Valuing Zero-coupon Convertible Bonds with the Laplace-Carson Transform*

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1 Introduction

Convertible bond (abbreviated to CB) is the most popular hybrid security with debt and equity-like features: A CB holder is entitled to receive fixed coupon payments as well as the principal repayment at maturity like a straight bond. He also has the right to forgo the fixed-income components and convert CB into the underlying common stock according to pre-specified conditions, i.e., a conventional CB may be converted at any time until a pre-specified maturity date into stocks at a fixed conversion ratio. Hence, it might be said that a CB is equivalent to a bond with an embedded American-style call option for conversion. CBs are attractive to investors due to their flexibility, and also to issuers due to the fact that CB yields are lower than those of equivalent straight bonds. This is the principal reason why CBs become an important segment of worldwide corporate bond markets.

The hybrid feature offers potentially unlimited gain to investors when the issuer's stock performs well. The investor is, however, exposed to the credit risk of default at the same time. Default occurs when the total firm value falls below the total redemption value of debt at maturity. Merton [25] applied the option pricing framework of Black and Scholes [10] and Merton [24] to develop a fundamental model for valuing a defaultable CB as a contingent claim on firm value, obtaining a partial differential equation (PDE) for the CB value. Ingersoll [14] used this PDE to discuss optimal policies for call and conversion, assuming that CB is the only senior debt in the firm's capital structure. Most of the structural models established previously are considered as extensions of the framework of Merton [25] and Ingersoll [14]; see Batten et al. [8] and Bhattacharya [9] for surveys.

To solve the PDE for CBs under more general assumptions, we need numerical methods such as finite difference and finite element methods. Brennan and Schwartz [11] adopted a finite-difference method to value callable CBs with discrete coupons and dividends. Barone-Adesi et al. [7] used a finite element method for solving a two-factor model of CBs under stochastic interest rate and volatility. See Table 3 of Batten et al. [8] for recent references on these numerical methods as well as lattice-based and simulation methods.

As another alternative, Fourier or Laplace transform (LT) method has been known as a powerful tool of solving general PDEs. In particular, Laplace-Carson transform (LCT), a minor variant of LT, has been extensively used in the context of option pricing [4, 12, 13, 16, 17, 18, 19, 20, 26, 28]. In the randomization of Carr [12], the LCT approach is used to obtain the

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*This is an early draft of my paper [22] in preparation. All of the proofs of Lemmas and Theorems are omitted here.

value of an American put option with random maturity distributed exponentially. For a general one-factor valuation problem, the basic LCT approach consists of the following three steps:

i) Taking the LCT of the PDE with respect to the remaining time to maturity, we have an associated ordinary differential equation (ODE).

ii) Solving this ODE together with appropriate boundary conditions, we obtain the LCT of the original value.

iii) The original value in the real-time domain can be computed by an algorithm for inverting LTs/LCTs numerically.

For multi-factor problems, multidimensional LTs and their numerical inversion are required in the steps i) and iii), respectively. For optimal stopping problems such as valuing American options, the LCT of the optimal stopping boundary can be jointly obtained in the step ii). For the step iii), various efficient methods have been developed for inverting LTs/LCTs; see, e.g., Abate and Whitt [1] for one-dimensional cases and Abate et al. [2] for multidimensional cases.

The expression for the solution obtained in the step ii) is seriously affected by the payoff function at maturity. If the payoff is a piecewise function defined on some separate intervals, then the LCT of the value becomes extremely complex. As we will see in Section 2, the payoff function of a typical defaultable CB is defined on three separate intervals. Hence, the plain LCT approach generates a cumbersome expression including six coefficients to be determined by boundary conditions. The purpose of this paper is to develop a refined LCT approach that generates a much simpler solution than the plain LCT approach. A primal target is the CB valuation problem. In recent years, various CBs have been issued with additional conversion provisions, including international CBs [5], CBs with reset clauses [15], reverse CBs [27], mandatory CBs [3] and so on; see Table 2 of Batten et al. [8] for detailed features of these variants. However, we only focus on a simple CB with no coupon payments, no call provision and voluntary conversion prior to maturity, because what we are actually aiming at is refining the LCT approach to meet more general claims.

