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Monodromy of confluent hypergeometric system with two irregular singular points

By

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Abstract

This paper studies the monodromy of a coonfluent hypergeometric system with two irregular singular points and regular singular points. By using the convergent semi-formal solution introduced in [1] we will show a concrete formula of the monodromy around irregular singular and regular singular points.

§1. Introduction

In [2] we gave the concrete formula of the monodromy for the confluent Hamiltonian system with an irregular singular point. The system was obtained by the confluence of regular singular points of a hypergeometric system. We used the expression of convergent semi-formal solutions given by first integrals of the Hamiltonian system. (See also [1]). In this note we continue to study the monodromy of confluent hypergeometric systems in a Hamiltonian form with two irregular singular points. We will give concrete formula of the monodromy by using the semi-formal theory. We will see that equations with two irregular singular points and a regular singular point have different monodromy about the irregular singular point compared to the equations with one irregular singular point.

This paper is organized as follows. In section 2 we study the convergent semi-formal solutions. In section 3 we introduce a class of confluent hypergeometric system written in a Hamiltonian form. In section 4 we construct functionally independent first integrals and calculate the monodromy for a certain example.

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§ 2. Semi-formal solution via first integrals

Let $n \geq 2$ and $\sigma \geq 1$ be integers. Consider the system

$$z^{2\sigma}\frac{dq}{dz} = \nabla_p \mathcal{H}(z,q,p), \quad z^{2\sigma}\frac{dp}{dz} = -\nabla_q \mathcal{H}(z,q,p), \tag{2.1}$$

where $q = (q_2, \ldots, q_n)$, $p = (p_2, \ldots, p_n)$, and where $\mathcal{H}(z,q,p)$ is analytic in $z \in \mathbb{C}$ in some neighborhood of the origin and entire in $(q,p) \in \mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$. We note that, by taking $q_1 = z$ as a new unknown function (2.1) is written in an equivalent form for the Hamiltonian function H, $H(q_1, q, p_1, p) := p_1 q_1^{2\sigma} + \mathcal{H}(q_1, q, p)$

$$\begin{aligned} \dot{q}_1 &= H_{p_1} = q_1^{2\sigma}, \quad \dot{p}_1 = -H_{q_1} = -2\sigma p_1 q_1^{2\sigma-1} - \partial_{q_1}\mathcal{H}(q_1,q,p), \\ \dot{q} &= \nabla_p H = \nabla_p \mathcal{H}(q_1,q,p), \quad \dot{p} = -\nabla_q H = -\nabla_q \mathcal{H}(q_1,q,p). \end{aligned} \tag{2.2}$$

The solution of (2.1) is given in terms of that of (2.2) by taking $q_1 = z$ as an independent variable.

Semi-formal solution. We define the semi-formal solution of (2.1) following [1]. Let $\mathcal{O}(\tilde{S}_0)$ be the set of holomorphic functions on $\tilde{S}_0$, where $\tilde{S}_0$ is the universal covering space of the punctured disk of radius r, $S_0 = \{|z| < r\} \setminus \{0\}$ for some $r > 0$. The $(2n-2)$-vector $\check{x}(z,c)$ of formal power series of c

$$\check{x}(z,c) := \sum_{|\nu| \geq 0} \check{x}_\nu(z) c^\nu = \check{x}_0(z) + X(z)c + \sum_{|\nu| \geq 2} \check{x}_\nu(z) c^\nu \tag{2.3}$$

is said to be a *semi-formal solution* of (2.1) if $\check{x}_\nu \in (\mathcal{O}(\tilde{S}_0))^{2n-2}$ and $(q(z,c), p(z,c)) := \check{x}(z,c)$ is the formal power series solution of (2.1). As for the properties of the semi-formal series (2.3) we refer to [1]. Here $X(z)$ is a $(2n-2)$ square matrix with component belonging to $\mathcal{O}(\tilde{S}_0)$. If $X(z)$ is invertible, then we say that $(q(z,c), p(z,c))$ is a *complete semi-formal solution.* We say that a semi-formal solution is a convergent semi-formal solution (at the origin) if the following condition holds. For every compact set K in $\tilde{S}_0$ there exists a neighborhood U such that the formal series converges for $q_1 \in K$ and $c \in U$. The semi-formal solution at the general point $z_0 \in \mathbb{C}$ is defined similarly.

