\zeta_0,\frac{\zeta_0}{1-\tau\zeta_0}).$$

By eliminating the parameters, we get the solution

$$\phi(t,x,z) = \frac{z_0}{1-tz_0},$$

which corresponds to the transformation $\Phi_{0\to\mathsf{G}}$ given by

$$\Phi_{0\to\mathsf{G}}[u](t,x) = \frac{u(t,x)}{1-tu(t,x)}.$$

We can see that a solution to (0) is written as $u = f(x)$ with an arbitrary function $f(x)$ in x, and that $v = \Phi_{0\to\mathsf{G}}[u](t,x) = f(x)/(1-tf(x))$ is a solution to (G) for any f.

Next, we study the reversed problem $(\mathsf{G}\to 0)_{z_0}$.

$$(\mathsf{G}\to 0)_{z_0} \qquad \begin{cases} \partial_t\psi + \sum_{m\in\mathbb{N}}\sum_{j=0}^{m}\binom{m}{j}z_jz_{m-j}\cdot\partial_{z_m}\psi = 0, \\ \psi(0,x,z) = z_0. \end{cases}$$

Here we used the calculation $D^m z_0^2 = \sum_{j=0}^{m}\binom{m}{j}z_jz_{m-j}$ in the derivation of the equation.

Assume that $\psi(t,x,z)$ is a solution. Then, by substituting $z_1 = z_2 = \cdots = 0$, we have

$$\begin{cases} \partial_t\psi(t,x,z_0,0,0,\dots) + z_0^2\cdot\partial_{z_0}\psi(t,x,z_0,0,0,\dots) = 0, \\ \psi(0,x,z_0,0,0,\dots) = z_0. \end{cases}$$

Here we used

$$\sum_{j=0}^{m}\binom{m}{j}z_jz_{m-j}\Bigg|_{z_1=z_2=\cdots=0} = \begin{cases} z_0^2, & (m=0), \\ 0, & (m\geq 1). \end{cases}$$

Therefore, by defining $\tilde{\psi}(t,x,z) := \psi(t,x,(z_0,0,0,\dots))$, we have $\partial_{z_m}\tilde{\psi} = 0$, $(m\geq 1)$, and

$$\begin{cases} \partial_t\tilde{\psi} + z_0^2\cdot\partial_{z_0}\tilde{\psi} = \partial_t\tilde{\psi} + \sum_{m\in\mathbb{N}}\sum_{j=0}^{m}\binom{m}{j}z_jz_{m-j}\cdot\partial_{z_m}\tilde{\psi} = 0, \\ \tilde{\psi}(0,x,z) = z_0. \end{cases}$$

These relations imply that $\tilde{\psi}$ is also a solution to $(\mathsf{G}\to 0)_{z_0}$. Using the uniqueness result, we can conclude that $\psi \equiv \tilde{\psi}$. Therefore, ψ is independent of z_i, $(i \geq 1)$ and satisfies

$$\begin{cases} \partial_t \psi + z_0^2 \cdot \partial_{z_0} \psi = 0, \\ \psi(0, x, z) = z_0. \end{cases}$$

We can also see from these formulas that ψ is independent of x, although we do not use it.

The problem above is an initial value problem of a partial differential equation in (t, x, z_0) of 1st order, and we solve it as follows.

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dz_0}{z_0^2} = \frac{d\psi}{0} = d\tau, \quad (t, x, z_0, \psi)|_{\tau=0} = (0, \xi, \zeta_0, \zeta_0)$$
$$\Longrightarrow \quad (t, x, z_0, \psi) = (\tau, \xi, \frac{\zeta_0}{1 - \tau\zeta_0}, \zeta_0).$$

By eliminating the parameters, we get the solution

$$\psi(t, x, z) = \frac{z_0}{1 + tz_0},$$

which corresponds to the transformation $\Phi_{\mathsf{G}\to 0}$ given by

$$\Phi_{\mathsf{G}\to 0}[v](t, x) = \frac{v(t, x)}{1 + tv(t, x)}.$$

We can see that solutions to (G) is written as $v = f(x)/(1 - tf(x))$ with an arbitrary function $f(x)$ in x, and that

$$u = \Phi_{\mathsf{G}\to 0}[v](t, x) = \frac{f(x)/(1 - tf(x))}{1 + tf(x)/(1 - tf(x))} = f(x)$$

is a solution to (0) for any f.

Also note that the reversibility of $\Phi_{0\to\mathsf{G}}$ and $\Phi_{\mathsf{G}\to 0}$ can be checked in a straightforward manner.

Example 2.3. Consider a pair of partial differential equations

$$\partial_t u = F(t, x, u, \partial_x u) = \partial_x u \quad (\mathsf{F}), \qquad \partial_t v = G(t, x, v, \partial_x v) = v^2 \quad (\mathsf{G}).$$

We expect $\Phi_{\mathsf{F}\to\mathsf{G}} = \Phi_{0\to\mathsf{G}} \circ \Phi_{\mathsf{F}\to 0}$ and $\Phi_{\mathsf{G}\to\mathsf{F}} = \Phi_{0\to\mathsf{F}} \circ \Phi_{\mathsf{G}\to 0}$. But let us try to solve the coupling equations directly.