2 The Plain LCT Approach

2.1 Assumptions

Following the framework of Merton [25] and Ingersoll [14], we consider a CB issued by a firm in frictionless markets, assuming that the CB is the only senior debt in the firm's capital structure except for common stock. Hence, a default would occur when the firm value falls below the total redemption value of the CBs. Let $(V_t)_{t \geq 0}$ denote the aggregate value process of the firm. Assume that $(V_t)_{t \geq 0}$ is a diffusion process with the Black-Scholes-Merton dynamics

$$dV_t = (r - \delta)V_t \, dt + \sigma V_t \, dW_t, \quad t \geq 0 \tag{1}$$

where $r > 0$ is the risk-free rate of interest, $\delta \geq 0$ is the instantaneous rate of the cash payments by the firm to either its shareholders or liabilities-holders (e.g., dividends or interest payments),

and $\sigma > 0$ is the volatility coefficient of the firm value, all of which are assumed to be constants. Suppose an economy with finite time period $[0, T]$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a filtration $\mathbb{F} \equiv (\mathcal{F}_t)_{t \in [0,T]}$. $W \equiv (W_t)_{t \in [0,T]}$ is a one-dimensional standard Brownian motion process defined on $(\Omega, \mathcal{F})$ and takes values in $\mathbb{R}$. The filtration $\mathbb{F}$ is the natural filtration generated by W and $\mathcal{F}_T = \mathbb{F}$. The firm value process defined in (1) is represented under the equivalent martingale measure $\mathbb{P}$, which implies that the firm value has mean rate of return r, and the conditional expectation $\mathbb{E}_t[\,\cdot\,] \equiv \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ is calculated under the measure $\mathbb{P}$.

Consider a defaultable CB with maturity date T and face value F. For simplicity, we focus on CBs with no coupon payments, no call provision and defaultable only at maturity. Assume that there are ℓ outstanding CBs of this firm in markets, and each CB is convertible into n shares. The holders who choose to convert their CBs into shares will dilute current shareholders' ownership. If there are m shares of common stock outstanding, the *conversion value* is given by γV_t, where γ is defined by

$$\gamma = \frac{n}{m + \ell n}, \tag{2}$$

for which $\gamma \ell$ (< 1) is called the *dilution factor*, indicating the fraction of the common stock held by the CB holders.

2.2   CB value and its LCT

Let $B(t, V_t)$ denote the CB value at time $t \in [0, T)$. From the assumptions on the capital structure and the default time, we see that there are three possible payoffs at maturity: CB holders receive either the conversion value γV_T if it exceeds the face value F, the face value F if it exceeds the conversion value γV_T, or the proportional firm value V_T/ℓ if the firm value is less than the par value of outstanding CBs, i.e.,

$$
\begin{aligned}
B(T, V_T) &= \max\left(\gamma V_T,\, \min\left(\frac{1}{\ell} V_T, F\right)\right) \\
&= \frac{1}{\ell} V_T \mathbf{1}_{\{V_T \leq F\ell\}} + F \mathbf{1}_{\{F\ell < V_T \leq \frac{F}{\gamma}\}} + \gamma V_T \mathbf{1}_{\{V_T > \frac{F}{\gamma}\}} \\
&= \frac{1}{\ell} V_T - \frac{1}{\ell}\left(V_T - F\ell\right)^+ + \gamma\left(V_T - \frac{F}{\gamma}\right)^+,
\end{aligned}
\tag{3}
$$

where $(x)^+ \equiv \max(x, 0)$ for $x \in \mathbb{R}$. Figure 1 illustrates the payoff value $B(T, V_T)$ at maturity as a function of the firm value V_T.

From the payoff value (3) and the theory of arbitrage pricing, the fair CB value at time t is given by solving the *optimal stopping problem*

$$B(t, V_t) = \operatorname*{ess\,sup}_{\tau_c \in [t,T]} \mathbb{E}_t\left[e^{-r(\tau_c - t)} \max\left(\gamma V_{\tau_c},\, \min\left(\frac{1}{\ell} V_{\tau_c}, F\right)\right)\right], \quad 0 \leq t \leq T, \tag{4}$$

where τ_c is a stopping time of the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. The random variable $\tau_c^* \in [t, T]$ is called the *optimal conversion time* if it gives the supremum value of the right-hand side of (4). Ingersoll [14] proved that $\tau_c = T$ (a.s.) if $\delta = 0$, i.e., it is *not* optimal for investors to convert early before maturity if there are no dividends; see Theorem 3.2 below.
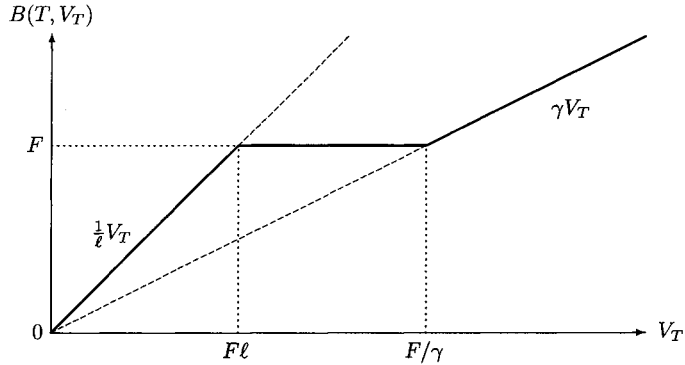
Figure 1: Payoff from CB at maturity.

