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An application of wave packet transform to scattering theory [1]

東京理科大学・理学研究科
米山 泰祐 (Taisuke Yoneyama)
Department of Mathematics,
Tokyo University of Science

1 Introduction

In this report, we consider the following Schrödinger equation with time-dependent short-range potentials:

$$(1)\qquad i\frac{\partial}{\partial t}u = H(t)u, \quad H(t) = H_0 + V(t), \quad H_0 = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} = -\frac{1}{2}\Delta$$

in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^n)$, where $V(t)$ is the multiplication operator of a function $V(t,x)$ and the domain $D(H_0) = H^2(\mathbb{R}^n)$ is the Sobolev space of order two. We give a characterization of the ranges of the wave operators for Schrödinger equation with time-dependent potentials which are short-range in space by using wave packet transform, which is different from the characterization in Kitada–Yajima [10]. We also give an alternative proof of the existence of the wave operators, which has been firstly proved by D. R. Yafaev [15].

We assume that $V(t,x)$ satisfies the following conditions, which is called short-range.

Assumption (A). (i) $V(t,x)$ is a real-valued Lebesgue measurable function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.

(ii) There exist real constants $\delta > 1$ and $C > 0$ such that

$$|V(t,x)| \le C(1+|x|)^{-\delta}$$

for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.

Assumption (B). There exists a family of unitary operators $(U(t,\tau))_{(t,\tau)\in\mathbb{R}^2}$ in $\mathscr{H}$ satisfying the following conditions.

(i) For $f \in \mathscr{H}$, $U(t,\tau)f$ is strongly continuous function with respect to t and satisfies

$$U(t,\tau')U(\tau',\tau) = U(t,\tau),\ U(t,t) = I \quad \text{for all } t,\tau',\tau \in \mathbb{R},$$

where I is the identity operator on $\mathscr{H}$.

(ii) For $f \in H^2(\mathbb{R}^n)$, $U(t,\tau)f$ is strongly continuously differentiable in $\mathscr{H}$ with respect to t and satisfies

$$\frac{\partial}{\partial t}U(t,\tau)f = -iH(t)U(t,\tau)f \quad \text{for all } t,\tau \in \mathbb{R}.$$

Remark 1. If Assumption (A) is satisfied and $V(t)f$ is strongly differentiable in $\mathscr{H}$ for $f \in H^2(\mathbb{R}^n)$, Assumption (B) is satisfied (c.f. T. Kato [9]).

[1] This is a joint work with Professor Keiichi Kato

Let $\mathscr{S}$ be the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^n$ and $\mathscr{S}'$ be the space of tempered distributions on $\mathbb{R}^n$. For positive constants a and R, we put $\Gamma_{a,R} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \big| |\xi| \le a \text{ or } |x| \ge R\}$ and $\mathscr{S}_0 = \left\{\Phi \in \mathscr{S} \Big| \|\Phi\|_{\mathscr{H}} = 1 \text{ and } \hat{\Phi}(0) \neq 0\right\}$.

Definition 2 (Wave packet transform ([3])). Let $\varphi \in \mathscr{S} \setminus \{0\}$ and $f \in \mathscr{S}'$. We define the wave packet transform $W_\varphi f(x,\xi)$ of f with the wave packet generated by a function φ as follows:

$$W_\varphi f(x,\xi) = \int_{\mathbb{R}^n} \overline{\varphi(y-x)} f(y) e^{-iy\xi} dy \quad \text{for } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Its inverse is the operator W_φ^{-1} which is defined by

$$W_\varphi^{-1} F(x) = \frac{1}{(2\pi)^n \|\varphi\|_{\mathscr{H}}^2} \int\int_{\mathbb{R}^{2n}} \varphi(x-y) F(y,\xi) e^{ix\xi} dy d\xi$$

for $x \in \mathbb{R}^n$ and a function $F(x,\xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 3. Let $\Phi \in \mathscr{S}_0$ and we put $\Phi(t) = e^{-itH_0}\Phi$. We define $\tilde{D}^{\pm,\Phi}_{scat}(\tau)$ by the set of all functions in $\mathscr{H}$ such that

$$\lim_{t \to \pm\infty} \left\| \chi_{\Gamma_{a,R}} W_{\Phi(t-\tau)}[U(t,\tau)f](x + (t-\tau)\xi, \xi) \right\|_{L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)} = 0$$

for some positive constants a and R. For $\tau \in \mathbb{R}$, $D^{\pm,\Phi}_{scat}(\tau)$ is defined by the closure of $\tilde{D}^{\pm,\Phi}_{scat}(\tau)$ in the topology of $\mathscr{H}$.

The main state of this report is the following.

Theorem 4 (Y. and K. Kato [16]). Suppose that (A) and (B) be satisfied. Then the wave operators

$$W_\pm(\tau) = \text{s-}\lim_{t \to \pm\infty} U(\tau,t) e^{-i(t-\tau)H_0}$$

exist for any $\tau \in \mathbb{R}$ and their ranges $\mathscr{R}(W_\pm(\tau))$ coincide with $D^{\pm,\Phi}_{scat}(\tau)$ for any $\Phi \in \mathscr{S}_0$. In particular, $D^{\pm,\Phi}_{scat}(\tau)$ is independent of Φ.