First, we study the problem $(\mathsf{F}\to\mathsf{G})_{z_0}$.

$$(\mathsf{F}\to\mathsf{G})_{z_0} \qquad \begin{cases} \partial_t \phi + \sum_{m\in\mathbb{N}} z_{m+1} \partial_{z_m} \phi = \phi^2, \\ \phi(0, x, z) = z_0. \end{cases}$$

The relations

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dz_i}{z_{i+1}} = \frac{d\phi}{\phi^2} = d\tau, \quad (t,x,z,\phi)|_{\tau=0} = (0,\xi,\zeta,\zeta_0)$$

imply

$$(t,x,z,\phi) = (\tau,\xi,\exp(\tau J)\zeta,\frac{\zeta_0}{1-\tau\zeta_0}),$$

where J denotes the same shift operator as in Example 2.1. By eliminating the parameters, we get the solution

$$\phi(t,x,z) = \frac{\sum_{j\in\mathbb{N}} z_j(-t)^j/j!}{1-t\sum_{j\in\mathbb{N}} z_j(-t)^j/j!},$$

which corresponds to the transformation $\Phi_{\mathsf{F}\to\mathsf{G}}$ given by

$$\Phi_{\mathsf{F}\to\mathsf{G}}[u](t,x) = \frac{u(t,x-t)}{1-tu(t,x-t)},$$

if $u(t,x)$ satisfies a similar regularity requirement as in Example 2.1.

Next, we study the reversed problem $(\mathsf{G}\to\mathsf{F})_{z_0}$.

$$(\mathsf{G}\to\mathsf{F})_{z_0} \qquad \begin{cases} \partial_t\psi + \sum_{m\in\mathbb{N}}\sum_{j=0}^{m}\binom{m}{j} z_j z_{m-j}\cdot\partial_{z_m}\psi = \partial_x\psi + \sum_{m\in\mathbb{N}} z_{m+1}\partial_{z_m}\psi, \\ \psi(0,x,z) = z_0. \end{cases}$$

The relations

$$\frac{dt}{1} = \frac{dx}{-1} = \frac{dz_i}{\sum_{j=0}^{i}\binom{i}{j} z_j z_{i-j} - z_{i+1}} = \frac{d\psi}{0} = d\tau, \quad (t,x,z,\psi)|_{\tau=0} = (0,\xi,\zeta,\zeta_0)$$

immediately imply

$$(t,x,\psi) = (\tau,-\tau+\xi,\zeta_0).$$

As for z, we use the following auxiliary function

$$X(\tau,\sigma) := \sum_{i\in\mathbb{N}} z_i(\tau)\frac{(\tau+\sigma)^i}{i!}.$$

(Precisely speaking, X is a formal power series in τ and σ.) Then, a straightforward calculation and the relations $\partial_\tau z_i = \sum_{j=0}^{i}\binom{i}{j} z_j z_{i-j}$ yield

$$\partial_\tau X(\tau,\sigma) = X(\tau,\sigma)^2,$$

which can be solved into

$$X(\tau,\sigma) = \frac{X(0,\sigma)}{1-\tau X(0,\sigma)}.$$

Since $X(0,\sigma) = \sum_{i\in\mathbb{N}} z_i(0)\sigma^i/i! = \sum_{i\in\mathbb{N}} \zeta_i\sigma^i/i!$, we have

$$X(\tau,\sigma) = \frac{\sum_{i\in\mathbb{N}} \zeta_i\sigma^i/i!}{1-\tau\sum_{i\in\mathbb{N}} \zeta_i\sigma^i/i!}.$$

This relation can be inverted into $\sum_{i\in\mathbb{N}} \zeta_i\sigma^i/i! = X(\tau,\sigma)/(1+\tau X(\tau,\sigma))$, and by substituting $\sigma = 0$, we get

$$\zeta_0 = \frac{\sum_{i\in\mathbb{N}} z_i\tau^i/i!}{1+\tau\sum_{i\in\mathbb{N}} z_i\tau^i/i!}.$$

They together give the solution

$$\psi(t,x,z) = \frac{\sum_{i\in\mathbb{N}} z_i t^i/i!}{1+t\sum_{i\in\mathbb{N}} z_i t^i/i!},$$

which corresponds to the transformation $\Phi_{\mathsf{G}\to\mathsf{F}}$ given by

$$\Phi_{\mathsf{G}\to\mathsf{F}}[v](t,x) = \frac{v(t,x+t)}{1+tv(t,x+t)},$$

if $v(t,x)$ satisfies a similar regularity requirement.

We can check our expectations $\Phi_{\mathsf{F}\to\mathsf{G}} = \Phi_{\mathsf{0}\to\mathsf{G}} \circ \Phi_{\mathsf{F}\to\mathsf{0}}$ and $\Phi_{\mathsf{G}\to\mathsf{F}} = \Phi_{\mathsf{0}\to\mathsf{F}} \circ \Phi_{\mathsf{G}\to\mathsf{0}}$ in a straightforward manner, and the same is true for the reversibility of $\Phi_{\mathsf{F}\to\mathsf{G}}$ and $\Phi_{\mathsf{G}\to\mathsf{F}}$.

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