Let $\mathcal{D} = [0, T] \times \mathbb{R}_+$. Solving the optimal stopping problem (4) is equivalent to finding the points (t, V_t) in $\mathcal{D}$ for which early conversion is optimal. Let $\mathcal{E}$ and $\mathcal{C}$ denote the *early conversion region* and *continuation region*, respectively. The early conversion region $\mathcal{E}$ is defined by

$$\mathcal{E} = \{(t, V_t) \in \mathcal{D} \mid B(t, V_t) = p(V_t)\},$$

where $p(V_t)$ is a virtual payoff at time t, which is defined by

$$p(V_t) = \max\left(\gamma V_t, \min\left(\frac{1}{\ell} V_t, F\right)\right). \tag{5}$$

No doubt, the continuation region $\mathcal{C}$ is the complement of $\mathcal{E}$ in $\mathcal{D}$. The boundary that separates $\mathcal{E}$ from $\mathcal{C}$ is referred to as the *early conversion boundary* (ECB), which is defined by

$$V_c(t) = \inf\{V_t \in \mathbb{R}_+ \mid B(t, V_t) = p(V_t)\}, \quad t \in [0, T].$$

For simplicity, let $V \equiv V_t$. In much the same way as in the valuation of American options, the value $B(t, V)$ and the ECB $V_c(t)$ can be jointly obtained by solving a *free boundary problem* [11], which is specified by the Black-Scholes-Merton PDE

$$\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 B}{\partial V^2} + (r - \delta)V\frac{\partial B}{\partial V} - rB = 0, \quad V < V_c(t), \tag{6}$$

together with the *boundary conditions*

$$\left|\begin{array}{l} \lim_{V \downarrow 0} B(t, V) = 0 \\[2mm] \lim_{V \uparrow V_c(t)} B(t, V) = \gamma V_c(t) \\[2mm] \lim_{V \uparrow V_c(t)} \frac{\partial B}{\partial V} = \gamma, \end{array}\right. \tag{7}$$

and the *terminal condition*

$$B(T, V) = p(V). \tag{8}$$

The second condition in (7) is often called the *value-matching condition*, while the third one is called the *smooth-pasting condition*.

With the change of variables $\tau = T - t$, let

$$\widetilde{B}(\tau, V) = B(T - \tau, V) = B(t, V) \quad \text{and} \quad \widetilde{V}_c(\tau) = V_c(T - \tau) = V_c(t), \quad \tau \geq 0.$$

For $\lambda \in \mathcal{C}$ ($\text{Re}(\lambda) > 0$), define the LCT of these time-reversed functions with respect to τ as

$$B^*(\lambda, V) = \mathcal{LC}[\widetilde{B}(\tau, V)](\lambda) \equiv \int_0^\infty \lambda e^{-\lambda \tau} \widetilde{B}(\tau, V) d\tau$$

and

$$V_c^*(\lambda) = \mathcal{LC}[\widetilde{V}_c(\tau)](\lambda) \equiv \int_0^\infty \lambda e^{-\lambda \tau} \widetilde{V}_c(\tau) d\tau.$$

Obviously, there is no essential difference between the LCT and the LT, i.e.,

$$\mathcal{L}[\widetilde{B}(\tau, V)](\lambda) = \frac{B^*(\lambda, V)}{\lambda}, \quad \text{Re}(\lambda) > 0.$$

Also, this relation implies that the LCT can be inverted by using previously established methods developed for inverting LTs; see Abate and Whitt [1].

Remark 2.1 In the context of option pricing, LCTs have been first adopted in the *randomization* of Carr [12] for valuing an American vanilla put option, of which maturity T is exponentially distributed random variable with mean $\mathbb{E}[T] = 1/\lambda$. The idea of randomization gives us another interpretation that the LCT $B^*(\lambda, V)$ can be regarded as an exponentially weighted sum (integral) of the time-reversed value $\widetilde{B}(\tau, V)$ for (infinitely many) different values of the maturity $T \in \mathbb{R}_+$, and hence for $\tau \in \mathbb{R}_+$, which makes LCTs be well defined. From the viewpoint of Carr's randomization, we assume λ is a positive real number.

