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混合整数非線形計画問題に対するDC計画法
DC programming approach for mixed-integer nonlinear programming problems *

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Abstract

In this paper, we consider a class of mixed integer programming problems (MIPs) whose objective functions are DC functions, that is, functions representable in terms of a difference of two convex functions, and particularly focus on the nonconvex case. Recently, Maehara, Marumo, and Murota provided a continuous reformulation without integrality gaps, for discrete DC programs having only integral variables. They also presented a new algorithm to solve the reformulated problem. Our aim is to extend their results to MIPs and further give a new algorithm to solve them. Specifically, we propose an algorithm based on DCA originally proposed by Pham Dinh and Le Thi, where convex MIPs are solved iteratively.

1 Introduction

Let us consider the following optimization problem:

$$\min\ f(x) \text{ sub.to } x \in S,\ x_N \in \mathbb{Z}^N.$$

Here $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a closed proper function, i.e., f is lower semicontinuous and its effective domain $\mathrm{dom}\, f := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$ is not empty. Moreover, $S \subseteq \mathbb{R}^n$ is a nonempty closed convex set, and $x_N = (x_i)_{i\in N}$ where $N \subseteq \{1, 2, \ldots, n\}$.

In the case where f is a linear or convex quadratic function and S is represented with only linear or convex quadratic inequalities, the branch-and-bound method and cutting plane techniques work nicely in the practical sense. Indeed, there are many commercial and free solvers implementing them, e.g. CPLEX [16], gurobi [13] and SCIP [1]. On the other hand, for the general nonlinear case, the above

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problem is extremely difficult to solve. There are a number of ways to approach such nonlinear mixed-integer problems. One method is to extend the framework of branch-and-bound to the continuous spaces [10, 25, 26]. Another is to utilize sequential quadratic programming (SQP) [9, 17, 7, 8]. These algorithms incorporate such techniques as trust regions, outer approximations and branch-and-bound techniques to solve quadratic problems approximating the original one. In particular, in applying these SQP-type algorithms to mixed-integer convex problems with continuously differentiable convex functions, global convergence to an optimum can be proved. There are also algorithms which deal solely with mixed-integer nonlinear programs with convex f [11, 12, 6, 28, 2]. See, for example, the surveys [3] and [5].

In this paper, we consider the case where f is a so-called *DC* function, that is, a function representable as the difference of two convex functions:

$$\begin{array}{ll} \min & f(x) = g(x) - h(x) \\ \text{sub.to} & x \in S,\ x_N \in \mathbb{Z}^N. \end{array} \tag{1.1}$$

where $g : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ and $h : \mathbb{R}^n \to \mathbb{R}$ are closed proper convex functions.

The class of DC functions covers are a very wide range of functions. For example, any twice continuously differentiable function is DC, moreover, functions generated by applying operators such as $\sum$, Π, $|\cdot|$, and $\max(\cdot,\cdot)$ to DC functions also belong to the class DC [14, 15]. Hence, the problem of our focus,(1.1) covers a wide class of nonlinear mixed integer programs. Note however, that given a DC function f, finding two explicit convex functions g and h representing f is a hard open problem. Among the functions for which a DC representation is easily found, perhaps the most common are the quadratic functions. In this paper, we assume that one DC representation is explicitly given; how we obtain it will not enter our discussion.

The DC programming in continuous variables is an important field of research in continuous optimization, and theoretical and practical aspects have been extensively studied [23, 24]. For example, the global optimality condition is completely characterized by the Toland-Singer duality theorem. This duality theorem in turn forms the basis for the fundamental DC programming algorithm known as DCA[23], which is known have nice convergence properties.

DC programming also has many useful applications. One example is in mixed-integer linear programs, where integer constraints on variables are incorporated into the objective functions via penalty functions [22]. Other notable results have been reported in sparse optimization [27, 19] and portfolio selection [18]. This is an active field, with remarkable recent progress in both theory and applications.

On the other hand, discrete DC programming, which concerns DC programs with integrally constrained variables, that is, (1.1) with $N = \{1, 2, \ldots, n\}$, is a still a relatively unexplored area, Recently, a promising approach was proposed by Maehara and Murota [20], who showed how the framework of discrete convex analysis can be applied, to export results in continuous DC theory to a discrete setting. This was further pursued in Maehara, Marumo and Murota [21], who proved a powerful result in constructing continuous relaxations of discrete DC programs. The simplest continuous relaxation for (1.1) may just replace $\mathbb{Z}^N$ by $\mathbb{R}^N$. As is well known, this does not work effectively in general, since an integrality gap usually occurs, that is, the optimal values of the original and relaxed problems do not coincide. On the other hand, the new continuous relaxation proposed in [21] replaces g with its closed convex closure (and h with an arbitrary relaxation). Its notable property is that no integrality gap is generated.