We use the following notations throughout the report. $i = \sqrt{-1}$, $n \in \mathbb{N}$. For a subset Ω in $\mathbb{R}^n$ or in $\mathbb{R}^{2n}$, the inner product and the norm on $L^2(\Omega)$ are defined by $(f,g)_{L^2(\Omega)} = \int_\Omega f\bar{g}dx$ and $\|f\|_{L^2(\Omega)} = (f,f)^{1/2}_{L^2(\Omega)}$ for $f,g \in L^2(\Omega)$, respectively. We write $\partial_{x_j} = \partial/\partial x_j$, $\partial_t = \partial/\partial t$, $L^2_{x,\xi} = L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, $(\cdot,\cdot) = (\cdot,\cdot)_{L^2_{x,\xi}}$, $\|\cdot\| = \|\cdot\|_{L^2_{x,\xi}}$, $\langle t\rangle = 1 + |t|$, $\|f\|_{\Sigma(l)} = \sum_{|\alpha+\beta|=l} \|x^\beta \partial_x^\alpha f\|_{\mathscr{H}}$ and $W_\varphi u(t,x,\xi) = W_\varphi[u(t)](x,\xi)$. $\|\cdot\|_{\mathscr{B}(X)}$ denotes the operator norm on the Hilbert space X. $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and the inverse Fourier transform defined by $\mathscr{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$ and $\mathscr{F}^{-1}f(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi$, respectively. We often write $\{\xi = 0\}$ as $\{(x,\xi) \in \mathbb{R}^{2n} | \xi = 0\}$. For sets A and B, $A \setminus B$ denotes the set $\{a \in A | a \notin B\}$. $\chi_A(x)$ the characterization function of a measurable set A, which is defined by $\chi_A(x) = 1$ on A and $\chi_A(x) = 0$ otherwise. $F(\cdots)$ denotes the multiplication operator of a function $\chi_{\{x \in \mathbb{R}^n | \cdots\}}(x)$. For an operator T on $\mathscr{H}$, $D(T)$ and $\mathscr{R}(T)$ denote the domain and the range of T, respectively. $\mathscr{H}_p(T)$ and $\mathscr{H}_p(T)^\perp$ denote pure point subspace of a self-adjoint operator T on $\mathscr{H}$ and its orthogonal complement space, respectively.

The proof of the existence of the wave operators is relied on Cook–Kuroda's method. That is, we prove

$$\int_0^\infty \|V(t)e^{-itH_0}f\|_{\mathscr{H}}dt$$

for any $f \in \mathscr{D}$, a dense set in $\mathscr{H}$. In our proof, taking $f \in W_\Phi^{-1}[C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\})]$ we have

$$\|V(t)e^{-itH_0}f\|_{\mathscr{H}} \leq \|V(t)e^{-itH_0}W_\Phi^{-1}\chi_{\Gamma_{a,R}^c}\|_{\mathscr{B}(\mathscr{H})}\|W_\Phi f\|_{\mathscr{H}} \tag{2}$$

for some positive numbers a and R. Using the representation

$$\begin{aligned} &W_{\varphi_0}[e^{itH_0}U(t,0)\psi](x,\xi) \\ &\quad = W_{\varphi_0}\psi(x,\xi) - i\int_0^t e^{i\frac{1}{2}s|\xi|^2}W_{\varphi(s)}[V(s)U(s,0)\psi](x+s\xi,\xi)ds, \end{aligned} \tag{3}$$

which is developed by K. Kato, M. Kobayashi and S. Ito ([7], [8]), we obtain

$$\|V(t)e^{-itH_0}W_\Phi^{-1}\chi_{\Gamma_{a,R}^c}\|_{\mathscr{B}(\mathscr{H})} = \|\chi_{\Gamma_{a,R}^c}W_\Phi[e^{-itH_0}V(t)]\|_{\mathscr{B}(L^2_{x,\xi})} \leq C\langle t\rangle^{-\delta}. \tag{4}$$

(See Lemma 8.)

V. Enss [4] and H. Kitada–K. Yajima [10] use the phase space decomposition operators $P_{\pm,R}$ and $P_{0,R}$ as follows: $P_{\pm,R}f(x) = \int\int\int_{|z|\geq R} e^{i(x-y)\xi}g_\pm^a(z,\xi)\eta(y-z)f(y)dzdyd\xi$ and $P_{0,R}f(x) = \int\int\int_{|z|<R} e^{i(x-y)\xi}\eta(y-z)f(y)dzdyd\xi$. Here $g_+^a(z,\xi)$ ($g_-^a(z,\xi)$) is smooth cut-off function whose support is contained in the set that $|\xi| \leq a$ and $z\cdot\xi < 0 (> 0)$ and η is a smooth function such that $\int \eta dx = 1$ and $\operatorname{supp}\hat{\eta}$ is included in a small ball in $\mathbb{R}^n$. Since $P_{-,R}U(t,0)f$, $P_{0,R}U(t,0)f \to 0$ as $t \to +\infty$ and $P_{+,R}U(t,0)f \sim P_{+,R}e^{-itH_0}f$ as $R \to \infty$, we have $U(t,0)f \sim P_{+,R}e^{-itH_0}f$ as $R, t \to \infty$. By using this formula, V. Enss [4] has proved that the ranges of the wave operators are the continuous spectral subspace of time-independent Hamiltonian $H = H_0 + V$ and H. Kitada–K. Yajima [10] has characterized the ranges of the wave operators for time-independent potentials.

On the contrary, our proof is simple. We decompose the phase space $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$ into only two parts $\Gamma_{a,R}$ and $\Gamma_{a,R}^c$, and estimate the wave packet transform of the solution in each part.

The above arguments are applied to the Laplacian with coefficients. The proof would be given in the forth-coming paper.

In the case that V does not depend on t, the following well-known theorem holds for $H = H_0 + V$. We give an alternative proof of the theorem by using our characterization.

