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DECAY OF EIGENFUNCTIONS OF ELLIPTIC PDE'S

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1. Introduction

We give an account of some results of our recent study on exponential decay rates of eigenfunctions of self-adjoint higher order elliptic operators on $\mathbb{R}^d$ [HS1, HS2]. Previously Agmon investigated the asymptotic behavior of the Green's function [Ag]. We are interested in decay rates as a function of direction and show that the possible decay rates are to a large extent determined algebraically. Consider a real elliptic polynomial Q of degree q on $\mathbb{R}^d$, $d \geq 2$. (Q elliptic means that for large $\xi \in \mathbb{R}^d, C|Q(\xi)| > |\xi|^q$ for some C.) We consider the operator $H = Q(p) + V(x)$, $p = -\mathrm{i}\nabla$, on $L^2 = L^2(\mathbb{R}^d)$ with V bounded and measurable. By the assumptions on Q the operator $Q(p)$ is self-adjoint with domain the standard Sobolev space of order q which consequently is also the domain of H. To simplify the presentation let us assume that $V = V_1 + V_2$ where V_1 is smooth and real-valued, and

$$\begin{aligned} &\forall\alpha: \ \partial^\alpha V_1(x) = o(|x|^{-|\alpha|}),\\ &V_2(x) = o(|x|^{-1/2}). \end{aligned} \tag{1.1}$$

We define the *global decay rate* of an L^2-eigenfunction ϕ of H with a real eigenvalue as

$$\sigma_g = \sup\{\sigma \geq 0 | \mathrm{e}^{\sigma|x|}\phi \in L^2\},$$

and assume

$$0 < \sigma_g < \infty. \tag{1.2}$$

We refer to [HS1] for sufficient conditions for (1.2).

It is intuitively clear that σ_g is determined by the directions of weakest exponential decay of ϕ. This statement can be made more precise by introducing more refined notions of exponential decay for a given $\phi \in L^2$ obeying (1.2) (not necessarily an eigenfunction). First let

$$\mathcal{E} = \{\eta \in \mathbb{R}^d | \mathrm{e}^{\eta\cdot x}\phi \in L^2\}.$$

Then we introduce three exponential decay rates depending on a direction $\omega \in S^{d-1}$:

$$\begin{aligned} &\sigma_c(\omega) = \sup\{\sigma|\mathrm{e}^{\sigma\omega\cdot x}\phi \in L^2\},\\ &\sigma_s(\omega) = \sup\{\eta\cdot\omega | \eta \in \mathcal{E}\},\\ &\sigma_{loc}(\omega) = \sup\{\sigma|\mathrm{e}^{\sigma|x|}\phi \in L^2(C) \text{ for some open cone C containing } \omega\}. \end{aligned}$$

It is easy to see that

$$\sigma_g \leq \sigma_c(\omega) \leq \sigma_s(\omega) \leq \sigma_{loc}(\omega), \quad \sigma_g = \inf_{\omega\in S^{d-1}} \sigma_{loc}(\omega).$$

We define the support function $\sigma_s(x)$ of the set $\mathcal{E}$ by $\sigma_s(t\omega) = t\sigma_s(\omega)$ for $t \geq 0$ (by definition $0\cdot\infty = 0$). Intuitively $\mathrm{e}^{-\sigma_s(x)}$ is an exponential upper bound of ϕ (assuming

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here that σ_s is finite at all points). Conversely this factor is an exponential lower bound in the following sense: Suppose $f : \mathbb{R}^d \to [0, \infty)$ is convex, $f(tx) = tf(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$, and in addition that $e^{tf}\phi \in L^2$ for all $t < 1$. Then $f(x) \leq \sigma_s(x)$ for all $x \in \mathbb{R}^d$, see [HS2].

