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Chapter 1. Introduction

The purpose of this expository article is to investigate the support problem for a special class of superprocesses in terms of random measure. In the theory of measure-valued stochastic processes, compact support problems have been discussed for many years. For instance, in the case of typical super-Brownian motion \( X = \{X_t; t \geq 0\} \), Iscoe (1988) proved that if the initial measure \( X_0(dx) \) has a compact support, then for every \( t > 0 \), \( X_t \) possesses a compact support. Let \( B_+ = B_+(\mathbb{R}^n) \) be the totality of nonnegative Borel measurable functions on \( \mathbb{R}^n \), and let \( L = L(dx) \) be a locally finite random measure on \( \mathbb{R}^n \). For \( B_+ \ni f \), we define \( \langle f, L \rangle := \int f(x)L(dx) \). Furthermore, \( M_F(\mathbb{R}^n) \) denotes the totality of finite Borel measures on \( \mathbb{R}^n \) equipped with weak convergence topology. We define a differential operator \( P \) by

\[
P:= \frac{1}{2} \sum_{k=1}^{n} a_k(x) \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{n} b_k(x) \frac{\partial}{\partial x_k} + c(x)(\cdot)
\]

where we assume that \( a_k, b_k, c \in C_b^\infty(\mathbb{R}^n) \) satisfy \( \exists \delta > 0 : a_j > \delta > 0 \). As a matter of fact, our target process \( X = (\{X_t, t \geq 0\}, P_\mu) \) in terms of measure \( L \) is an \( M_F(\mathbb{R}^n) \)-valued Markov process, and its Laplace transition functional is given by

\[
\mathbb{E}_\mu[e^{-\langle \varphi, X_t \rangle}] = e^{-\langle u(t), \mu \rangle}.
\]

Here the function \( u(t) \equiv u(t, x) \) satisfies

\[
\begin{cases}
\partial_t u = Pu - \dot{L}(dx)u^2 \\
u(t, x)|_{t=0+} = \varphi(x)
\end{cases}
\]

where the symbol \( \dot{L}(dx) \) means \( \frac{L(dx)}{dx} \). For brevity's sake, in what follows we shall proceed the argument simply for \( d = 1 \). Our discussion on construction of superprocesses can be extended up to multi-dimensional case. However, the argument on the compact support problem for superprocesses is restricted to one-dimensional case.
§2. Main result

For $\mu \in M_F(\mathbb{R})$, the support of $\mu$, say, $\text{supp} (\mu)$ is defined by

$$\text{supp} (\mu) := \{ A \in \mathcal{B}(\mathbb{R}) : \mu (A^c) = 0 \}.$$ (4)

While, the global support of superprocess $X(\cdot)$, say, $\text{Gsupp} (X)$ is defined by

$$\text{Gsupp} (X) := \bigcup_{t \geq 0} \text{supp} (X_t (dx)).$$ (5)

It is a key point that we relate the support $\text{Gsupp} (X)$ of superprocess $X_t$ in terms of locally finite measure $L = L(dx)$ on $\mathbb{R}$ to a nonlinear singular elliptic boundary problem.

Let $d = 1, a(x) > 0$. We consider the associated boundary problem: for a differential operator $P = \frac{1}{2} a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x)$,

$$\begin{cases}
Pv = v^2(x) \frac{L(dx)}{dx}, & a_1 < x < a_2 \\
v(a_1) = \beta_1, & v(a_2) = \beta_2.
\end{cases}$$ (6)

When we denote the solution of (6) by $v(x; \beta_1, \beta_2)$, since $\exists \{\beta_1^{(n)}\}_n \nearrow \infty, \exists \{\beta_2^{(n)}\}_n \nearrow \infty$, the problem (6) possesses a unique solution $v(x; \beta_1^{(n)}, \beta_2^{(n)})$. Note that the solution $v(x)$ is a continuous convex function defined on the interval $I = [a_1, a_2]$.