Monodromy function. We consider (2.1). Let z_0 be any point in $\mathbb{C}$ and let q and p be semi-formal solutions of (2.1) around z_0. We define the monodromy function $v(c)$ around z_0 by

$$(q,p)((z - z_0)e^{2\pi i} + z_0, v(c)) = (q,p)(z,c), \tag{2.4}$$

where $v(c) = (v_j(c))$. The existence of $v(c)$ in the class of formal power series of c is proved in [1]. If we denote the linear part of $v(c)$ by $M^{-1}c$, then by considering the linear part of the monodromy relation we have $X((z-z_0)e^{2\pi i} + z_0) = X(z)M$. Hence M is the so-called monodromy factor.

In the following we will show that the convergent semi-formal solutions of (2.1) can be obtained by solving certain system of nonlinear equations given by first integrals. We consider (2.2). Given functionally independent first integrals $H(q_1, q, p_1, p)$ and $\psi_j \equiv \psi_j(q_1, q, p)$ $(j = 1, 2, \ldots, 2n-2)$ of (2.2), where the functional independentness means that there exists a neighborhood V of the origin of (q, p, p_1) such that the matrix

$$(2.5) \qquad {}^t(\nabla_{q,p,p_1} H, \nabla_{q,p,p_1} \psi_j)_{j\downarrow 1,2,\ldots,2n-2}$$

has full rank $2n-1$ on $(q_1, p_1, q, p) \in \tilde{S}_0 \times V$. We assume that every coefficient of ψ_j expanded in the power series of q, p is holomorphic with respect to q_1 on $\tilde{S}_0$.

Let the point $(q_{1,0}, p_{1,0}, q_0, p_0)$ and the values $c_{j,0}$ $(j = 1, 2, \ldots, 2n-2)$ satisfy that

$$(2.6) \qquad H(q_{1,0}, p_{1,0}, q_0, p_0) = 0, \ \psi_j(q_{1,0}, q_0, p_0) = c_{j,0}, \ (j = 1, 2, \ldots, 2n-2).$$

For $c_j = \tilde{c}_j + c_{j,0}$, $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{2n-2}) \in \mathbb{C}^{2n-2}$ we consider the system of equations of p_1, q and p

$$(2.7) \qquad H(q_1, p_1, q, p) = 0, \ \psi_j(q_1, q, p) = c_j, \ (j = 1, 2, \ldots, 2n-2).$$

If (2.7) has a solution, then we denote it by $q \equiv q(q_1, c)$, $p \equiv p(q_1, c)$, $p_1 \equiv p_1(q_1, c)$. We see that q, p and p_1 are holomorphic functions of q_1 in $\tilde{S}_0$ and $\tilde{c}$ in some neighborhood of the origin if we assume (2.5). We have

Theorem 2.1. *Suppose that $H(q_1, q, p_1, p)$ and $\psi_j \equiv \psi_j(q_1, q, p)$ $(j = 1, 2, \ldots, 2n-2)$ be functionally independent. Assume (2.6). Then the solution of (2.7) gives the convergent complete semi-formal solution $(q(z, c), p(z, c))$ $(q_1 = z)$ of (2.1) provided q or p is not a constant function.*

The proof of Theorem 2.1 was given in [2].

§ 3. Confluent hypergeometric equation

We consider a class of hypergeometric system

$$(3.1) \qquad (z - C)\frac{dv}{dz} = Av,$$

where C is a diagonal matrix and A is a constant matrix. The system has only regular singular points on $\mathbb{C} \cup \{\infty\}$. Set $v = {}^t(q, p) \in \mathbb{C}^n$ and assume that C and A are block diagonal matrices

$$(3.2) \qquad C = \mathrm{diag}(\Lambda_1, \Lambda_1), \ A = \mathrm{diag}(A_1, -{}^tA_1)$$

where Λ_1 and A_1 are $n-1$ square diagonal and constant matrices, respectively such that $\Lambda_1 A_1 = A_1 \Lambda_1$. Define