Theorem 5 (J. Cook [2], S. T. Kuroda [11], E. Mourre [13] and V. Enss [4]). Suppose that (A) be satisfied and that V do not depend on t. Then the wave operators $W_\pm = \text{s-}\lim_{t\to\pm\infty} e^{i(t-\tau)H}e^{-i(t-\tau)H_0}$ exist, are independent of τ and are strongly complete:

$$\mathscr{R}(W_\pm) = \mathscr{H}_p^\perp(H). \tag{5}$$

The plan of this report is as follows. In section 2, we recall the properties of the wave packet transform. In section 3, we give a proof of Theorem 4). In section 4 we prove Theorem 5 by using our characterization.

2 Properties of wave packet transform

In this section, we explain the properties of the wave packet transform and give the representation of solutions to (1) via wave packet transform, which is introduced in [7], [8].

Proposition 6. Let $\varphi, \psi \in \mathscr{S} \setminus \{0\}$ and $f \in \mathscr{S}'$. Then the wave packet transform $W_\varphi f(x,\xi)$ has the following properties:
(i) $W_\varphi f(x,\xi) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$.
(ii) If $f, g \in \mathscr{H}$, we have

$$(W_\varphi f, W_\psi g) = \overline{(\varphi,\psi)_{\mathscr{H}}}(f,g)_{\mathscr{H}} = (\psi,\varphi)_{\mathscr{H}}(f,g)_{\mathscr{H}}. \tag{6}$$

(iii) The inversion formula $(\psi,\varphi)^{-1}W_\psi^{-1}[W_\varphi f] = f$ holds for $f \in \mathscr{S}'$ if $(\psi,\varphi) \neq 0$.

Proof. See [5]. □

Let $\varphi_0 \in \mathscr{S} \setminus \{0\}$, $\varphi(t,x) = e^{-itH_0}\varphi_0(x)$ and $\psi \in \mathscr{H}$. Integrating by parts, we have

$$\begin{aligned} W_{\varphi(t)}[\Delta u](t,x,\xi) &= \int \overline{\varphi(y-x)}\Delta u(t,y)e^{-iy\xi}dy \\ &= \int \Delta\overline{\varphi(y-x)}u(t,y)e^{-iy\xi}dy - 2i\int \xi\cdot\nabla_y\overline{\varphi(y-x)}u(t,y)e^{-iy\xi}dy \\ &\qquad - |\xi|^2\int \overline{\varphi(y-x)}u(t,y)e^{-iy\xi}dy \\ &= W_{\Delta\varphi(t)}u(t,x,\xi) + 2i\xi\cdot\nabla_x W_{\varphi(t)}u(t,x,\xi) - |\xi|^2 W_{\varphi(t)}u(t,x,\xi). \end{aligned}$$

Since $W_{\varphi(t)}[i\partial_t u](t,x,\xi) = i\partial_t W_{\varphi(t)}u(t,x,\xi) + W_{i\partial_t\varphi(t)}u(t,x,\xi)$ and $i\partial_t\varphi(t) = H_0\varphi(t)$, we transform (1) to

$$\left(i\partial_t + i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right)W_{\varphi(t)}u(t,x,\xi) = W_{\varphi(t)}\left[V(t)u\right](t,x,\xi),.$$

We have by the method of characteristic curve that

$$\begin{aligned} W_{\varphi(t)}[U(t,t_0)\psi](x,\xi) =& e^{-i\frac{1}{2}(t-t_0)|\xi|^2}W_{\varphi_0}\psi(x-(t-t_0)\xi,\xi) \\ &- i\int_{t_0}^t e^{-i\frac{1}{2}(t-s)|\xi|^2}W_{\varphi(s)}\left[V(s)U(s,t_0)\psi\right](x-(t-s)\xi,\xi)ds. \end{aligned} \tag{7}$$

In particular, we have for the case that $V \equiv 0$

$$W_{\varphi(t)}[e^{-itH_0}\psi](x,\xi) = e^{-i\frac{1}{2}t|\xi|^2}W_{\varphi_0}\psi(x-t\xi,\xi). \tag{8}$$

By (7) and (8), we have (3) and

$$\begin{aligned} &W_{\varphi(t)}[U(t,t')e^{-it'H_0}\psi](x+t\xi,\xi) \\ &= W_{\varphi_0}\psi(x,\xi) + i\int_t^{t'} e^{-i\frac{1}{2}(t-s)|\xi|^2}W_{\varphi(s)}\left[V(s)U(s,t')e^{-it'H_0}\psi\right](x+s\xi,\xi)ds. \end{aligned} \tag{9}$$

3 Proof of Theorem 4

In this section, Theorem 4 by using the wave packet transform which is defined in the previous section.

The following well-known propagation estimate is used in the proof of Lemma 8.

Lemma 7. Let $f \in \mathscr{S}$. Suppose that $\operatorname{supp} \hat{f} \subset K$ with some compact set K which does not contain the origin. For any open set $K' \supset K$ and any non-negative integer l, there exists a positive constant C such that

$$|e^{-itH_0} f(x)| \leq C\langle x\rangle^{-l} \|f\|_{\Sigma(l)}$$

for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ with $x/t \notin K'$ and $t \neq 0$.

Proof. See [12] or [6]. □

Using the above lemma, we obtain the following lemma.

Lemma 8. Suppose that (A) be satisfied. Let a and R be positive constants. Then for any $L \in (0, a/6]$ and $\varphi_0 \in \mathscr{S} \setminus \{0\}$ with $\operatorname{supp} \hat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n | \, L/2 < |\xi| < L\}$, there exists a positive constant C satisfying

$$\left\| W_{\varphi(s)} \left[V(s)\psi\right](x + s\xi, \xi) \right\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \leq C\langle s\rangle^{-\delta} \|\psi\|_{\mathscr{H}} \tag{10}$$

for any $s \geq 0$ and any $\psi \in \mathscr{H}$.