Moreover, for $\forall a_1 \leq x_0 \leq x \leq a_2$, $v(x)$ satisfies

$$v(x) = v(x_0) + \Phi_0(x_0)(x - x_0) + \int_{x_0}^{x} \Phi_1(y)v(y)dy + \int_{x_0}^{x} dy \int_{x_0}^{y} \Phi_2(z)v(z)dz + \int_{x_0}^{x} dy \int_{x_0}^{y} \frac{2v^2(z)}{a(z)}L(dz),$$ (7)

where

$$\Phi_0(x) = v'(x+) + \frac{2b(x)}{a(x)}, \quad \Phi_1(x) = \frac{2b(x)}{a(x)},$$

$$\Phi_2(x) = \frac{2b(x)a'(x) - 2b'(x)a(x) + 2a(x)c(x)}{a(x)^2}.$$ (8)

Then we can obtain an explicit expression of the approximate solution. For $\psi \in C^+(\mathbb{R}), \text{supp} (\psi) \subset (-K, K), \theta > 0$, when we denote by $u_K(t, x; \theta \psi)$ the solution of

$$u(t, x) = 0, \quad x \in (-K, K)^c$$

$$u(t, x) = \theta \int_0^t \int_{-K}^{K} p_K(t - s, x, y)\psi(y)dyds$$

$$- \int_0^t \int_{-K}^{K} p_K(t - s, x, y)u^2(s, y)L(dy)ds, \quad x \in (-K, K),$$ (8)
then a simple fact $v_K \geq 0$ yields concurrently to $v_K \nearrow$ in $t \psi$ and $v_K \nearrow$ in $\psi$, and furthermore it follows immediately that

$$v_K(t, x; \theta \psi) \leq \sup_{t, x} \int_0^t \int_{-K}^K p_K(t - s, x, y) \theta \psi(y) dy ds < \infty.$$ 

On the other hand, $v_K(\theta, t, x; a_1, a_2)$ denote the solution of (8) with the test function replaced by $\psi = 1_{[a_1, a_2]}$.

For simplicity, we assume henceforth that $\text{supp}(X_0) \subset [a_1, a_2] \subset (-K, K)$, $b(x) = 0, c(x) > 0$. We shall represent the positive support probability of superprocess $X_t$ by the solution of (6). The argument of Iscoe (1988) for occupation time processes $\int_0^t X_s^K ds$ or $\int_0^t X_s ds$ implies that

$$E_{X_0}^L \{ \exp \left\{ -\theta \int_0^\infty X_s^K([a_1, a_2]^c) ds \right\} \} = \exp \left\{ -\int_{-\infty}^\infty v_K(\theta, x; a_1, a_2)X_0(dx) \right\}$$

holds. And besides we have

$$v_K(\theta, x; a_1, a_2) = \lim_{t \to \infty} \left( \lim_{n \to \infty} v_K(\theta \psi_n; t, x) \right),$$

and we can deduce that $v(x) \equiv v_K(\theta, x; a_1, a_2)$ satisfies that its second derivative $v''$ is a signed measure, and also that for $x \in (-K, K),

$$\frac{dv}{dx}(x \pm) = \int_{x_0}^{x \pm} \frac{2c(y)v(y)}{a(y)} dy + \int_{x_0}^{x \pm} \frac{2v^2(y)}{a(y)} L(dy) - 2\theta \int_{x_0}^{x \pm} 1_{[a_1, a_2]^c}(y) dy + (\text{Constant}).$$

Thus the representation of probability for the support can be derived.

$$P_{X_0}^L(\text{supp}(X_t) \cap [a_1, a_2]^c = \emptyset, \forall t \geq 0) = \lim_{K \to \infty} P_{X_0}^L(\text{supp}(X^K_t) \cap [a_1, a_2]^c = \emptyset, \forall t \geq 0)$$

$\Leftarrow$ by virtue of the right continuity of the path $X^K_t(\omega)$

$$= \lim_{K \to \infty} P_{X_0}^L \left( \int_0^\infty X_s^K([a_1, a_2]^c)ds = 0 \right)$$

$\Leftarrow$ by the expression of the occupation time process (9)

$$= \lim_{K \to \infty} \lim_{\theta \to \infty} \exp \left\{ -\int_{-\infty}^\infty v_K(\theta, x; a_1, a_2)X_0(dx) \right\}$$

$$= \lim_{n \to \infty} \exp \left\{ -\int_{a_1}^{a_2} v(x; \beta_1^{(n)}, \beta_2^{(n)})X_0(dx) \right\}$$

(10)
By virtue of the above-mentioned facts we can get the following principal result, the theorem for compact support.