$$H := \langle (z-\Lambda_1)^{-1}p, A_1 q\rangle. \tag{3.3}$$

Then one can write (3.1) in the Hamiltonian form

$$\frac{dq}{dz} = H_p(z,q,p), \quad \frac{dp}{dz} = -H_q(z,q,p). \tag{3.4}$$

First, by setting $z = 1/\zeta$ in (3.4) we have

$$-\zeta^2 \frac{dq}{d\zeta} = (\frac{1}{\zeta} - \Lambda_1)^{-1} A_1 q, \quad -\zeta^2 \frac{dp}{d\zeta} = -{}^t A_1 (\frac{1}{\zeta} - \Lambda_1)^{-1} p. \tag{3.5}$$

Subsitute $\zeta = \varepsilon^{-1}\eta$ in (3.5). Replace λ_ν with $\varepsilon\lambda_\nu$ if $\nu \in J$ and multiply the μ-th row of A_1 with ε^{-1} if $\mu \in J'$. Then we let $\varepsilon \to 0$. Define the diagonal matrix $\mathfrak{A}$ by $\mathfrak{A} := \text{diag }(\mathfrak{A}_1, \dots, \mathfrak{A}_n)$ where $\mathfrak{A}_\nu$ is given by $-\lambda_\nu^{-1}$ if $\nu \in J'$ and $(\eta^{-1} - \lambda_\mu)^{-1}$ if $\mu \in J$, respectively. Then we obtain

$$-\eta^2 \frac{dq}{d\eta} = \mathfrak{A} A_1 q, \quad -\eta^2 \frac{dp}{d\eta} = -{}^t A_1 \mathfrak{A} p. \tag{3.6}$$

We will write (3.6) in a Hamiltonian form. Set $\eta = q_1$, and define H by

$$H(q_1, p_1, q, p) := p_1 q_1^2 - \langle \mathfrak{A}(q_1) A_1 q, p\rangle. \tag{3.7}$$

One can easily see that $\dot{q} = \eta^2 \frac{dq}{d\eta}$ and $\dot{p} = \eta^2 \frac{dp}{d\eta}$. Because $-\mathfrak{A} A_1 q = H_p$ and $-{}^t A_1 \mathfrak{A} p = H_q$, one easily sees that (3.6) is equivalent to the Hamiltonian system with the Hamiltonian function (3.7).

We will introduce the irregular singularity by the confluence of singularities. We assume $\lambda_j \neq 0$ for all j. Suppose that λ_j's are mutually different. Then it follows from $\Lambda_1 A_1 = A_1 \Lambda_1$ that A_1 is a diagonal matrix. Denote the diagonal entries of A_1 by τ_j. Let J, J' and J'' be the nonempty disjoint subsets of $\{2, 3, \dots, n\}$ such that $J \cup J' \cup J'' = \{2, 3, \dots, n\}$. The characteristic roots λ_j corresponding to $j \in J$ and $j \in J'$ merge to 1 and 0, respectively. Then, by the confluence of singular points we obtain the Hamiltonian

$$H(q_1, p_1, q, p) = p_1 q_1^2 + \sum_{j \in J'} \frac{\tau_j}{\lambda_j} q_j p_j + \frac{q_1^2}{(q_1-1)^2} \sum_{j \in J} \frac{\tau_j}{\lambda_j} q_j p_j + q_1 \sum_{j \in J''} \frac{\tau_j}{\lambda_j} \frac{q_j p_j}{q_1 - \lambda_j^{-1}}. \tag{3.8}$$

§ 4. Calculation of monodromy

In this section we will calculate the monodromy for the Hamiltonian (3.8) via first integrals. We assume that λ_j's are mutually different. First, we construct first integrals of the Hamiltonian vector field

$$\begin{aligned}\chi_H := q_1^2\frac{\partial}{\partial q_1} - 2q_1p_1\frac{\partial}{\partial p_1} - \sum_{j\in J''}\frac{\tau_j}{\lambda_j}\frac{q_jp_j}{(q_1-\lambda_j^{-1})^2}\frac{\partial}{\partial p_1} \\ + \sum_{j\in J'}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right) + \sum_{j\in J''}\frac{\tau_j}{\lambda_j}\frac{q_1}{q_1-\lambda_j^{-1}}\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right) \\ + \frac{2q_1}{(q_1-1)^3}\left(\sum_{j\in J}\frac{\tau_j}{\lambda_j}q_jp_j\right)\frac{\partial}{\partial p_1} + \frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\left(q_j\frac{\partial}{\partial q_j} - p_j\frac{\partial}{\partial p_j}\right).\end{aligned} \tag{4.1}$$