In this paper we extend the theorem of Maehara, Marumo and Murota to mixed integer DC programs, and propose a new algorithm based on the DCA originally proposed by Pham Dinh and Le Thi [23], where we iteratively solve a sequence of convex mixed integer programs. This paper is organized as follows. In Section 2 we briefly describe existing results in continuous and discrete DC programming, and in Section 3, we show how to extend the theorem of Maehara, Marumo and Murota to obtain a continuous relaxation of (1.1) with no integrality gap. Next, in Section 4 we describe our algorithm.

Throughout the paper, we will use the following notations: For any nonempty set $X \subseteq \mathbb{R}^n$, we denote the convex hull and closure of X by $\mathrm{co}\, X$ and $\mathrm{cl}\, X$, respectively. Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. For $x \in \mathrm{dom}\, \varphi$, the subdifferential of φ at x, that is, the set of all subgradients of φ at x, is denoted by $\partial\varphi(x)$. We write the conjugate of φ as φ^*, that is, a function $\varphi^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $\varphi^*(y) = \sup_{x\in\mathbb{R}^n} \{\langle y, x\rangle - \varphi(x)\}$ where $\langle y, x\rangle$ represents the inner product of y and x, i.e., $\langle y, x\rangle = y^{\mathrm{T}}x$. Recall that the function φ^* is convex, moreover if φ is a closed proper convex function, then $(\varphi^*)^* = \varphi$. For $\varphi_2 : \mathbb{Z}^n(\text{resp., } \mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ the epigraph of φ_2 is the set $\mathrm{epi}\, \varphi_2 := \{(x, x_{n+1}) \mid x_{n+1} \geq \varphi_2(x),\, x \in \mathbb{Z}^n(\text{resp., } \mathbb{R}^n)\} \subseteq \mathbb{R}^{n+1}$.

2 A brief review of continuous and discrete DC programming

We begin by considering (1.1) with $S = \mathbb{R}^n$ and $N = \emptyset$, more specifically,

$$\min_{x\in\mathbb{R}^n}\{g(x) - h(x)\}. \tag{2.1}$$

Then, the following proposition holds.

Proposition 2.1 ([23]). *Suppose that the DC program (2.1) has an optimal solution x^*. Then, we have*

1. $\partial g(x^*) \supseteq \partial h(x^*)$,
2. $\bar{y} \in \partial h(x^*) \Leftrightarrow x^* \in \partial h^*(\bar{y})$, *and*
3. $\bar{y} \in \partial h(x^*) \Rightarrow \bar{y}$ *is an optimal solution to* $\inf_{y\in\mathbb{R}^n}\{h^*(y) - g^*(y)\}$.

The following theorem is known as Toland-Singer duality, and forms the basis for DC minimization algorithms.

Theorem 2.2. *(Toland-Singer duality)*

$$\inf_{x\in\mathbb{R}^n}\{g(x) - h(x)\} = \inf_{y\in\mathbb{R}^n}\{h^*(y) - g^*(y)\}$$

We next define stationary points for the DC program that contains the global optima.

Definition 2.3. *A stationary point for $g - h$ is a point x^* such that*

$$\partial g(x^*) \cap \partial h(x^*) \neq \emptyset.$$

Let us introduce an existing algorithm for solving the DC program which will become the base of our proposed algorithms and cite its convergence results. For details we refer the reader to [23].

<u>SIMPLIFIED DC ALGORITHM(DCA)</u>

Step 0: Choose $x^0 \in \mathbb{R}^n$. Set $k = 0$

Step 1: Choose $y^k \in \partial h(x^k)$ and $x^{k+1} \in \partial g^*(y^k)$

Step 2: If stopping criterion is satisfied stop,

else set $k = k + 1$ and go to Step 1

Theorem 2.4 ([23]). *Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by the simplified DCA. Then, the following statements hold.*

1. $g(x^{k+1}) - h(x^{k+1}) \leq g(x^k) - h(x^k)$.
2. $h^*(y^{k+1}) - g^*(y^{k+1}) \leq h^*(y^k) - g^*(y^k)$.
3. *Every accumulation point x^* (y^*) of the sequence $\{x^k\}$ ($\{y^k\}$) is a stationary point of $g - h$ ($h^* - g^*$).*

We now turn to DC programs with discrete variables. Before introducing the results of Maehara, Marumo and Murota, we define some concepts related to discrete functions. Consider a function on discrete variables, $\varphi : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$.