The following result gives a criterion for having $\sigma_s = \sigma_{loc}$ at a given direction. Note that $\mathcal{E}$ is convex and therefore $\mathcal{E}$ has a supporting hyperplane at every $\eta_0 \in \partial\mathcal{E}$. If there is exactly one supporting hyperplane passing through $\eta_0 \in \partial\mathcal{E}$ we call η_0 a *regular point* of $\partial\mathcal{E}$.

Lemma 1.1. *Suppose $\omega_0 \in S^{d-1}$ is given so that for some (possibly not unique) regular point $\eta_0 \in \partial\mathcal{E}$ the supporting hyperplane at this point has normal ω_0. Then*

$$\sigma_{loc}(\omega_0) = \sigma_s(\omega_0) = \eta_0 \cdot \omega_0.$$

If $\mathcal{E}$ is bounded then all points in $\partial\mathcal{E}$ are regular if and only if $\partial\mathcal{E}$ is C^1, see [Ro, p. 246]. It turns out that for an L^2-eigenfunction with (1.2) the possible values of σ_s are to a large extent determined algebraically, see Theorem 2.1. In combination with Lemma 1.1 we would then get information on the *local decay rate* σ_{loc} for an L^2-eigenfunction which is more related to asymptotics than our other notions of decay rates, although in comparison with actual asymptotics being a somewhat rough measure. We give an example in Section 3 of an L^2-eigenfunction such that the assumption of Lemma 1.1 is fulfilled for some values of ω_0 while for other values of ω_0 the conclusion of the lemma is false, that is $\sigma_{loc}(\omega) > \sigma_s(\omega)$ for some ω.

2. Calculating the decay rate, $H\phi = \lambda\phi$

Our main result is the following theorem:

Theorem 2.1. *Suppose* (1.1), *$(H - \lambda)\phi = 0$ with $\lambda \in \mathbb{R}$ and $\phi \in L^2$, and that* (1.2) *are fulfilled.*

For $\omega_0 \in S^{d-1}$ with $\sigma_0 := \sigma_c(\omega_0) < \infty$ let $\eta_0 = \sigma_0\omega_0$ and $\hat{C}_0 = \{\hat{x} \in S^{d-1} | \sigma_s(\hat{x}) = \eta_0 \cdot \hat{x}\}$ (note that $\eta_0 \in \partial\mathcal{E}$ and that $\hat{C}_0$ is a parametrization of the set of supporting hyperplanes at this point in terms of normals). For any such ω_0 there exists $(\xi, \theta, \beta) \in \mathbb{R}^d \times \hat{C}_0 \times \mathbb{C}$ solving the pair of equations

$$Q(\xi + i\eta_0) = \lambda, \tag{2.1a}$$

$$\nabla Q(\xi + i\eta_0) = \beta\theta. \tag{2.1b}$$

If the set of η_0's which occur in the set of all solutions $(\xi, \theta, \beta, \eta_0) \in \mathbb{R}^d \times S^{d-1} \times \mathbb{C} \times \mathbb{R}^d$ to the pair of equations (2.1a) and (2.1b) is bounded, then $\sigma_c(\omega) < \infty$ for all $\omega \in S^{d-1}$.

Clearly the system of equations (2.1a) and (2.1b) is a system of algebraic equations. It would make sense for certain real-analytic symbols, and in fact our proof of Theorem 2.1 is rather robust and applies with modifications to certain elliptic variable coefficient differential operators and certain pseudodifferential operators with elliptic symbol being uniformly real-analytic in the ξ-variable assuming $\sigma_c(\omega_0) < \sigma_a$ for the given eigenfunction where σ_a is the uniform analyticity radius. For example our proof works for the symbol $(|\xi|^2 + s^2)^{1/2} + V(x)$ assuming $0 < \sigma_g \leq \sigma_c(\omega_0) < s$.