For $k = 2, \ldots, n$ we will construct the first integrals in the form $q_kw_k(q_1)$. We see that w_k satisfies

$$\begin{cases} \left((q_1-1)^2\frac{\partial}{\partial q_1} + \frac{\tau_k}{\lambda_k}\right)w_k = 0 & \text{if } k \in J, \\ \left(q_1^2\frac{\partial}{\partial q_1} + \frac{\tau_k}{\lambda_k}\right)w_k = 0 & \text{if } k \in J' \\ \left(q_1^2\frac{\partial}{\partial q_1} + \frac{\tau_k}{\lambda_k}\frac{q_1}{q_1-\lambda_k^{-1}}\right)w_k = 0 & \text{if } k \in J''. \end{cases} \tag{4.2}$$

Hence we have

$$w_k(q_1) = \begin{cases} \exp\left(\frac{\tau_k}{\lambda_k(q_1-1)}\right) & \text{if } k \in J \\ \exp\left(\frac{\tau_k}{\lambda_kq_1}\right) & \text{if } k \in J' \\ \left(\frac{q_1}{q_1-\lambda_k^{-1}}\right)^{\tau_k} & \text{if } k \in J''. \end{cases} \tag{4.3}$$

Next we consider the first integrals $w := p_ku_k(q_1)$. By (4.1) the equation $\chi_Hw = 0$ can be written in a similar form as in the above. Hence we have $u_k(q_1) = \left(\frac{q_1}{q_1-\lambda_k^{-1}}\right)^{-\tau_k}$ if $k \in J''$, and $= \exp\left(-\frac{\tau_k}{\lambda_kq_1}\right)$ if $k \in J'$, $= \exp\left(-\frac{\tau_k}{\lambda_k(q_1-1)}\right)$ if $k \in J$. Hence we have

$$u_j(q_1) = w_j(q_1)^{-1}, \quad j = 2, \ldots, n. \tag{4.4}$$

By (4.3) and (4.4) we have the first integrals Ψ_j $(j = 2, \ldots, n)$ and $\tilde{\Psi}_j$ $(j = 2, \ldots, n)$

$$\Psi_j = q_jw_j(q_1), \qquad \tilde{\Psi}_j = p_jw_j(q_1)^{-1}. \tag{4.5}$$

Summing up the above we have

Theorem 4.1. *Assume $\lambda_j \neq 0, 1$ for all j and that λ_j's are mutually different. Then the Hamitonian vector field (4.1) has $2n-1$ functionally independent first integrals H, Ψ_j's and $\tilde{\Psi}_j$'s $(j = 2, \ldots, n)$ given by (4.5).*

We will determine monodromy using first integral. We take the convergent non constant semi-formal solution $q(q_1,c)$, $p(q_1,c)$ and $p_1(q_1,c)$ defined by (2.7). The monodromy function $v(c)$ around z_0 is defined by (2.4). In view of the argument in section 2, we will study the monodromy around the origin $z_0=0$ or around $z_0=\lambda_k^{-1}$ for some $k\in J$. Note that λ_k^{-1} is a regular singular point of the our equation which remains unchanged under the confluence procedure.

First we consider the case $z_0=0$. In order to determine the monodromy function $v(c)$, we first note $H(q_1e^{2\pi i},p_1,q,p)=H(q_1,p_1,q,p)$. For $2\le j\le n-1$ we have

$$\begin{aligned}\Psi_j(q_1e^{2\pi i},q,p)&=q_jw_j(q_1e^{2\pi i})=\\&=\begin{cases}e^{2\pi i\tau_j}q_jw_j(q_1)=c_je^{2\pi i\tau_j} & \text{if } j\in J''\\ q_jw_j(q_1)=c_j & \text{if } j\in J\cup J'.\end{cases}\end{aligned}\tag{4.6}$$