Definition 2.5. *A convex function $\hat{\varphi} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex extension of φ if*

$$\hat{\varphi}(x) = \varphi(x) \quad (x \in \mathbb{Z}^n).$$

The convex closure of φ is the function $\varphi^{\mathrm{cl}} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ whose epigraph is equal to the closed convex hull of the epigraph of φ.

While the convex closure can be defined for any φ, clearly, not all discrete functions have convex extensions. If the discrete function φ does have a convex extension $\hat{\varphi}$, then we always have

$$\varphi^{\mathrm{cl}}(x) = \hat{\varphi}(x) \quad (x \in \mathbb{Z}^n).$$

As, in this paper, we will be concerned only with discrete functions which are the restrictions of continuous convex functions on $\mathbb{R}^n$ to $\mathbb{Z}^n$, all discrete functions will trivially have convex extensions.

Let us consider the DC program (1.1) with $S = \mathbb{R}^n$ in which all variables are restricted to integer values, i.e., $N = \{1, 2, \ldots, n\}$:

$$\min_{x \in \mathbb{Z}^n} \{g(x) - h(x)\}. \tag{2.2}$$

If we define the discrete functions $g_{\mathbb{Z}}$, $h_{\mathbb{Z}} : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ as the restrictions of g and h to $\mathbb{Z}^n$:

$$g_{\mathbb{Z}}(x) = g(x),\ h_{\mathbb{Z}}(x) = h(x) \quad (x \in \mathbb{Z}^n) \tag{2.3}$$

and let $\hat{g}$, $\hat{h} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any convex extensions of $g_{\mathbb{Z}}$ and $h_{\mathbb{Z}}$, then the following continuous DC program is clearly a relaxation of (2.2)

$$\min_{x \in \mathbb{R}^n} \{\hat{g}(x) - \hat{h}(x)\}. \tag{2.4}$$

The original functions g and h are obvious candidates for the convex extensions $\hat{g}$ and $\hat{h}$, but this is usually a poor choice as the two optimal values of (2.2) and (2.4) generally do not coincide. Maehara, Marumo and Murota [21] proved that the appropriate choice of $\hat{g}$ ensures this will not happen.

Theorem 2.6 ([21]). *If $\hat{g}$ is the convex closure of $g_{\mathbb{Z}}$, then the optimal values of the two problems* (2.2) *and* (2.4) *coincide.*

We now turn to our main concern, mixed integer DC programs.

3 Continuous relaxation with no integrality gap

We begin by rephrasing problem (1.1). By using the indicator function of set S, that is, the function $\delta_S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\delta_S(x) = \begin{cases} 0 & (x \in S) \\ +\infty & (x \notin S) \end{cases},$$

(1.1) can be written as

$$\begin{array}{ll} \min & (\delta_S(x) + g(x)) - h(x) \\ \text{sub.to} & x_N \in \mathbb{Z}^N. \end{array} \tag{3.1}$$

Since S is a closed convex set, δ_S, and hence δ_S+g are closed proper convex functions.

For convenience of notations, we set $M = \{1, 2, \ldots, n\} \setminus N$, and express $x \in \mathbb{R}^n$ as $x = (x_M, x_N)$. Now define $\tilde{g}$ and $\tilde{h}$ respectively as the restrictions of $\delta_S + g$ and h to $\mathbb{R}^M \times \mathbb{Z}^N$. We also denote the convex closure of $\tilde{g}$ by $\tilde{g}^{\mathrm{cl}}$. Convex extensions, epigraphs, and convex closures of $\tilde{g}$ and $\tilde{h}$ are defined in a manner analogous to the discrete functions in Section 1; for example, the epigraph of $\tilde{g}$ is defined as the set $\{(x_M, x_N, x_{n+1}) \in \mathbb{R}^M \times \mathbb{Z}^N \times \mathbb{R} \mid x_{n+1} \geq g(x_M, x_N)\}$.

In the rest of this section, we extend the theorem of Maehara, Marumo and Murota, to mixed-integer DC programs (1.1), that is, DC programs involving both integer-valued and continuous variables. More precisely, we can prove the following theorem. Here, we just state it without the proof.