From the PDE (6) with the conditions (7) and (8), we see that the LCT $B^*(\lambda, V)$ satisfies the ODE

$$\frac{1}{2} \sigma^2 V^2 \frac{d^2 B^*}{dV^2} + (r - \delta) V \frac{dB^*}{dV} - (\lambda + r) B^* + \lambda p(V) = 0, \quad V < V_c^*, \tag{9}$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim_{V \downarrow 0} B^*(\lambda, V) = 0 \\[2mm] \lim_{V \uparrow V_c^*} B^*(\lambda, V) = \gamma V_c^* \\[2mm] \lim_{V \uparrow V_c^*} \frac{dB^*}{dV} = \gamma. \end{array} \right. \tag{10}$$

It is straightforward but cumbersome to solve (9) with the boundary conditions (10) and the continuity conditions of $B^*(\lambda, V)$ and its first derivatives at $V = F\ell$, $V = F/\gamma$ and $V = V_c^*$. By

this plain LCT approach, Hayashi et al. [13] obtained

$$
B^*(\lambda, V) = \begin{cases} A_1 V^{\theta_1} + \dfrac{1}{\ell}\dfrac{\lambda V}{\lambda+\delta}, & V \le F\ell \\[2mm] A_2 V^{\theta_1} + A_3 V^{\theta_2} + \dfrac{\lambda F}{\lambda+r}, & F\ell < V \le \dfrac{F}{\gamma} \\[2mm] A_4 V^{\theta_1} + A_5 V^{\theta_2} + \dfrac{\lambda\gamma}{\lambda+\delta}, & \dfrac{F}{\gamma} < V < V_c^* \\[2mm] \gamma V, & V \ge V_c^*, \end{cases} \tag{11}
$$

where A_i $(i = 1, \dots, 5)$ are constants given by

$$
\begin{aligned}
A_1 &= \frac{\lambda\big(\lambda+r+(\delta-r)\theta_2\big)(\gamma^{\theta_1}-\ell^{-\theta_1})F^{1-\theta_1}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)} \\
&\quad - \frac{\theta_2\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_1})F^{1-\theta_2}}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}(V_c^*)^{\theta_2-\theta_1} + \frac{\gamma\delta}{\theta_1(\lambda+\delta)}(V_c^*)^{1-\theta_1}, \\
A_2 &= \frac{\lambda\big(\lambda+r+(\delta-r)\theta_2\big)\gamma^{\theta_1}F^{1-\theta_1}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)} \\
&\quad - \frac{\theta_2\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})F^{1-\theta_2}}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}(V_c^*)^{\theta_2-\theta_1} + \frac{\gamma\delta}{\theta_1(\lambda+\delta)}(V_c^*)^{1-\theta_1}, \\
A_3 &= -\frac{\lambda\big(\lambda+r+(\delta-r)\theta_1\big)\ell^{-\theta_2}F^{1-\theta_2}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}, \\
A_4 &= -\frac{\theta_2\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})F^{1-\theta_2}}{\theta_1(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}(V_c^*)^{\theta_2-\theta_1} + \frac{\gamma\delta}{\theta_1(\lambda+\delta)}(V_c^*)^{1-\theta_1}, \\
A_5 &= \frac{\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})F^{1-\theta_2}}{(\theta_1-\theta_2)(\lambda+\delta)(\lambda+r)}.
\end{aligned}
$$

The parameters $\theta_1 \equiv \theta_1(\lambda) > 1$ and $\theta_2 \equiv \theta_2(\lambda) < 0$ are two real roots of the quadratic equation

$$
\tfrac{1}{2}\sigma^2\theta^2 + \big(r-\delta-\tfrac{1}{2}\sigma^2\big)\theta - (\lambda+r) = 0. \tag{12}
$$

From the value-matching condition in (10), the LCT V_c^* is given by

$$
V_c^*(\lambda) = \left[\frac{\gamma\delta(\theta_1-1)(\lambda+r)F^{\theta_2-1}}{\lambda\big(\lambda+r+(\delta-r)\theta_1\big)(\gamma^{\theta_2}-\ell^{-\theta_2})} \right]^{\frac{1}{\theta_2-1}}. \tag{13}
$$

3   A Refined LCT Approach

From the complex solutions (11) and (13), it is really hard to have any prospect of further analysis. To refine these solutions, we will use the notion of premium decomposition: For the CB value $B(t, V)$, we can decompose it into two parts, i.e.,

$$
B(t, V) = b(t, V) + \pi(t, V), \quad t \in [0, T], \tag{14}
$$

where $b(t, V)$ is the European CB value and $\pi(t, V)$ is the premium for early conversion. Note that both $B(t, V)$ and $b(t, V)$ have the same terminal value at $t = T$, i.e.,

$$
b(T, V) = B(T, V) = p(V), \tag{15}
$$

which is an important key of our refinement. Applying the standard risk-neutral valuation method to $b(t, V_t)$, we obtain

$$
\begin{aligned}
b(t, V) &= \mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}p(V)\right] \\
&= \frac{1}{\ell}\mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}V\right] - \frac{1}{\ell}\mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}(V - F\ell)^+\right] + \gamma\mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}\left(V - \frac{F}{\gamma}\right)^+\right] \\
&= \frac{1}{\ell}c(t, V; 0) - \frac{1}{\ell}c(t, V; F\ell) + \gamma c(t, V; F/\gamma),
\end{aligned}
\tag{16}
$$