We have the following application for rotationally invariant Q:

Theorem 2.2. *Under the conditions of Theorem 2.1 for Q rotationally invariant let G be a polynomial of degree $q/2$ so that $G(\xi^2) = Q(\xi)$. We assume all the zeros* *of $G-\lambda$ have multiplicity one. (There are at most $\frac{q}{2}-1$ values of λ for which this is not the case.) Then there are at most $q/2$ positive numbers σ_0 (being independent of $\omega_0=\eta_0/|\eta_0|$) for which there is a solution to the pair of equations (2.1a) and (2.1b) with $|\eta_0|=\sigma_0$, and σ_g is one of them. In addition, $\sigma_{loc}(\omega)=\sigma_g$ for all $\omega\in S^{d-1}$.*

Conversely if $G(\xi^2)=Q(\xi)$, $\lambda\in\mathbb{R}$ and all the zeros of $G-\lambda$ have multiplicity one, then for any solution to the system of equations (2.1a) and (2.1b) with $\eta_0\neq 0$ the number $|\eta_0|$ is the global decay rate of some L^2-eigenfunction for some real $V\in C_c^\infty$.

3. An example, $\sigma_{loc}\neq\sigma_s$

In this section we consider for $\epsilon\in(0,1/2)$ the polynomial

$$Q(\xi)=|\xi|^4+2\epsilon\xi_d+\epsilon^2\xi_d^2$$

in dimension $d\geq 2$. A crude estimate gives $Q(\xi)\geq -2\epsilon^{4/3}$. We take $\lambda=-1$ so that $\lambda<\inf\sigma(Q(p))$, and we note

$$Q(\xi)-\lambda=(|\xi|^2+\mathrm{i}(1+\epsilon\xi_d))(|\xi|^2-\mathrm{i}(1+\epsilon\xi_d)). \tag{3.1}$$

We first solve the system

$$\begin{aligned} Q(\xi+\mathrm{i}\sigma\omega)&=\lambda,\\ \nabla Q(\xi+\mathrm{i}\sigma\omega)&=\beta\theta \end{aligned}$$

for $\sigma>0$ given $\omega\in S^{d-1}$.

The result is that for $\omega_d\neq 0$

$$\sigma=\pm\epsilon\omega_d/2+\sqrt{\lambda_0^2-\epsilon^2(1-\omega_d^2)/4}, \tag{3.2}$$

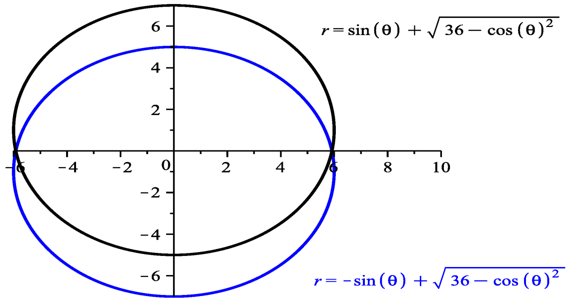
where $2\lambda_0^2=(\epsilon/2)^2+\sqrt{1+(\epsilon/2)^4}$, $\lambda_0>0$.

In addition there is another set of solutions which are only valid for $\omega_d=0$. Namely for $d=2$, $\sigma=1/\epsilon$ and for $d\geq 3$, any $\sigma\geq 1/\epsilon$ independent of $\omega\in S^{d-1}$ such that $\omega_d=0$.

Now suppose $(Q(p)+V+1)\phi=0$ for some $V\in C_c^\infty$ and for a nonzero $\phi\in L^2$. Combining the computation (3.2) with some general results [HS1, HS2] we then conclude that $0<\sigma_g<\infty$ and that $\sigma_c(\omega)<\infty$ for all $\omega\in S^{d-1}$ (near $\omega_d=0$ the choices (3.2) stay well below $1/\epsilon$ making the choice $\sigma\geq 1/\epsilon$ irrelevant). Thus σ must be given by one of (3.2), and there are the following possibilities for $\sigma(\omega)=\sigma_c(\omega)$ (only continuous $\sigma(\cdot)$ are relevant):

1) $\sigma(\omega)=\epsilon\omega_d/2+\sqrt{\lambda_0^2-\epsilon^2(1-\omega_d^2)/4}$,
2) $\sigma(\omega)=-\epsilon\omega_d/2+\sqrt{\lambda_0^2-\epsilon^2(1-\omega_d^2)/4}$,
3) $\sigma(\omega)=-\epsilon|\omega_d/2|+\sqrt{\lambda_0^2-\epsilon^2(1-\omega_d^2)/4}$,
4) $\sigma(\omega)=\epsilon|\omega_d/2|+\sqrt{\lambda_0^2-\epsilon^2(1-\omega_d^2)/4}$.