Theorem 3.1. *Let $\hat{h} : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary convex extension of $\tilde{h}$. Then, the following (continuous) DC program:*

$$\min_{x \in \mathbb{R}^n} \{\tilde{g}^{\mathrm{cl}}(x) - \hat{h}(x)\}. \tag{3.2}$$

has the same optimal value as the mixed-integer DC program (3.1), *i.e., as* (1.1). *In particular, the optimal set of* (3.1) *is contained in* (3.2).

By the above result, we have only to solve (3.2) instead of (1.1). In the remainder of the paper, we propose a specific algorithm for solving (3.2). In our algorithm, we choose h as $\hat{h}$, a convex extension of $\tilde{h}$. Therefore, our target is to solve the following problem

$$\min_{x\in\mathbb{R}^n} \{\tilde{g}^{\mathrm{cl}}(x) - h(x)\}. \tag{3.3}$$

4 A basic algorithm for the mixed integer DC program

In this section, we formulate a basic algorithm based on the DCA of Section 2, for solving (3.3). For the DC program (3.3), recall that the DCA involves finding x_k and y_k with

$$y^{k+1} \in \partial h(x^k), \text{ and } x^{k+1} \in \partial(\tilde{g}^{\mathrm{cl}})^*(y^{k+1}).$$

Finding $x^{k+1} \in \partial(\tilde{g}^{\mathrm{cl}})^*(y^{k+1})$ can be accomplished by using the following relations:

$$\begin{aligned} x^{k+1} \in \partial(\tilde{g}^{\mathrm{cl}})^*(y^{k+1}) &\Leftrightarrow \partial\tilde{g}^{\mathrm{cl}}(x^{k+1}) \ni y^{k+1} \\ &\Rightarrow x^{k+1} \text{ is a solution of } \sup_{w\in\mathbb{R}^n} \left(\langle y^{k+1}, w\rangle - \tilde{g}^{\mathrm{cl}}(w)\right) \\ &\Leftrightarrow x^{k+1} \text{ is a solution of } \inf_{w\in\mathbb{R}^n} \left(\tilde{g}^{\mathrm{cl}}(w) - \langle y^{k+1}, w\rangle\right) \end{aligned}$$

The rightmost optimization problem involves minimizing a convex function. However, this cannot be solved by using standard convex optimization methodologies such as the interior point method, since we do not have an explicit expression of $\tilde{g}^{\mathrm{cl}}$ in general. This can be overcome by using Theorem 3.1 to note that it corresponds to solving the following convex mixed integer program:

$$\begin{array}{ll} \min & g(x) - \langle y^k, x\rangle \\ \text{sub.to} & x \in S,\ x_N \in \mathbb{Z}^N. \end{array} \tag{4.1}$$

Hence, by replacing $x^{k+1} \in \partial(\tilde{g}^{\mathrm{cl}})^*(y^{k+1})$ with (4.1) in Step 2 of the simplified DCA, we gain a specific algorithm for solving (1.1) as below:

SEQUENTIAL CONVEX MIXED-INTEGER PROGRAMMING METHOD (SCMIP)

Step 0: Choose $x^0 \in \mathbb{R}^n$. Set $k = 0$.

Step 1: Choose $y^{k+1} \in \partial h(x^k)$ and solve (4.1) to obtain x^{k+1}.

Step 2: If stopping criterion is satisfied, stop,

else set $k = k + 1$ and goto step 1

Obviously, each iteration point x^k is feasible to (1.1). By applying existing results on convergence for the DCA, we can make some observations for the case that at least one of $\tilde{g}^{\mathrm{cl}}$ and h is a strongly convex function.

- both $\tilde{g}^{\mathrm{cl}}(x^k) - h(x^k)(= f(x^k))$ and $h^*(y^k) - (\tilde{g}^{\mathrm{cl}})^*(y^k)$ strictly decrease

- if x^*(resp., y^*) is an accumulation points of $\{x^k\}$ (resp., $\{y^k\}$), then x^*(resp.,y^*) is a stationary point of $\min \tilde{g}^{\mathrm{cl}}(x) - h(x)$ (resp., $\min h^*(y) - (\tilde{g}^{\mathrm{cl}})^*(y)$). That is to say, $y^* \in \partial \tilde{g}^{\mathrm{cl}}(x^*) \cap \partial h(x^*)$ and $x^* \in \partial(\tilde{g}^{\mathrm{cl}})^*(y^*) \cap \partial h^*(y^*)$ hold.
- the x^N-part of x^k converges to some point within finitely many iterations.