where $c(t, V; K)$ denotes the value of a European vanilla call option with maturity T and strike price K $(K = 0, F\ell, F/\gamma)$. This value is well known and is given by

$$
c(t, V; K) = V\mathrm{e}^{-\delta(T-t)}\Phi\big(d_+(V, K, T-t)\big) - K\mathrm{e}^{-r(T-t)}\Phi\big(d_-(V, K, T-t)\big),
\tag{17}
$$

where $\Phi(\,\cdot\,)$ is the standard normal cumulative distribution function defined by

$$
\Phi(x) = \int_{-\infty}^{x}\phi(y)\mathrm{d}y \quad\text{with}\quad \phi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}, \qquad x \in \mathbb{R},
$$

and

$$
d_\pm(x, y, \tau) = \frac{\log(x/y) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.
$$

Clearly, $c(t, V; 0) = V\mathrm{e}^{-\delta(T-t)}$. With the change of variables $\tau = T - t$, let $\tilde{c}(\tau, V; K) = c(T - \tau, V; K) = c(t, V; K)$ and $\tilde{b}(\tau, V) = b(T - \tau, V) = b(t, V)$. For $\lambda > 0$, define the LCTs $c^*(\lambda, V; K) = \mathcal{LC}[\tilde{c}(\tau, V; K)](\lambda)$ and $b^*(\lambda, V) = \mathcal{LC}[\tilde{b}(\tau, V)](\lambda)$. Then, from (16), the LCT $b^*(\lambda, V)$ can be represented as

$$
b^*(\lambda, V) = \frac{1}{\ell}\frac{\lambda V}{\lambda + \delta} - \frac{1}{\ell}c^*(\lambda, V; F\ell) + \gamma c^*(\lambda, V; F/\gamma).
\tag{18}
$$

In order to carry out a further analysis, we need the following lemmas:

Lemma 3.1

$$
\begin{cases}
\lambda + r = -\frac{1}{2}\sigma^2\theta_1\theta_2, \\
\lambda + \delta = -\frac{1}{2}\sigma^2(\theta_1 - 1)(\theta_2 - 1).
\end{cases}
$$

Lemma 3.2

$$
c^*(\lambda, V; K) = \begin{cases}
\xi_1(V), & V < K \\
\xi_2(V) + \dfrac{\lambda V}{\lambda + \delta} - \dfrac{\lambda K}{\lambda + r}, & V \geq K,
\end{cases}
$$

where for $i = 1, 2$,

$$
\xi_i(V) \equiv \xi_i(V; K) = \frac{2}{\sigma^2}\frac{\lambda K}{\theta_i(\theta_i - 1)(\theta_1 - \theta_2)}\left(\frac{V}{K}\right)^{\theta_i}.
$$

For ease of exposition, for $i = 1, 2$, we write

$$
\eta_i \equiv \eta_i(V) = \xi_i(V; F\ell) \quad\text{and}\quad \zeta_i \equiv \zeta_i(V) = \xi_i(V; F/\gamma).
$$

Then, from (18) and Lemma 3.2, we obtain

$$
b^*(\lambda, V) = \begin{cases} -\dfrac{1}{\ell}\eta_1(V) + \gamma\zeta_1(V) + \dfrac{1}{\ell}\dfrac{\lambda V}{\lambda+\delta}, & V \le F\ell \\[2mm] -\dfrac{1}{\ell}\eta_2(V) + \gamma\zeta_1(V) + \dfrac{\lambda F}{\lambda+r}, & F\ell < V \le \dfrac{F}{\gamma} \\[2mm] -\dfrac{1}{\ell}\eta_2(V) + \gamma\zeta_2(V) + \gamma\dfrac{\lambda V}{\lambda+\delta}, & V > \dfrac{F}{\gamma}. \end{cases} \tag{19}
$$

As we saw in Section 2, the LCT $B^*(\lambda, V)$ satisfies the boundary conditions in (10), from which the corresponding boundary conditions for the LCT $\pi^*(\lambda, V) = \mathcal{LC}[\tilde\pi(\tau, V)](\lambda)$ for $\tilde\pi(\tau, V) = \pi(T - \tau, V) = \pi(t, V)$ can be written as

$$
\begin{vmatrix} \lim_{V\downarrow 0} \pi^*(\lambda, V) = 0 \\[2mm] \lim_{V\uparrow V_c^*} \pi^*(\lambda, V) = \gamma V_c^* - b^*(\lambda, V_c^*) \\[2mm] \lim_{V\uparrow V_c^*} \dfrac{d\pi^*}{dV} = \gamma - \dfrac{db^*}{dV}\Big|_{V=V_c^*}. \end{vmatrix} \tag{20}
$$