The case 4) cannot actually be $\sigma_c(\omega)$ for the eigenfunction ϕ because it does not describe the boundary of a convex set. For 1) and 2) the set $\partial\mathcal{E}$ is C^1 and Lemma 1.1 applies. For 3) we cannot apply Lemma 1.1 to all ω due to the wedge at $\omega_d=0$ while indeed it applies near $\omega_d=\pm1$ for example. The sets $\partial\mathcal{E}$ are depicted for the cases 1) and 2) for $d=2$ by polar plots (this is for $2\lambda_0/\epsilon=6$ and in terms of the unit $\epsilon/2$):



Note that in this picture $\partial\mathcal{E}$ for the case 3) is the union of the (closed) upper blue arch and the (closed) lower black arch, whence there is a wedge at $\omega_d = 0$ (in fact defined by $\sin\psi = \frac{\epsilon}{2\lambda_0} = \frac{1}{6}$ where $\pi - 2\psi$ is the apex angle). In general a computation for case 3) shows that Lemma 1.1 applies if and only if $|\omega_d| > \frac{\epsilon}{2\lambda_0}$ and in this case

$$\sigma_{loc}(\omega) = \sigma_s(\omega) = \lambda_0 - \tfrac{\epsilon}{2}|\omega_d|.$$

If on the other hand $|\omega_d| \leq \frac{\epsilon}{2\lambda_0}$ we compute for case 3)

$$\sigma_s(\omega) = (1 - \omega_d^2)^{1/2}\sqrt{\lambda_0^2 - \epsilon^2/4}. \tag{3.3}$$

We present below an example of case 3) where $\sigma_{loc}(\omega) > \sigma_s(\omega)$ for $|\omega_d| < \frac{\epsilon}{2\lambda_0}$.

Let us first note that there are examples of 1) and 2): Indeed (motivated by (3.1)) we take

$$\phi_\pm = \chi + (1-\chi)(p^2 \pm \mathrm{i}(1+\epsilon p_d))^{-1}\delta,$$

where $\chi \in C_c^\infty$, $0 \leq \chi \leq 1$, χ is 1 in a small neighbourhood of 0 and has small support, and δ is the delta function at 0. If $g_\pm(x)$ denotes the Green's function $(p^2 \pm \mathrm{i} + \epsilon^2/4)^{-1}(x,0)$ then

$$\phi_\pm(x) = \chi(x) + \big(1 - \chi(x)\big)\mathrm{e}^{\pm x_d\epsilon/2} g_\pm(x). \tag{3.4}$$

Using properties of $g_\pm(x)$ (see the discussion in [HS1, Subsection 1.2] and note that $\sqrt{\mp\mathrm{i} - \epsilon^2/4} = \mathrm{i}\lambda_0 \mp (2\lambda_0)^{-1}$) we deduce that each choice $\phi = \phi_\pm$ fulfills $(Q(p) + V + 1)\phi = 0$ for some $V \in C_c^\infty$. The choice ϕ_- is an example of the case 1) while the choice ϕ_+ is an example of the case 2). In general for these cases we have that for all $\omega \in S^{d-1}$

$$\sigma_{loc}(\omega) = \sigma_s(\omega) = \lambda_0 \mp \tfrac{\epsilon}{2}\omega_d,$$

respectively.