In the above discussion, the assumption that at least one of $\tilde{g}^{\mathrm{cl}}$ and h is strongly convex is crucial. We place emphasis on the "at least one" phrase.

In problem (1.1), we did not assume strong convexity of either g or h. Thus, at first glance, the above results may seem inapplicable, however, it can be easily overcome by considering the equivalent problem for fixed $\rho > 0$

$$\min \ \left(g(x) + \rho\frac{\|x\|^2}{2}\right) - \left(h(x) + \rho\frac{\|x\|^2}{2}\right) \text{ sub. to } x^N \in \mathbf{Z}^N, \ x \in S. \tag{4.2}$$

We note here that the convex closure of $g(\cdot) + \rho\|\cdot\|^2/2$ restricted to $\mathbf{Z}^N \times \mathbf{R}^M$ is usually not strongly convex, whereas $h(\cdot) + \rho\|\cdot\|^2/2$ always is. Thus it is important that we do not need the strong convexity of both g and h.

Before ending this section, we make an important remark concerning the drawbacks of transforming (1.1) to (4.2). Consider two different DC-decompositions (g_1, h_1) and (g_2, h_2) for f, i.e., $f = g_1 - h_1 = g_2 - h_2$, and corresponding continuous DC programs of the form (3.3). Their two optimal sets are exactly the same. However, their sets of stationary points may possibly differ. This phenomenon does not occur in continuous DC programs without discrete variables, and thus it is characteristic of (3.3). To illustrate it, let us consider the following trivial mixed integer program:

$$\min \ x \ \text{ s.t. } x \in \{-1, 0, 1\}. \tag{4.3}$$

Choose two DC-decompositions $(g_1, h_1) = (x, 0)$ and $(g_2, h_2) = (x^2 + x, x^2)$. Then, $\mathrm{dom}\ \tilde{g}_1^{\mathrm{cl}} = \mathrm{dom}\ \tilde{g}_2^{\mathrm{cl}} = [-1,\ 1]$, $\tilde{g}_1^{\mathrm{cl}}(x) = x$, and $\tilde{g}_2^{\mathrm{cl}}$ is the polygonal line connecting the three points $(-1, 0), (0, 0)$ and $(1, 2)$. Thus the resulting optimization problem of the form (3.3) are:

$$\min \begin{cases} \infty & (x < -1) \\ x & (-1 \le x \le 1) \\ \infty & (1 \le x) \end{cases} \qquad \text{and} \qquad \min \begin{cases} \infty & (x < -1) \\ -x^2 & (-1 \le x \le 0) \\ 2x - x^2 & (0 \le x \le 1) \\ \infty & (1 \le x) \end{cases}$$

The set of stationary points of the former problem is nothing but the optimal set $\{-1\}$ of (4.3), while that of the latter is $\{0, -1\}$. This example indicates that the choice of DC decomposition may affect efficiency in finding the optima.

5 Concluding remarks

In this paper, we have considered mixed integer programs having DC objective functions and closed convex constraints. For these problems, we have extended the result of Maehara, Marumo, and Murota concerning continuous relaxations of discrete DC programs and obtained a continuous DC program whose optimal value is exactly equal to the original one. We have also proposed an algorithm to solve the obtained relaxed problem.

Our contribution can be summarized as follows:

- We have proposed a new framework for solving nonconvex mixed integer problems, which are notorious as being extremely difficult. Although our method involves the computationally costly routine of repeatedly solving convex MIPs, it still has significant merit, since it provides a practical way of dealing with these tough problems.

- We have theoretically proved convergence of generated sequences, thus the solutions provided by our algorithm are stationary points which have good chances of being the global optimum.

We conclude this paper by mentioning that while our method does not obtain polynomial complexity, there may be some specific problems for which it is tractable. For example, Maehara, Marumo and Murota [21] showed that their DCA-based algorithm can efficienly solve the degree-concentrated spanning tree problem. The search for such problems is a possible direction for future work.

References

[1] T. Achterberg, SCIP: solving constraint integer programs, Math. Prog. Comp. **1** (2009) 1–41.

[2] P. Bonami, L. T. Biegler, A. R. Conn, G. Cornéjols, I. E. Grossmann, C. D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya, A. Wăchter, An algorithmic framework for convex mixed integer nonlinear programs, Discrete Optimization **53** (2009) 78–87.