The LCT $\pi^*(\lambda, V)$ satisfies the ODE

$$
\frac{1}{2}\sigma^2 V^2 \frac{d^2\pi^*}{dV^2} + (r - \delta)V\frac{d\pi^*}{dV} - (\lambda + r)\pi^* = 0, \quad V > 0. \tag{21}
$$

From the first boundary condition $\lim_{V\downarrow 0}\pi^*(\lambda, V) = 0$, we have

$$
\pi^*(\lambda, V) = A_0 V^{\theta_1}, \quad V \ge 0, \tag{22}
$$

where A_0 is a constant. Applying the smooth-pasting condition in (20) to $\pi^*(\lambda, V)$ and using $b^*(\lambda, V)$ for $V > F/\gamma$, we obtain

$$
A_0 = \frac{1}{\theta_1}\left[\frac{\gamma\delta V_c^*}{\lambda+\delta} + \theta_2\left\{\frac{1}{\ell}\eta_2(V_c^*) - \gamma\zeta_2(V_c^*)\right\}\right](V_c^*)^{-\theta_1},
$$

so that for $V < V_c^*$

$$
\begin{aligned}
\pi^*(\lambda, V) &= \frac{1}{\theta_1}\left[\frac{\gamma\delta V_c^*}{\lambda+\delta} + \theta_2\left\{\frac{1}{\ell}\eta_2(V_c^*) - \gamma\zeta_2(V_c^*)\right\}\right]\left(\frac{V}{V_c^*}\right)^{\theta_1} \\[2mm]
&= \frac{1}{\theta_1}\left[\frac{\gamma\delta V_c^*}{\lambda+\delta} + \frac{\theta_1 - 1}{\theta_1 - \theta_2}\frac{\lambda F}{\lambda+\delta}\left(1 - (\gamma\ell)^{-\theta_2}\right)\left(\frac{\gamma V_c^*}{F}\right)^{\theta_2}\right]\left(\frac{V}{V_c^*}\right)^{\theta_1}.
\end{aligned} \tag{23}
$$

In addition, from the value-matching condition in (20), we obtain the LCT V_c^* for the conversion boundary as

$$
V_c^*(\lambda) = \frac{F}{\gamma}\left[-\frac{\delta\theta_2}{\lambda\left(1 - (\gamma\ell)^{-\theta_2}\right)}\right]^{\frac{1}{\theta_2-1}}, \tag{24}
$$

which enables us to simplify $\pi^*(\lambda, V)$ in (23) down to

$$
\begin{aligned}
\pi^*(\lambda, V) &= \frac{\gamma V_c^*}{\theta_1} \frac{1}{\lambda + \delta} \left[\delta + \frac{\theta_1 - 1}{\theta_1 - \theta_2} \lambda \left(1 - (\gamma \ell)^{-\theta_2} \right) \left(\frac{\gamma V_c^*}{F} \right)^{\theta_2 - 1} \right] \left(\frac{V}{V_c^*} \right)^{\theta_1} \\
&= \frac{\gamma \delta V_c^*}{\theta_1} \frac{1}{\lambda + \delta} \left(1 - \frac{\theta_1 - 1}{\theta_1 - \theta_2} \theta_2 \right) \left(\frac{V}{V_c^*} \right)^{\theta_1} = \frac{\gamma \delta V_c^*}{\lambda + \delta} \frac{1 - \theta_2}{\theta_1 - \theta_2} \left(\frac{V}{V_c^*} \right)^{\theta_1} \\
&= \frac{2}{\sigma^2} \frac{\gamma \delta V_c^*}{(\theta_1 - 1)(\theta_1 - \theta_2)} \left(\frac{V}{V_c^*} \right)^{\theta_1} .
\end{aligned}
$$

Since the European CB value $b(t, V)$ is explicitly given in (16) and (17), it would suffice to invert $\pi^*(\lambda, V)$ for obtaining the target CB value $B(t, V)$. Hence, we summarize the results as

Theorem 3.1 *The value $B(t, V)$ of the CB with voluntary conversion prior to maturity and no coupon payments is given by*

$$
B(t, V) = \begin{cases} b(t, V) + \mathcal{LC}^{-1}[\pi^*(\lambda, V)](T - t), & V < \mathcal{LC}^{-1}[V_c^*(\lambda)](T - t) \\ \gamma V, & V \geq \mathcal{LC}^{-1}[V_c^*(\lambda)](T - t), \end{cases} \tag{25}
$$

where $b(t, V)$ is the associated European CB value given by

$$
b(t, V) = \frac{1}{\ell} V e^{-\delta(T-t)} - \frac{1}{\ell} c(t, V; F\ell) + \gamma c(t, V; F/\gamma),
$$