Proof. Let $\rho = a/6$ and let l be an integer satisfying $l \geq \delta + (n+1)/2$. We put $V_\rho(t, x) = \chi_0(\frac{1}{\rho\langle t\rangle} x) V(t, x)$, where $\chi_0 \in C^\infty(\mathbb{R}^n)$ satisfies $\chi_0(x) = 1$ for $|x| \geq 1$ and $\chi_0(x) = 0$ for $|x| \leq 1/2$. Thus there exists a positive constant C such that $|V_\rho(t, x)| \leq C\langle t\rangle^{-\delta}$ for any $t \in \mathbb{R}$ and any $x \in \mathbb{R}^n$.

For $(x, \xi) \in \mathbb{R}^{2n} \setminus \Gamma_{a,R}$, we decompose $\mathbb{R}^n$ into $\mathscr{O}_1 \equiv \{y \in \mathbb{R}^n | |y - (x + s\xi)| \leq as/3\}$ and $\mathscr{O}_2 \equiv \{y \in \mathbb{R}^n | |y - (x + s\xi)| > as/3\}$. Then the inequality that $|y| \geq |x + s\xi| - |y - (x + s\xi)| \geq (as - R) - \frac{as}{3} \geq \rho\langle s\rangle$ for $(x, \xi) \in \mathbb{R}^{2n} \setminus \Gamma_{a,R}$ and $y \in \mathscr{O}_1$ and the equality that $V_\rho(s, y) = V(s, y)$ for $|y| \geq \rho\langle s\rangle$ imply

$$\begin{aligned}
&\left\| \int_{y \in \mathscr{O}_1} \overline{e^{-isH_0} \varphi_0(y - (x + s\xi))} V(s, y)\psi(y) e^{-i\xi y} dy \right\|_{L^2(\mathbb{R}^{2n} \setminus \Gamma_{a,R})} \\
&\qquad \leq \left\| \int \overline{e^{-isH_0} \varphi_0(y - x)} V_\rho(s, y)\psi(y) e^{-i\xi y} dy \right\| \\
&\qquad \leq \left\| \overline{e^{-isH_0} \varphi_0(y - x)} V_\rho(s, y)\psi(y) \right\|_{L^2(\mathbb{R}^n_x \times \mathbb{R}^n_y)} \\
&\qquad \leq \|V_\rho\|_{L^\infty(\mathbb{R}^n)} \|\varphi_0\|_{\mathscr{H}} \|\psi\|_{\mathscr{H}} \\
&\qquad \leq C\langle s\rangle^{-\delta} \|\psi\|_{\mathscr{H}}.
\end{aligned}$$

On the other hand, Lemma 7 shows that

$$
\begin{aligned}
&\left\|\int_{y\in\mathscr{O}_2}\overline{e^{-isH_0}\varphi_0(y-(x+s\xi))}V(s,y)\psi(y)e^{-i\xi y}dy\right\| \\
=&\left\|\left(\chi_{\{|y-x|>\frac{as}{3}\}}\overline{e^{-isH_0}\varphi_0(y-x)}\right)V(s,y)\psi(y)\right\|_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_y)} \\
\leq& C\langle s\rangle^{-l+(n+1)/2}\|\varphi_0\|_{\Sigma(l)}\left\|\langle y-x\rangle^{-(n+1)/2}V(s,y)\psi(y)\right\|_{L^2(\mathbb{R}^n_x\times\mathbb{R}^n_y)} \\
=& C\langle s\rangle^{-l+(n+1)/2}\|\varphi_0\|_{\Sigma(l)}\left\|\langle x\rangle^{-(n+1)/2}\right\|_{\mathscr{H}}\|\psi\|_{\mathscr{H}} \\
\leq& C\langle s\rangle^{-\delta}\|\psi\|_{\mathscr{H}},
\end{aligned}
$$

since $\operatorname{supp}\hat{\varphi}_0\subset\{\xi\in\mathbb{R}^n|\,0<|\xi|<a/6\}$.

Hence (10) is obtained. □

We shall use the following lemma in Section 4. Let $\Gamma_a^{b,\pm}=\{(x,\xi)\in\mathbb{R}^{2n}|\,|\xi|\geq a, |x|\geq b \text{ and } \pm x\cdot\xi\geq 0\}$.

Lemma 9. Suppose that (A) be satisfied. Let a and b be positive constants. Then for any $L\in(0,a/6]$ and $\varphi_0\in\mathscr{S}\setminus\{0\}$ with $\operatorname{supp}\hat{\varphi}_0\subset\{\xi\in\mathbb{R}^n|\,L/2<|\xi|<L\}$, there exists a positive constant C independent of b satisfying

$$\left\|W_{\varphi(s)}\left[V(s)\psi\right](x\pm s\xi,\xi)\right\|_{L^2(\Gamma_a^{b,\pm})}\leq C\langle s+b\rangle^{-\delta}\|\psi\|_{\mathscr{H}}$$

for any $s\geq 0$ and any $\psi\in\mathscr{H}$.

Proof. The proof is obtained by the similar argument as in the proof of Lemma 8. □

Now we give an alternative proof of the existence of the wave operators $W_\pm(\tau)$ by using wave packet transform.

Proposition 10 (D. R. Yafaev [15]). Suppose that (A) and (B) be satisfied. Then the wave operators $W_\pm(\tau)$ exist for any $\tau\in\mathbb{R}$.

Proof. Substituting $V(t-\tau,x)$ for $V(t,x)$, it suffices to show the case $\tau=0$. We prove the existence in the case $t\to+\infty$ only. Let $\Phi\in\mathscr{S}_0$ and $u_0\in\mathscr{H}$.