Similarly we have

$$\begin{aligned}\tilde{\Psi}_j(q_1e^{2\pi i},q,p)&=p_jw_j(q_1e^{2\pi i})^{-1}=\\&=\begin{cases}e^{-2\pi i\tau_j}p_jw_j(q_1)^{-1}=c_je^{-2\pi i\tau_j} & \text{if } j\in J''\\ p_jw_j(q_1)^{-1}=c_j & \text{if } j\in J\cup J'.\end{cases}\end{aligned}\tag{4.7}$$

We define $v(c)=(v_j(c))_j$ by

$$v_j(c)=\begin{cases}c_je^{2\pi i\tau_j} & \text{if } 2\le j\le n,\ j\in J''\\ c_j & \text{if } 2\le j\le n,\ j\in J\cup J'.\end{cases}\tag{4.8}$$

Similarly we define $\tilde{v}(c)=(\tilde{v}_j(c))_j$ by the right-hand side of (4.8) with τ_j in (4.8) replaced by $-\tau_j$.

As for the monodromy function around λ_k^{-1} ($k\in J''$) we define the monodromy function $w^{(k)}(c)$ by the right hand side of (4.8) with τ_j replaced by $-\tau_j\delta_{k,j}$. Here $\delta_{k,j}$ is Kronecker's delta. We similarly define $\tilde{w}^{(k)}(c)$ by replacing τ_j with $\tau_j\delta_{k,j}$.

Let q and p satisfy (2.7) with ψ_j's given by (4.5). Then we easily see that

$$H(q_1e^{2\pi i},p_1,q,p)=0,\quad \psi_j(q_1e^{2\pi i},q,p)=v_j(c),\quad 1\le j\le 2n-2.\tag{4.9}$$

By the uniqueness of semi-formal solution we obtain $q(q_1e^{2\pi i},v(c))=q(q_1,c)$ and $p(q_1e^{2\pi i},v(c))=p(q_1,c)$. This implies that $v(c)$ is the monodromy function as desired. In the case of other regular singular points we may argue in the same way as in the case of the origin. Thus we have proved

Theorem 4.2. *Assume $\lambda_j\neq 0,1$ for all j and that λ_j's are mutually different. Then the monodromy functions $v(c)$ about the origin corresponding to the semi-formal solution of (2.1) defined by (2.7) is given by $v(c)$ and $\tilde{v}(c)$. On the other hand, the the monodromy functions about $q_1=1$ is the identity function, while those around λ_k^{-1} ($k\in J''$) are given by $w^{(k)}(c)$ and $\tilde{w}^{(k)}(c)$.*

§ 5. Nonlinear case

Consider the Hamiltonian $H+H_1$, where H and H_1 are given, respectively, by (3.8) and

$$H_1=\sum_{j=2}^{n}q_j^2B_j(q_1,q), \tag{5.1}$$

where $B_j(q_1,q)$'s are holomorphic at the origin with respect to $(q_1,q)\in\mathbb{C}\times\mathbb{C}^{n-1}$. In order to give the formula of the monodromy we will construct first integrals of the Hamiltonian vector field $\chi_H+\chi_{H_1}$ in the forms $q_kw_k(q_1)$ $(k=2,\ldots,n)$ and $p_ku_k(q_1)+W_k(q_1,q)$ $(k=2,\ldots,n)$. Note that χ_{H_1} is given by

$$\chi_{H_1}=\sum_{j=2}^{n}\left(-2q_jB_j\frac{\partial}{\partial p_j}-q_j^2\sum_{\nu=2}^{n}\partial_{q_\nu}B_j\frac{\partial}{\partial p_\nu}-q_j^2(\partial_{q_1}B_j)\frac{\partial}{\partial p_1}\right). \tag{5.2}$$

As for the first integrals $q_kw_k(q_1)$ we have $\chi_{H_1}(q_kw_k(q_1))=0$ because the first integrals do not contain p and p_1. Hence $q_kw_k(q_1)$'s are first integrals of $\chi_H+\chi_{H_1}$, where w_k is given by (4.3).