**Theorem 1.** (Main Result) Let $\mu \in M_F(\mathbb{R})$ and $\text{supp}(\mu) \subset [a_1, a_2]$. Suppose that $d = 1$, $a(x) > 0$, $b(x) = 0$, $c(x) > 0$. For $\forall \varepsilon > 0$ ($\varepsilon << 1$ : sufficiently small), there exist proper real numbers $\underline{x} = \underline{x}(\varepsilon) < a_1$, $\overline{x} = \overline{x}(\varepsilon) > a_2$ such that $v$ is a nonnegative solution of (7) on the interval $(\underline{x}, \overline{x})$, i.e. $v(x) \geq 0$ for $x \in (\underline{x}, \overline{x})$. If $v$ satisfies the conditions

\[
\sup_{a_1 \leq x \leq a_2} v(x) \leq \varepsilon, \quad \lim_{x \to \underline{x}} v(x) = \lim_{x \to \overline{x}} v(x) = \infty,
\]

then the superprocess $X = \{X_t, t \geq 0\}$ has the compact support.

§3. Formulation of superprocess by admissible functional

Let us denote by $X = \{X_t, t \geq 0\}$ the measure-valued branching process corresponding to a locally finite random measure $L$, and $P^L_\mu$ denotes the probability law of the measure-valued process $X$. Then a measure-valued process $(X_t, P^L_\mu)$ in terms of random measure $L$ is given by the following Laplace transition functional.

\[
E^L_\mu[e^{-\langle \varphi, X_t \rangle}] = e^{-\langle u(t), \mu \rangle} \quad \text{with} \quad X_0 = \mu \in M_F(\mathbb{R}).
\]

Here the function $u(t, x)$ satisfies the following Cauchy problem.

\[
\begin{cases}
\partial_t u = Pu - \frac{L(dx)}{dx} u^2, \\
u(0, x) = \varphi \in C^+_b(\mathbb{R}).
\end{cases}
\]

Now, suggested by a formulation by Dawson-Fleischmann (1995), we shall consider the above initial value problem as an integral equation. As a matter of fact, when we write the fundamental solution to the aforementioned Cauchy problem by $p$, then we have

\[
u(t, x) = \int p(t, x, y)\varphi(y)dy - \int_0^t \int p(t-s, x, y)u^2(s, y)L(dy)ds.
\]

This means that we consider the mild solution to the above Cauchy problem. We shall assume henceforth:

[Assumption] For any $c > 0$,

\[
\int_{-\infty}^{\infty} e^{-cx^2} L(dx) < \infty, \quad \text{a.s.}
\]
case that the branching rate term $\gamma$ in the super-Brownian motion or the Dawson-Watanabe superprocess would be changed into a general additive functional which does not always possess its density. For a finite measure $\tilde{L}$ on $\mathbb{R}$ and a local time $\ell_{t,x}(\omega)$ of Brownian motion $B_s$, we define the additive functional $K_t^{[\tilde{L}]}(\omega)$ by

$$K_t^{[\tilde{L}]}(\omega) := \int \ell_{t,x}(\omega) \tilde{L}(dx). \quad (16)$$

We shall impose the following admissible conditions.

[Dynkin's Admissibility] For a Brownian motion $(B_t, \Pi_{0,x})$,

(i) $\Pi_{r,x}[K^{[\tilde{L}]}(r, t)] < \infty$, for $\forall r < t, x$

(ii) $\Pi_{r,x}[K^{[\tilde{L}]}(r, t)] \rightarrow 0$ uniformly in $x$ $(r, t \rightarrow s)$ $\forall s$

**Theorem 2.** (Dynkin, 1994) *If the transition function $\mathcal{P}(r, \mu; t, C) = P_{r, \mu}(X_t \in C)$ satisfied the following two conditions, then the measure-valued Markov process named $(\xi, K, \psi)$-superprocess with parameters $X = (X_t, P_{r, \mu})$ can be determined.*

\begin{align*}
\int \mathcal{P}(r, \mu; t, dv)e^{-\langle f, \nu \rangle} &= \exp\{-\langle v(r), \mu \rangle\}, \quad (17) \\
v(r, x) + \Pi_{r,x} \int_r^t \psi(s, v(s))(\xi_s)dK_s &= \Pi_{r,x} f(\xi_t). \quad (18)
\end{align*}