First, we show the existence of $W_+(0)u_0$ for $W_\Phi u_0\in C_0^\infty(\mathbb{R}^{2n}\setminus\{\xi=0\})$. Let a and R be positive constants satisfying

$$\text{(11)}\qquad \operatorname{supp}W_\Phi u_0\subset\mathbb{R}^{2n}\setminus\Gamma_{a,R}$$

and $\varphi_0\in\mathscr{S}\setminus\{0\}$ satisfying

$$\text{(12)}\qquad \operatorname{supp}\hat{\varphi}_0\subset\left\{\xi\in\mathbb{R}^n\,\middle|\,\frac{L}{2}<|\xi|<L\right\}\text{ with }0<L\leq\frac{a}{6}\quad\text{and}\quad|(\Phi,\varphi_0)_{\mathscr{H}}|>0.$$

By (2), (3), (4), (6), (11) and Lemma 8, we have

$$\|V(t)e^{itH_0}u_0\|\leq\frac{1}{|(\varphi_0,\Phi)_{\mathscr{H}}|}\|\chi_{\Gamma^c_{a,R}}W_\Phi[e^{-itH_0}V(t)]\|_{\mathscr{B}(L^2_{x,\xi})}\leq C\langle t\rangle^{-\delta}$$

for $t \geq 0$ The above inequality implies the existence of $W_+(0)u_0$ for $u_0 \in W_\Phi^{-1}(C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$.

For $u_0 \in \mathscr{H}$, the existence of $W_+(0)u_0$ follows from the fact that $W_\Phi^{-1}(C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$ is dense in $\mathscr{H}$. Indeed, let ε be a fixed positive number. Since $C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\})$ is dense in $L^2(\mathbb{R}^{2n})$, there exists $\omega \in C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\})$ satisfying $\|W_\Phi u_0 - \omega\| \leq \varepsilon$. Putting $\tilde{u_0} = W_\Phi^{-1}\omega$, we have $\|U(0,t')e^{-it'H_0}u_0 - U(0,t)e^{-itH_0}u_0\|_{\mathscr{H}} \leq \|U(0,t')e^{-it'H_0}\tilde{u_0} - U(0,t)e^{-itH_0}\tilde{u_0}\|_{\mathscr{H}} + 2\varepsilon$ for any $t' \geq t > 0$. $(U(0,t)e^{-itH_0}\tilde{u_0})$ is a Cauchy sequence with respect to t as $t \to \infty$ in $\mathscr{H}$, so is $(U(0,t)e^{-itH_0}u_0)$. □

Next, we characterize the ranges of the wave operators by the wave packet transform.

Proposition 11. Suppose that (A) and (B) be satisfied. Then we have

$$\mathscr{R}(W_\pm(\tau)) = D_{scat}^{\pm,\Phi}(\tau)$$

for any $\Phi \in \mathscr{S}_0$.

Proof. It suffices to prove that $\mathscr{R}(W_+(0)) = D_{scat}^{+,\Phi}(0)$.

Let $\Phi \in \mathscr{S}_0$ and ε be a fixed positive number. Until the end of the proof, we abbreviate $W_+ = W_+(0)$, $D_{scat}^+ = D_{scat}^{+,\Phi}(0)$ and $\tilde{D}_{scat}^+ = \tilde{D}_{scat}^{+,\Phi}(0)$.

We first prove that $\mathscr{R}(W_+) \subset D_{scat}^+$. Let $f \in \mathscr{R}(W_+)$ and we fix $g \in W_\Phi^{-1}(C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$ satisfying

$$\|f - W_+ g\|_{\mathscr{H}} \leq \varepsilon. \tag{13}$$

Then there exist positive constants a and R such that $\chi_{\Gamma_{a,R}}(x,\xi)W_\Phi g(x,\xi) = 0$ for all $(x,\xi) \in \mathbb{R}^{2n}$. By (8) and the definition of W_+, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \|\chi_{\Gamma_{a,R}} W_{\Phi(t)}[U(t,0)W_+ g](x+t\xi,\xi)\| \\
&\leq \lim_{t\to\infty} \left(\|\chi_{\Gamma_{a,R}} W_{\Phi(t)}[e^{-itH_0}g](x+t\xi,\xi)\| + \|W_{\Phi(t)}[U(t,0)W_+ g - e^{-itH_0}g]\|\right) \\
&= \|\chi_{\Gamma_{a,R}} W_\Phi g\| + \lim_{t\to\infty} \|U(t,0)W_+ g - e^{-itH_0}g\|_{\mathscr{H}} \\
&= 0.
\end{aligned}$$

Hence we have $W_+ g \in \tilde{D}_{scat}^+$, which and (13) show $\mathscr{R}(W_+) \subset D_{scat}^+$.

Next we prove $\mathscr{R}(W_+) \supset D_{scat}^+$. If the inverse wave operator

$$W_+^{-1}u_0 = \lim_{t\to+\infty} e^{itH_0}U(t,0)u_0 \tag{14}$$

exists for any $u_0 \in D_{scat}^+$, we obtain $\mathscr{R}(W_+) \supset D_{scat}^+$. It suffices to prove that (14) exists for $u_0 \in \tilde{D}_{scat}^+$, since $D_{scat}^\pm$ is the closure of $\tilde{D}_{scat}^\pm$.