$c(t, V; K)$ is the value of the associated vanilla call option with strike price K ($K = F\ell, F/\gamma$) given by (17), and

$$
\pi^*(\lambda, V) = \frac{2}{\sigma^2} \frac{\gamma \delta V_c^*}{(\theta_1 - 1)(\theta_1 - \theta_2)} \left(\frac{V}{V_c^*} \right)^{\theta_1}, \quad V < V_c^*.
$$

The LCT $V_c^* \equiv V_c^*(\lambda)$ for the early conversion boundary is given by

$$
V_c^*(\lambda) = \frac{F}{\gamma} \left[-\frac{\delta \theta_2}{\lambda \left(1 - (\gamma \ell)^{-\theta_2} \right)} \right]^{\frac{1}{\theta_2 - 1}} .
$$

Theorem 3.2 *If $\delta = 0$, then it is not optimal for investors to convert early before maturity.*

Assume hereafter that $\delta > 0$.

Theorem 3.3 *For the time-reversed early conversion boundary $\left(\widetilde{V}_c(\tau) \right)_{\tau \geq 0}$, we have*

$$
\lim_{\tau \to 0} \widetilde{V}_c(\tau) = \lim_{t \to T} V_c(t) = \frac{F}{\gamma}. \tag{26}
$$

Theorem 3.4 *For the time-reversed early conversion boundary $\left(\widetilde{V}_c(\tau) \right)_{\tau \geq 0}$, we have*

$$
\lim_{\tau \to \infty} \widetilde{V}_c(\tau) = \lim_{T \to \infty} V_c(t) = 0, \tag{27}
$$

i.e., for the perpetual case, it is optimal for investors to convert quickly after purchase.

Remark 3.1 In general, the assertion of Theorem 3.4 does *not* hold for conventional CBs. The quick conversion of the perpetual CB is primarily due to the assumption of no-coupon payments. There would be no rational reason for holding bonds with neither redemption nor coupons.

4 Broad Applicability

A remarkable feature of our refined LCT approach is that the early conversion premium does *not* depend on the complex terminal condition of the value, because the premium is defined by the difference of the American and European values, both of which are equal at maturity. As a result, the ODE for the LCT of the premium is given in a very concise form, together with a little bit modified boundary conditions. We immediately see that this idea is also applicable to other American-style claims, provided that the associated European claims have closed-form solutions for the value and its LCT. For example, consider an American vanilla call option with maturity T and strike price K, written on a dividend-paying asset. Let $(S_t)_{t \geq 0}$ be the price process of the underlying asset, and assume that $(S_t)_{t \geq 0}$ is a geometric Brownian motion process with the same dynamics as (1). Then, the value of the American vanilla call option at time $t \in [0, T]$, $C(t, S_t)$, is decomposed as

$$C(t, S_t) = c(t, S_t) + \pi_c(t, S_t), \quad t \in [0, T], \tag{28}$$

where $c(t, S_t) = c(t, S_t; K)$ is the value of the associated European vanilla call option given in (17) with $V := S_t$, and $\pi_c(t, S_t)$ is the early exercise premium. Applying the refined LCT approach to this pricing problem, we see that the LCT $\pi_c^*(\lambda, S) = \mathcal{LC}[\widetilde{\pi}_c(\tau, S)](\lambda)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 S^2 \frac{\mathrm{d}^2 \pi_c^*}{\mathrm{d}S^2} + (r - \delta)S \frac{\mathrm{d}\pi_c^*}{\mathrm{d}S} - (\lambda + r)\pi_c^* = 0, \quad S < S_c^*, \tag{29}$$

with the boundary conditions

$$\left|\begin{array}{l} \displaystyle\lim_{S \downarrow 0} \pi_c^*(\lambda, S) = 0 \\[2mm] \displaystyle\lim_{S \uparrow S_c^*} \pi_c^*(\lambda, S) = S_c^* - K - c^*(\lambda, S_c^*) \\[2mm] \displaystyle\lim_{S \uparrow S_c^*} \frac{\mathrm{d}\pi_c^*}{\mathrm{d}S} = 1 - \frac{\mathrm{d}c^*}{\mathrm{d}S}\bigg|_{S = S_c^*}, \end{array}\right. \tag{30}$$

where c^* is given in Lemma 3.2 and $S_c^* \equiv S_c^*(\lambda) = \mathcal{LC}[\widetilde{S}_c(\tau)](\lambda)$ is the LCT of the early exercise boundary $(S_c(t))_{t \in [0, T]}$. Solving the ODE (29) with (30), we obtain

$$\begin{aligned} \pi_c^*(\lambda, S) &= \frac{1}{\theta_1}\left\{\frac{\delta}{\lambda + \delta}S_c^* - \theta_2 \xi_2(S_c^*)\right\}\left(\frac{S}{S_c^*}\right)^{\theta_1} \\ &= \frac{2}{\sigma^2}\frac{\delta}{(\theta_1 - 1)(\theta_1 - \theta_2)}\left(S_c^* - \frac{rK}{\delta}\frac{\theta_1 - 1}{\theta_1}\right)\left(\frac{S}{S_c^*}\right)^{\theta_1}, \quad S < S_c^*, \end{aligned}$$

where the LCT $S_c^* (\geq K)$ is a unique positive solution of the functional equation

$$\lambda\left(\frac{S_c^*}{K}\right)^{\theta_2} + \delta\theta_2 \frac{S_c^*}{K} + r(1 - \theta_2) = 0.$$