§4. Construction of sequence of approximate measure-valued processes

In this section we shall construct a basic process as a limit of increasing sequence of finite measure $M_F(\mathbb{R})$-valued processes realized on the common basic probability space. This provides us with a proto-type in the construction of our target superprocess. For each $K \in \mathbb{N}$, we put

$$E_K := \bigcup_{n=1}^{K} \{n\} \times (-n, n), \quad (19)$$

and we denote by $\tilde{X}_t^K \equiv \tilde{X}_t^K(dx)$ an $M_F(E_K)$-valued process. We shall first of all construct this measure-valued basic process $\tilde{X}_t^K$ in what follows. For $x \in (-n, n)$, a Markov process $w_K$ on $E_K$ starting at a point $(n, x)$ can be defined as

$$w_K(t) := \{n\}, w(t), \quad \text{for} \quad 1 \leq t \leq \tau_n$$

$$w_K(\tau_n) := \{n+1\}, w(\tau_n), \quad \tau_n = \inf\{t > 0 : w(t) = \pm n\}$$

where $w$ is a $P$-diffusion starting at a point $x$. Notice that the stochastic process $w_K$ dies out finally at time $\tau_K$. Next we consider a random measure $L_K$. In fact, we define

$$L_K(\{n\} \times (a, b)) := L((-n, n) \cap (a, b)), \quad \text{for} \quad n \leq K.$$
On this account, we can define the admissible additive functional $\mathcal{K}_{t}^{[L_{K}]}(w_{K})$ by making use of this random measure $L_{K}$, i.e.
\[
\mathcal{K}_{t}^{[L_{K}]}(w_{K}) := \int \tilde{\ell}_{t,y}(w_{K})L_{K}(dy)
\]  
(20)

where $\tilde{\ell}_{t,x}$ is a positive random variable given by
\[
\tilde{\ell}_{t,x}(w) := \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{0}^{t} 1_{(a-\epsilon,a+\epsilon)}(w(s))ds.
\]  
(21)

Then an application of the previous Dynkin's existence theorem (Theorem 2) with this admissible additive functional $\mathcal{K}_{t}^{[L_{K}]}$ gives us a superprocess, which we denote by $\tilde{X}_{t}^{K} = \tilde{X}_{t}^{K}(dx)$. That is to say,
\[
E_{r,x}^{(L_{K})}e^{-\langle \varphi, \tilde{X}_{t}^{K} \rangle} = \exp\{-\langle v(r), \mu \rangle \}
\]  
(22)

\[
v(r,x) + \Pi_{r,x}^{P} \int_{r}^{t} v(s,w_{K}(s))^{2}d\mathcal{K}_{t}^{[L_{K}]} = \Pi_{r,x}^{P} \varphi(w_{K}(t)).
\]  
(23)

Next we shall construct a new approximate sequence of branching measure-valued processes by employing the above-mentioned process, and shall give its characterization. Before constructing the superprocess in question, we consider first the initial measure as its initial value. We choose a finite measure $\mu \in M_{F}(\mathbb{R})$ as a candidate of the initial measure for our measure-valued process $\tilde{X}_{t}^{K}$. For $n \geq 1$, for each subset $B \subset \mathbb{R}$ we define
\[
\tilde{X}_{0}^{K}(\{n\} \times B) := \mu(B \cap \{(n-1,n) \cup (-n,-n+1)).
\]  
(24)

Then, if it is the case of the number $M \in \mathbb{N}$ satisfying $M > K$, the law of the process $\tilde{X}_{t}^{M}$ restricted to a set $E_{K} = \bigcup_{n=1}^{K} \{n\} \times (-n,n)$ is equivalent to the law of the process $\tilde{X}_{t}^{K}$. In other words,
\[
\mathcal{L}(\tilde{X}_{t}^{M} \mid E_{K}) = \mathcal{L}(\tilde{X}_{t}^{K}), \quad \text{for } \forall M > K.
\]  