Let $u_0 \in \tilde{D}_{scat}^+$ and let a and R be positive constants satisfying

$$\lim_{t\to\infty} \|\chi_{\Gamma_{a,R}} W_{\Phi(t)}[U(t,0)u_0](x+t\xi,\xi)\| = 0. \tag{15}$$

Until the end of the proof, we abbreviate $\Gamma = \Gamma_{a,R}$ and $\Gamma^c = \mathbb{R}^{2n} \setminus \Gamma$. Take $\varphi_0 \in \mathscr{S} \setminus \{0\}$ satisfying (12). By (6), we have for $t' \geq t > 0$

$$
\begin{aligned}
\left(e^{itH_0}U(t,0)u_0, \psi\right)_{\mathscr{H}} &= \left(U(t,0)u_0, e^{-itH_0}\psi\right)_{\mathscr{H}} \\
&= \frac{1}{(\varphi(t), \Phi(t))_{\mathscr{H}}} \left(W_{\Phi(t)}[U(t,0)u_0], W_{\varphi(t)}[e^{-itH_0}\psi]\right) \\
&= \frac{1}{(\varphi_0, \Phi)_{\mathscr{H}}} \left(\chi_\Gamma(x - t\xi, \xi) W_{\Phi(t)}[U(t,0)u_0], W_{\varphi(t)}[e^{-itH_0}\psi]\right) \\
&+ \frac{1}{(\varphi_0, \Phi)_{\mathscr{H}}} \left(\chi_{\Gamma^c}(x - t\xi, \xi) W_{\Phi(t)}[U(t,0)u_0], W_{\varphi(t)}[e^{-itH_0}\psi]\right)
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
&(e^{it'H_0}U(t',0)u_0, \psi)_{\mathscr{H}} \\
&= \left(U(t,0)u_0, U(t,t')e^{-it'H_0}\psi\right)_{\mathscr{H}} \\
&= \frac{1}{(\varphi_0, \Phi)_{\mathscr{H}}} \left((\chi_\Gamma + \chi_{\Gamma^c})(x - t\xi, \xi) W_{\Phi(t)}[U(t,0)u_0], W_{\varphi(t)}[U(t,t')e^{-it'H_0}\psi]\right).
\end{aligned} \tag{17}
$$

By (9), we have

$$
\begin{aligned}
&W_{\varphi(t)}[e^{-itH_0}\psi - U(t,t')e^{-it'H_0}\psi](x,\xi) \\
&= -i \int_t^{t'} e^{-i\frac{1}{2}(t-s)|\xi|^2} W_{\varphi(s)} \left[V(s)U(s,t')e^{-it'H_0}\psi\right] (x - (t-s)\xi, \xi) ds.
\end{aligned} \tag{18}
$$

By (15), (16), (17), (18) and Lemma 8 show that

$$
\begin{aligned}
&\sup_{\|\psi\|_{\mathscr{H}}=1} \left| \left|(\varphi_0, \Phi)_{\mathscr{H}} \left(e^{itH_0}U(t,0)u_0 - e^{it'H_0}U(t',0)u_0, \psi\right)_{\mathscr{H}}\right| \right. \\
&= \|\chi_\Gamma W_{\Phi(t)}[U(t,0)u_0](x + t\xi, \xi)\| \\
&+ \int_t^{t'} \left\| W_{\varphi(s)} \left[V(s)U(s,t')e^{-it'H_0}\psi\right] (x + s\xi, \xi) \right\|_{L^2(\Gamma^c)} ds \\
&\to 0 \quad \text{as } t, t' \to \infty.
\end{aligned} \tag{19}
$$

(14) follows from (19). □

Theorem 4 is obtained by Proposition 10 and 11.

4 Proof of Theorem 5

In this section, we give an alternative proof of Theorem 5 by using our characterization.

Proof. We shall only prove for $\tau = 0$ and for the case that $t \to +\infty$. We fix $\Phi \in \mathscr{S}_0$. We use the same notations $D^+_{scat} = D^{+,\Phi}_{scat}(0)$ and $\tilde{D}^+_{scat} = \tilde{D}^{+,\Phi}_{scat}(0)$ as in the proof of Proposition 11.

Firstly, we prove $\mathscr{H}_p(H)^\perp \supset D^+_{scat}$. For $u_0 \in \tilde{D}^+_{scat}$, we have

$$
\lim_{t\to\infty} \|\chi_{\Gamma_{a,R}} W_{\Phi(t)}[e^{-itH}u_0](x + t\xi, \xi)\| = 0 \tag{20}
$$

for some positive constants a and R. On the other hand, for $\omega \in \mathscr{H}_p(H)$, without loss of generality, $e^{-itH}\omega = e^{-it\lambda}\omega$ for some $\lambda \in \mathbb{R}$ and for any $t \in \mathbb{R}$. Taking $\varphi_0 \in \mathscr{S}$ satisfying (12), we get for $t \geq 0$

$$
\begin{aligned}
&(e^{-itH}u_0, e^{-it\lambda}\omega)_{\mathscr{H}} \\
(21) \qquad &= \frac{1}{(\varphi_0, \Phi)_{\mathscr{H}}} \left(\chi_{\Gamma_{a,R}} W_{\Phi(t)}[e^{-itH}u_0](x+t\xi, \xi), e^{-it\lambda} W_{\varphi(t)}\omega(x+t\xi,\xi)\right) \\
&+ \frac{1}{(\varphi_0, \Phi)_{\mathscr{H}}} \left(W_{\Phi(t)}[e^{-itH}u_0](x+t\xi,\xi), \chi_{\Gamma^c_{a,R}} e^{-it\lambda} W_{\varphi(t)}\omega(x+t\xi,\xi)\right).
\end{aligned}
$$

By (20), the first term of (21) is estimated by

$$
\begin{aligned}
&\lim_{t\to+\infty} \left| \left(\chi_{\Gamma_{a,R}} W_{\Phi(t)}[e^{-itH}u_0](x+t\xi,\xi), e^{-it\lambda} W_{\varphi(t)}\omega(x+t\xi,\xi)\right) \right| \\
&\qquad \leq C\|\omega\|_{\mathscr{H}} \lim_{t\to+\infty} \|\chi_{\Gamma_{a,R}} W_{\Phi(t)}[e^{-itH}u_0](x+t\xi,\xi)\| = 0.
\end{aligned}
$$