Hence, the American vanilla call option has the value

$$C(t, S) = \begin{cases} c(t, S) + \mathcal{LC}^{-1}[\pi_c^*(\lambda, S)](T - t), & S < \mathcal{LC}^{-1}[S_c^*(\lambda)](T - t) \\ S - K, & S \geq \mathcal{LC}^{-1}[S_c^*(\lambda)](T - t). \end{cases}$$

As another example, consider a European continuous-installment option with maturity T and strike price K, written on a dividend-paying asset $(S_t)_{t\in[0,T]}$; see Kimura [19]. Installment options are path-dependent claims in which a small amount of up-front premium instead of a lump sum is paid at the time of purchase, and then a sequence of installments are paid up to maturity. If the installments are paid at a certain rate, say $q > 0$, per unit time, it is referred to as a continuous-installment option. The holder has the right of stopping payments at any time, thereby terminating the option contract. Hence, an optimal stopping problem similar to American-style options arises for the installment option even in European style. Let t (≥ 0) be the purchase time and let $c_i(t, S_t; q)$ denote the initial premium of the continuous-installment call option, assuming the same framework as that for the vanilla call option in this paper. Then, the value $c_i(t, S_t; q)$ can be decomposed as

$$c_i(t, S_t; q) = c(t, S_t) + \pi_i(t, S_t; q) - K_t, \quad t \in [0, T], \tag{31}$$

where

$$K_t = \frac{q}{r}\big(1 - e^{-r(T-t)}\big), \quad t \in [0, T],$$

is the NPV of the future payment stream at time t, and

$$\pi_i(t, S_t; q) = \operatorname*{ess\,sup}_{s\in[t,T]} \mathbb{E}_t\left[e^{-r(s-t)}\big(K_s - c(s, S_s)\big)^+\right],$$

represents the American compound put option maturing in time T written on the vanilla call option. Using the refined LCT approach combined with the decomposition (31), we have

$$\pi_i^*(\lambda, S; q) \equiv \mathcal{LC}[\widetilde{\pi}_i(\tau, S; q)](\lambda) = -\frac{2}{\sigma^2}\frac{q}{\theta_2(\theta_1 - \theta_2)}\left(\frac{S}{S_i^*}\right)^{\theta_2}, \quad S > S_i^*,$$

where $S_i^* \equiv S_i^*(\lambda) = \mathcal{LC}[\widetilde{S}_i(\tau)](\lambda)$ is the LCT of the optimal stopping boundary $\big(S_i(t)\big)_{t\in[0,T]}$, which is given by

$$S_i^*(\lambda) = K\left[\frac{q(\theta_1 - 1)}{\lambda K}\right]^{\frac{1}{\theta_1}}.$$

Hence, the initial premium of the continuous-installment option is given by

$$c_i(t, S; q) = \begin{cases} 0, & S \leq \mathcal{LC}^{-1}[S_i^*(\lambda)](T - t) \\ c(t, S) + \mathcal{LC}^{-1}[\pi_i^*(\lambda, S; q)](T - t) - K_t, & S > \mathcal{LC}^{-1}[S_i^*(\lambda)](T - t). \end{cases}$$

Note that the solutions for $\pi_i^*(\lambda, S; q)$ and $S_i^*(\lambda)$ have much simpler expressions as compared with those in Kimura [19, Equations (32) and (25)].

Due to the simplicity of the idea, the refined LCT approach is broadly applicable to, e.g., American continuous-installment options [17], American fractional lookback options [20], American exchange options [21] and so on. Of course, the approach is also applicable to put counterparts in the same way.

Finally, we make a brief remark about a certain similarity between our LCT approach and the so-called *quadratic approximation* (QA) developed by MacMillan [23] and Barone-Adesi and Whaley [6]. In the QA for an American vanilla option, the focus is also on the early exercise

premium, for which an *approximate* ODE is derived. The option value is given by the sum of the associated European value and an approximate solution for the early exercise premium. The significant difference between these two approaches is that the ODE derived from the LCT approach is exact in the Laplace domain, whereas the ODE derived from the QA approach is an approximation in the real-time domain. It has been known from numerical experiences that the LCT approach generates almost exact values, while the QA approach generates less accurate values, in particular, for options with long maturity.

Acknowledgments

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