Let us now denote by $P_{X_{0}}^{L,K}$ the probability law of the measure-valued process $\tilde{X}^{K}$, and we put $E_{\infty} := \bigcup_{n=1}^{\infty} \{n\} \times (-n,n)$ and $\tilde{X}^{\infty}$ denotes an $M(E_{\infty})$-valued process. Then note that since the law $\{P_{X_{0}}^{L,K}\}_{K}$ of $\tilde{X}^{K}$ becomes a consistent family, its projective limit induces the law of $M(E_{\infty})$-valued process $\tilde{X}^{\infty}$. Hence, if we define a new $M_{F}((-K,K))$-valued process $X_{t}^{K}$ as
\[
X_{t}^{K}(B) := \sum_{n=1}^{K} \tilde{X}_{t}^{\infty}(\{n\} \times B),
\]  
(25)
then an increasing sequence of stochastic processes \( \{X^K_t(B)\}_{K} \nearrow \) is obtained.

**Proposition 3.** (Characterization) Let \( u_K(t, x) \) be a log-Laplace function of \( X^K_t \). Then \( X^K_t \) satisfies the following

\[
E_{X^K_0}[e^{-\langle \varphi, X^K_t \rangle}] = e^{-\langle u_K(t), \mu \rangle}, \quad \text{with } X^K_0(dx) = \mu(dx).
\]  

(26)

Moreover, the function \( u_K(t, x) \) satisfies uniquely the following integral equation:

\[
u K(t, x) = \int_{-K}^{K} p_K(t, x, y) \varphi(y) dy - \int_{0}^{t} \int_{-K}^{K} p_K(t-s, x, y) u_K^2(s, y) L(dy) ds,
\]

(27)

\[
E[X^K_t(B)] = \int_{-K}^{K} \int_{B} p_K(t, x, y) \mu(dx) dy,
\]

(28)

where \( p_K(t, x, y) \) is the fundamental solution of the Dirichlet boundary value problem:

\[
\partial_t u - Pu = 0, \quad u|_{\partial(-K,K)} = 0
\]

(29)

§5. Existence of superprocess in terms of finite measure

Therefore \( M_F(\mathbb{R}) \)-valued process \( X = \{X_t, t \geq 0\} \) with the initial measure \( \mu \in M_F(\mathbb{R}) \) can be defined by the following limit

\[
X_t(dx) := \lim_{K \to \infty} X^K_t(dx).
\]

(30)

We call this stochastic process \( X_t \) a superprocess in terms of random measure \( L \) which represents a random media. Next we shall extend \( p_K(t, \cdot, \cdot) \) onto \( \mathbb{R} \times \mathbb{R} \). Namely,

\[
p_K(t, x, y) = 0 \quad \text{if } x \text{ or } y \notin (-K, K).
\]

Then, since \( p_K(t, \cdot, \cdot) \nearrow p(t, \cdot, \cdot) \), we may apply the monotone convergence theorem to obtain

\[
E[X_t(B)] = \int_{-\infty}^{\infty} \int_{B} p(t, x, y) \mu(dx) dy, \quad \forall B \in B(\mathbb{R}).
\]

(31)

On the other hand, since we have \( \{X^K_t(\cdot)\}_{K} \nearrow \) in \( K \), the sequence of log-Laplace functions \( \{u_K(t, \cdot)\}_{K} \) associated with the sequence of those measure-valued processes is also increasing \( \nearrow \) in \( K \). As a consequence, by using the monotone convergence theorem again, the log-Laplace function \( u(t, x) \) of the above-mentioned limit process \( X_t(dx) \) can also be obtained by

\[
u(t, x) = \lim_{K \to \infty} u_K(t, x).
\]

(32)
Finally, an application of the monotone convergence theorem again leads to the following:

\[
u(t, x) = \lim_{K \to \infty} u_K(t, x) = \lim_{K \to \infty} \int_{-K}^{K} p_K(t, x, y) \varphi(y) dy - \lim_{K \to \infty} \int_{0}^{t} \int_{-K}^{K} p_K(t-s, x, y) u_K^2(s, y) L(dy) ds
= \int_{-\infty}^{\infty} p(t, x, y) \varphi(y) dy - \int_{0}^{t} \int_{-\infty}^{\infty} p(t-s, x, y) u^2(s, y) L(dy) ds.
\] (33)

**Remark.** It is interesting to note that the above construction requires us only local finiteness of the random measure \( L(dx) \).

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**References.**


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