By Lemma 7, the second term of (21) is estimated by

$$
\begin{aligned}
&\lim_{t\to+\infty} \left| \left(W_{\Phi(t)}[e^{-itH}u_0](x+t\xi,\xi), \chi_{\Gamma^c_{a,R}} e^{-it\lambda} W_{\varphi(t)}\omega(x+t\xi,\xi)\right) \right| \\
&\leq \|u_0\|_{\mathscr{H}} \lim_{t\to+\infty} \|\chi_{\Gamma^c_{a,R}} W_{\varphi(t)}\omega(x+t\xi,\xi)\| \\
&\leq \|u_0\|_{\mathscr{H}} \lim_{t\to+\infty} \left\| \chi_{\left\{(x,\xi) \middle| |x+t\xi| \geq at/2\right\}}(x,\xi) \int_{\mathbb{R}^n} \varphi(t, y-(x+t\xi))\omega(y) e^{-iy\xi} dy \right\| \\
&\leq \|u_0\|_{\mathscr{H}} \lim_{t\to+\infty} \left(\|F(|x| > at/3)\varphi(t)\|_{\mathscr{H}} \|\omega\|_{\mathscr{H}} + \|\varphi_0\|_{\mathscr{H}} \|F(|x| > at/6)\omega\|_{\mathscr{H}} \right) \\
&= 0.
\end{aligned}
$$

Thus we obtain $\mathscr{H}_p(H)^\perp \supset D^+_{scat}$.

Secondly, we prove $\mathscr{H}_p(H)^\perp \subset D^+_{scat}$. It suffices to prove $\tilde{\mathscr{H}_p}(H)^\perp \subset \tilde{D}^+_{scat}$, where $\tilde{\mathscr{H}_p}(H)^\perp = \{E_H((a',b'))f | f \in \mathscr{H}_p(H)^\perp \text{ and } 0 < a' < b'\}$. Let $f \in \tilde{\mathscr{H}_p}(H)^\perp$ and be a positive constant d and $\phi \in C_0^\infty([0,\infty))$ satisfying that $\phi(H)f = f$ and that $\phi \equiv 0$ on $[0, d^2/2]$. Since w-$\lim_{t\to\infty} e^{-itH}f = 0$ in $\mathscr{H}$ and $(\phi(H) - \phi(H_0))$ is a compact operator on $\mathscr{H}$, we have

$$
(22) \qquad \lim_{t\to\infty} \left\| (\phi(H) - \phi(H_0))\, e^{-itH} f \right\|_{\mathscr{H}} = 0.
$$

Let $\Xi_d = \{(x,\xi) \in \mathbb{R}^{2n} | \, |\xi| \leq d\}$ and $\Xi^r_d = \{(x,\xi) \in \mathbb{R}^{2n} | \, |\xi| > d \text{ and } |x| \leq r\}$. Then we obtain

$$
(23) \qquad (\chi_{\Xi_d} + \chi_{\Xi^r_d} + \chi_{\Gamma^{r,-}_d} + \chi_{\Gamma^{r,-}_d})(x,\xi) = 1
$$

for almost all $(x,\xi) \in \mathbb{R}^{2n}$, where $\Gamma^{r,\pm}_d$ are defined in Section 2. Since $\phi(|\xi|^2/2)\chi_{\Xi_d}(y,\xi) \equiv 0$ for any $(y,\xi) \in \mathbb{R}^{2n}$, we obtain

$$
\begin{aligned}
\phi(H_0) W_\Phi^{-1}[\chi_{\Xi_d}\Psi] &= \mathscr{F}^{-1}\left[\phi\left(\frac{|\xi|^2}{2}\right) \mathscr{F} W_\Phi^{-1} \chi_{\Xi_d}\Psi\right] \\
&= \mathscr{F}^{-1}\left[\phi\left(\frac{|\xi|^2}{2}\right) \left[\int \Phi(x-y)\chi_{\Xi_d}(y,\xi)\Psi(y,\xi)dy\right]\right] = 0.
\end{aligned}
$$

Thus we have for $t \geq 0$

$$
\begin{aligned}
&(\chi_{\Xi_d} W_\Phi[e^{-itH} f](x,\xi), \Psi) \\
&\quad = \left(\phi(H) e^{-itH} f, W_\Phi^{-1} \chi_{\Xi_d} \Psi\right)_{\mathscr{H}} \\
&\quad = \left(e^{-itH} f, \phi(H_0) W_\Phi^{-1} \chi_{\Xi_d} \Psi\right)_{\mathscr{H}} + \left((\phi(H) - \phi(H_0)) e^{-itH} f, W_\Phi^{-1} \chi_{\Xi_d} \Psi\right)_{\mathscr{H}} \\
&\quad = \left((\phi(H) - \phi(H_0)) e^{-itH} f, W_\Phi^{-1} \chi_{\Xi_d} \Psi\right)_{\mathscr{H}}.
\end{aligned} \tag{24}
$$