Now for an example of case 3), we consider

$$g(x) = (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}x\cdot\xi}(1 + \epsilon\xi_d)(Q(\xi) + 1)^{-1} d\xi;\ x \neq 0.$$

It is well-defined and smooth, and introducing as above

$$\phi(x) = \chi(x) + \big(1 - \chi(x)\big)g(x), \tag{3.5}$$

for this ϕ we have $\sigma_g \in (0,\infty)$, cf. Paley-Wiener theory. We claim that $(Q(p)+V+1)\phi = 0$ for some $V \in C_c^\infty$ and that this is an example of case 3). Since $g(x) = \overline{g(-x)}$ we have $\sigma_c(\omega) = \sigma_c(-\omega)$ is valid for all ω. This excludes the cases 1) and 2) and we are left with case 3). Whence it remains to construct V. We use the function $g_-(x) = (p^2 - z)^{-1}(x,0)$, $z = \mathrm{i} - \epsilon^2/4$, from above and represent

$$2\mathrm{i}g(x) = \mathrm{e}^{-x_d\epsilon/2}g_-(x) - \mathrm{e}^{x_d\epsilon/2}\overline{g_-(x)} \text{ and } \operatorname{Re} g(x) = \cosh(x_d\epsilon/2)\operatorname{Im} g_-(x).$$

Next we use that $\operatorname{Im}\left((p^2 - z)^{-1}(x,0)\right) > 0$ for all $|x| > 0$ small enough (this is valid for any $d \geq 2$ and for any $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$). For example in dimension $d = 3$ explicitly for $z = \mathrm{i} - \epsilon^2/4$ this property holds for $|x| < \pi(2\lambda_0)^{-1}$. Whence by possibly adjusting the support of χ we can safely define

$$V = -\{(Q(p)+1)\phi\}/\phi \in C_c^\infty. \tag{3.6}$$

Finally from the asymptotics of g_- we obtain $\sigma_{loc}(\omega) = \lambda_0 - \frac{\epsilon}{2}|\omega_d|$. Comparing with (3.3) we see that for the eigenfunction (3.5) indeed $\sigma_{loc}(\omega) > \sigma_s(\omega)$ for $|\omega_d| < \frac{\epsilon}{2\lambda_0}$.

Remarks. It is easy to check that the potential V of (3.6) satisfies $\overline{RV} = V$ where $Rf(x_\perp, x_d) = f(x_\perp, -x_d)$ using the fact that also $Q(p)$ has conjugate reflected symmetry. However there is no reason to believe that V is real-valued. If on the other hand we pick an arbitrary real nonzero $V \in C_c^\infty$, $V \geq 0$, the variational principle shows that for some $\kappa < 0$ the energy $\lambda = -1$ is an eigenvalue of $H = Q(p) + \kappa V$. If furthermore $RV = V$ then we can pick a corresponding eigenfunction ϕ obeying $\overline{R\phi} = \phi$. This ϕ is an example of case 3) with a real potential in C_c^∞. However it appears difficult to compute asymptotics for $|\omega_d| \leq \frac{\epsilon}{2\lambda_0}$. We claim that for $d = 3$ at least $\sigma_{loc}(\omega) > \sigma_s(\omega)$ when $\omega_d = 0$. This can be done by first representing the Green's function without potential as

$$\left(Q(p)+1\right)^{-1}(x,0) = e^{-x_d\epsilon/2}\int g_-(x-y)e^{y_d\epsilon}g_+(y)dy,$$

where $g_\pm$ are given as above. For $d = 3$ we may use the familiar expression $(p^2 - z)^{-1}(x,0) = (4\pi)^{-1}\mathrm{e}^{\mathrm{i}\sqrt{z}|x|}/|x|$, $\operatorname{Im}\sqrt{z} > 0$, and estimate this integral explicitly (after a suitable deformation of contour) and show that indeed $\sigma_{loc}(\omega) > \sqrt{\lambda_0^2 - \epsilon^2/4}$ when $\omega_d = 0$.

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