We will construct the first integrals $p_ku_k(q_1)+W_k(q_1,q)$ by solving $(\chi_H+\chi_{H_1})(p_ku_k+W_k)=0$, where $u_k=w_k^{-1}$ and $k=2,\ldots,n$. We compare the coefficients of p_k in the equation. Because no term containing p_k appears from $\chi_{H_1}(p_ku_k+W_k)$, we may consider $\chi_H(p_ku_k)=0$. We easily see that u_k is given by $u_k=w_k^{-1}(q_1)$, where $w_k(q_1)$ is given by (4.3). Next we construct W_k by comparing the coefficients of the powers of $p_k^0=1$ in the equation $(\chi_H+\chi_{H_1})(p_ku_k+W_k)=0$. Because $\chi_{H_1}W_k=0$ by definition, it follows that W_k is determined by the equation

$$\chi_HW_k=-\chi_{H_1}(p_ku_k)=u_k\left(2q_kB_k+\sum_{j=2}^{n}q_j^2\partial_{q_k}B_j\right).$$

By expanding $B_j(q_1,q)=\sum_\ell B_j^{(\ell)}(q_1)q^\ell$ and $W_k(q_1,q)=\sum_\ell W_k^{(\ell)}(q_1)q^\ell$ and setting

$$\mathcal{R}^{(\ell)}(q_1)=\left(2B_k^{(\ell-e_k)}(q_1)+\sum_{j=2}^{n}(\ell+e_k-2e_j)B_j^{(\ell+e_k-2e_j)}(q_1)\right),$$

where e_k is the k-th unit vector, we see that $W_k^{(\ell)}(q_1)$ satisfies

$$\left(q_1^2\frac{d}{dq_1}+\sum_{j\in J'}\frac{\tau_j}{\lambda_j}\ell_j+\sum_{j\in J''}\frac{\tau_j}{\lambda_j}\frac{\ell_jq_1}{q_1-\lambda_j^{-1}}+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\frac{\tau_j}{\lambda_j}\ell_j\right)W_k^{(\ell)}=w_k(q_1)^{-1}\mathcal{R}^{(\ell)}(q_1). \tag{5.3}$$

In view of (4.2) we can easily see that the solution of the inhomogeneous equation is given by $\prod_{j=2}^{n} w_j(q_1)^{\ell_j}$. Hence $W_k^{(\ell)}$ is given by

$$W_k^{(\ell)} = \left(\prod_{\nu=2}^{n} w_\nu(q_1)^{\ell_\nu}\right) \int_a^{q_1} t^{-2} w_k(t)^{-1} \mathcal{R}^{(\ell)}(t) \prod_{\nu=2}^{n} w_\nu(t)^{-\ell_\nu} dt, \tag{5.4}$$

where $a \in \mathbb{C} \setminus 0$ is some fixed point. Note that $W_k^{(\ell)}$ is analytic on the universal covering space of $\mathbb{C}\setminus\{0, \lambda_j^{-1}(j \in J)\}$. The series $\sum_\ell W_k^{(\ell)}(q_1)q^\ell$ converges if q_1 is on some compact set in the universal covering space of $\mathbb{C} \setminus \{0, \lambda_j^{-1}(j \in J)\}$ and q is in some neighborhood of the origin. Note that $\sum_\ell W_k^{(\ell)}(q_1)q^\ell$ is the convergent semi-formal series. Summing up the above we have

Theorem 5.1. *The Hamiltonian system with the Hamiltonian function $H + H_1$ has $(2n-1)$ functionally independent first integrals of the form, $H + H_1$, $q_k w_k(q_1)$, $p_k w_k(q_1)^{-1} + W_k(q_1, q)$ $(k = 2, \ldots, n)$.*

We expect that one can calculate the monodromy of the Hamiltonian system with the Hamiltonian function $H + H_1$ by using the first integrals in Theorem 5.1, which is left for the future problem.

References

[1] Balser, W., Semi-formal theory and Stokes' phenomenon of nonlinear meromorphic systems of ordinary differential equations, Formal and analytic solutions of differential and difference equations, *Banach Center Publications*, **97** (2012), 11-28.

[2] Yoshino, M., Semi-formal solution and monodromy of some confluent hypergeometric equations, *RIMS Kôkyûroku Bessatsu*, **52** (2015), 255-262.