(22), (24) and the fact that $\|W_\Phi^{-1} \chi_{\Xi_d} \Psi\|_{\mathscr{H}} \leq \|\Psi\|$ yield that there exists a sequence (t_N) tending to ∞ as $N \to \infty$ such that

$$
\lim_{N \to \infty} \|\chi_{\Xi_d} W_\Phi[e^{-it_N H} f]\| = 0 \tag{25}
$$

and

$$
\lim_{N \to \infty} \|\chi_{\Xi_d^r} W_\Phi[e^{-it_N H} f]\| = 0 \tag{26}
$$

for any positive constant r. Fix $\Phi' \in C_0^\infty(\mathbb{R}^n)$ with $|(\Phi, \Phi')_{\mathscr{H}}| > 0$. For any positive constant r, there exists a positive constant r' satisfying $\chi_{\Xi_d^r} W_{\Phi'}[e^{-itH} f](x,\xi) = \chi_{\Xi_d^r} W_{\Phi'}[F(|x| < r') e^{-itH} f](x,\xi)$ for any $t \geq 0$. By the RAGE theorem ([1], [14]), there exists a sequence (t_N) tending to ∞ as $N \to \infty$ such that $\lim_{N\to\infty} \|F(|x| < r') e^{-it_N H} f\|_{\mathscr{H}} = 0$. Hence we get (26). For any positive number ε, we take a positive constant r sufficiently large so that $C\|f\|_{\mathscr{H}} \langle r \rangle^{1-\delta} \leq \varepsilon/4$. By (23), we have for $t, t' \geq t_N$

$$
\begin{aligned}
&\left(e^{it'H_0} e^{-it'H} f - e^{itH_0} e^{-itH} f, \psi\right)_{\mathscr{H}} \\
&= \Big((\chi_{\Xi_d} + \chi_{\Xi_d^r} + \chi_{\Gamma_d^{r,-}}) W_\Phi[e^{-it_N H} f], W_\Phi\big[\big(e^{-i(t_N - t')H} e^{-it'H_0} \\
&\qquad\qquad - e^{-i(t_N - t)H} e^{-itH_0}\big)\psi\big]\Big) \\
&\quad + \Big(W_\Phi[e^{-it_N H} f], \chi_{\Gamma_d^{r,+}} \left(I(t' - t_N; x, \xi) - I(t - t_N; x, \xi)\right)\Big),
\end{aligned}
$$

where $I(t; x, \xi) = -i \int_0^t e^{\frac{i}{2}(s-t)|\xi|^2} W_{\Phi(s)}[V e^{-i(s-t)H} e^{-i(t+t_N)H_0} \psi](x + s\xi, \xi) ds$.
By (7) with $t = 0$ and $t_0 = -t_N$, we obtain

$$
\begin{aligned}
&\chi_{\Gamma_d^{r,-}} W_\Phi[e^{-it_N H} f](x,\xi) \\
&\quad = \chi_{\Gamma_d^{r,-}} W_\Phi[U(0, -t_N) f](x,\xi) \\
&\quad = \chi_{\Gamma_d^{r,-}} W_{\Phi(-t_N)} f(x - t_N \xi, \xi) \\
&\qquad - i \int_0^{t_N} e^{\frac{i}{2}(-s+t_N)|\xi|^2} \chi_{\Gamma_d^{r,-}} W_{\Phi(-s)}[V e^{i(s+t_N)H} f](x - s\xi, \xi) ds.
\end{aligned} \tag{27}
$$

Lemma 9 implies that

$$
\|\chi_{\Gamma_d^{r,+}} I(t - t_N; x, \xi)\| \leq C' \|\psi\|_{\mathscr{H}} \int_0^{t - t_N} \langle s + r \rangle^{-\delta} ds \leq C \|\psi\|_{\mathscr{H}} \langle r \rangle^{1-\delta} \tag{28}
$$

and

$$
\int_0^{t_N} \|\chi_{\Gamma_d^{r,-}} W_{\Phi(-s)}[V e^{i(s+t_N)H} f](x - s\xi, \xi)\| ds \leq C \|f\|_{\mathscr{H}} \langle r \rangle^{1-\delta}, \tag{29}
$$

where C and C' is a positive constant and independent of r and N. Taking $\varphi_0 \in \mathscr{S}$ satisfying (12) with $a = d$, we have by Lemma 7

$$\begin{aligned}
&\lim_{N\to\infty} \|\chi_{\Gamma_d^{r,-}} W_{\Phi(-t_N)} f(x - t_N\xi, \xi)\| \\
(30) \qquad &\le \lim_{N\to\infty} C \left(\|F(|x| > dt_N/3)\varphi(-t_N)\|_{\mathscr{H}} \|f\|_{\mathscr{H}} + \|\varphi_0\|_{\mathscr{H}} \|F(|x| > dt_N/6) f\|_{\mathscr{H}} \right) \\
&= 0.
\end{aligned}$$

By (27), (29) and (30), we obtain

$$\overline{\lim_{N\to\infty}} \|\chi_{\Gamma_d^{r,-}} W_\Phi [e^{-it_N H} f]\| \le \varepsilon. \tag{31}$$

Thus (25), (26), (28) and (31) imply that $\overline{\lim}_{t,t'\to\infty} \|e^{it'H_0} e^{-it'H} f - e^{itH_0} e^{-itH} f\|_{\mathscr{H}} \le \varepsilon$. Hence there exists $\Omega \in \mathscr{H}$ such that

$$\lim_{t\to\infty} \|\chi_{\Gamma_{a,R}} W_\Phi [e^{itH_0} e^{-itH} f - \Omega]\| \le \lim_{t\to\infty} \|e^{itH_0} e^{-itH} f - \Omega\|_{\mathscr{H}} = 0. \tag{32}$$

Since $W_\Phi^{-1}(C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$ is dense in $\mathscr{H}$, for any positive number ε' there exist $\Omega' \in W_\Phi^{-1}(C_0^\infty(\mathbb{R}^{2n} \setminus \{\xi = 0\}))$ and positive constants a and R satisfying

$$\|\Omega - \Omega'\|_{\mathscr{H}} \le \varepsilon' \text{ and } \chi_{\Gamma_{a,R}} W_\Phi \Omega' \equiv 0. \tag{33}$$

(32) and (33) yield $\mathscr{H}_p(H)^\perp \subset D^+_{scat}$, This and the former part implies (5). □

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