[3] P. Bonami, M. Kilinç, J. Linderoth, Algorithms and software for mixed integer nonlinear programs, in Mixed integer nonlinear programming, The IMA volumes in mathematics and its applications (J. Lee and S. Leyffer eds.) **154** (2012).

[4] P. Briest, P. Krysta, Single-minded unlimited supplou pricing on sparse instances, Proc. 17th Annu ACM-SIAM Symp Discrete Algorithms, (2006) 1093–1102.

[5] S. Burer and A. N. Letchford, Non-convex mixed-integer nonlinear programming: A survey, Surveys in Operations Research and Management Science, **17** (2012) 97–106.

[6] M. A. Duran, I. E. Grossmann, An outer-approximation algorithm for a class of mixed-integer nonlinear programs, Math. Prog. A **36** (1986) 307–339.

[7] O. Exler, T. Lehmann, K. Schittkowski, A trust region SQP algorithm for mixed integer nonlinear programming, Optim. Lett. **1** (2007) 269–280.

[8] O. Exler, T. Lehmann, K. Schittkowski, A comparative study of SQP-type algorithms for nonlinear and nonconvex mixed-integer optimization, Math. Prog. Comp. **4** (2012) 383–412.

[9] R. Fletcher, S. Leyffer, Solving mixed integer nonlinear programs by outer approximation, Math. Prog. A **66** (1994) 327–349.

[10] C. A. Floudas, Deterministic global optimization: Theory, algorithms and applications, Kluwer Academic Publishers, Dordrecht (1999).

[11] A. M. Geoffrion, Generalized Benders decomposition, Journal of Optimization, Theory and Applications, **4** (1972) 237–260.

[12] O. K. Gupta and A. Ravindran, Branch and bound experiments in convex nonlinear integer programming, Management Science,

[13] Gurobi optimizer reference manual, Optimization, Gurobi and others, URL:http://www.gurobi.com **2** (2012) 1–3.

[14] R. Horst, N. V. Thoai, DC programming: overview, J. of Optimization Theory and Applications **103** (1999) 1–43.

[15] R. Horst, P. M. Pardalos, and N. V. Thoai, Introduction to Global Optimization 2nd Edition, Springer Science+Business Media,B.V., 2000.

[16] *ILOG CPLEX V12.1, User's Manual for CPLEX*, International Business Machines Corporation, **46** (2009) 157.

[17] S. Leyffer, Integrating SQP and branch-and-bound for mixed integer nonlinear programming, Comput. Optim. Appl. **18** (2001), 295–309.

[18] H. A. Le Thi, H. M. Le, T. Pham Dingh, Portfolio selection under downside risk measures and cardinality constraints based on DC programming and DCA, Computational Management Science **6** (2009) 459–475.

[19] H. A. Le Thi, H. M. Le, T. Pham Dingh, Feature selection in machine learning: an exact penalty approach using a Difference of Convex function algorithm, Machine Learning **101** (2015) 163–186.

[20] T. Maehara, K. Murota, A framework of discrete DC programming by discrete convex analysis, Math. Prog. A **152** (2015) 435–466.

[21] T. Maehara, N. Marumo, K. Murota, Continuous relaxation for discrete DC programming, in Modelling, computation and optimization in information systems and management sciences, AISC **359** (2015) 181–190.

[22] Y. S. Niu, T. Pham Dinh, A DC programming approach for mixed-integer linear programs, in Modelling, computation and optimization in information systems and management sciences, CCIS **14** (2008) 244–253.

[23] T. Pham Dingh, H. A. Le Thi, Convex analysis approach to D. C. programming: theory, algorithms and applications, Acta Mathematica Vietnamica **22** (1997) 289–355.

[24] T. Pham Dingh, H. A. Le Thi, Recent advances in DC programming and DCA, in Lecture Notes in Computer Science **8342** (2014) 1–37.

[25] H. D. Sherali, H. Wang, Global optimization of nonconvex factorable programming problems, Math. Prog. A **89** (2001) 459–478.

[26] M. Tawarmalani, N. V. Sahinidis, Global optimization of mixed–integer nonlinear programs: a theoretical and computational study, Math. Prog. A **99** (2004) 563–591.

[27] M. Thiao, T. Pham Dingh, H. A. Le Thi, A DC programming approach for space eigenvalue problem, Proc. ICML2010 (2010), 1063–1070.

[28] T. Westerlund, F. Pettersson, An extended cutting plane method for solving convex MINLP problems, Computers and Chemical Engineering **19** (1995) 131–136.