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Finiteness of certain products of algebraic groups over a finite field

By

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Abstract

Let $G_1, \dots, G_n$ be smooth connected and commutative algebraic groups over a finite field F. We show the finiteness of the tensor product $(G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(\operatorname{Spec} F)$ of $G_1, \dots, G_n$ in the category of Mackey functors and also Ivorra-Rülling's K-group $T(G_1, \dots, G_n)(\operatorname{Spec} F)$ associated with those algebraic groups as reciprocity functors. We apply this to prove that, for a product of open curves, the finiteness of the relative Chow group and an abelian fundamental group which classifies abelian coverings with bounded ramification along the boundary.

§ 1. Introduction

A Mackey functor over a perfect field F (a finite Mackey functor in the sense of [4], see Def. 2.1 for the definition) is a co- and contravariant functor from the category of étale schemes over F to the category of abelian groups. A smooth connected and commutative algebraic group G over the field F is regarded as a Mackey functor by the correspondence $x \mapsto G(x)$. Such an algebraic group G can be extended to a Nisnevich sheaf with transfers on the category of regular schemes over F with dimension ≤ 1. Furthermore, it satisfies the following condition which is the so-called reciprocity law (Rosenlicht's theorem when the base field F is algebraically closed [12], Chap. III, Sect. 3, Thm. 1; [3], Prop. 2.2.2): For any open (=non-proper) regular connected curve C over F and a section $a \in G(C)$ there exists an effective Weil divisor $D = \sum_P n_P P$ on the smooth compactification $\overline{C}$ of C such that its support is the boundary $|D| = \overline{C} \smallsetminus C$

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and

$$\sum_{P\in C} v_P(f)\,\mathrm{Tr}_{P/x_C}\, s_P(a) = 0$$

for any $f \neq 0$ in the function field $F(C)$ of C with $f \equiv 1 \bmod D$, that is, $v_P(f-1) \geq n_P$ for any $P \in |D|$, where v_P is the valuation at P, $s_P : G(C) \to G(P)$ is the pull-back along the natural inclusion $P \hookrightarrow C$ and $\mathrm{Tr}_{P/x_C} : G(P) \to G(x_C)$ is the push-forward along the finite map $P \to x_C := \operatorname{Spec} H^0(\overline{C}, \mathscr{O}_{\overline{C}})$. F. Ivorra and K. Rülling [3] have introduced the notion of a *reciprocity functor* as a Nisnevich sheaf with transfers on the category of regular schemes over F with dimension ≤ 1 satisfying several axioms. One of the axioms is the reciprocity law as above. They have also introduced a "product" $T(\mathscr{M}_1,\dots,\mathscr{M}_n)$ associated with reciprocity functors $\mathscr{M}_1,\dots,\mathscr{M}_n$ in the category of reciprocity functors (for the precise definition of the "product", see [3] Def. 4.2.3). By the very construction of the product T, as a Mackey functor, $T(\mathscr{M}_1,\dots,\mathscr{M}_n)$ is a quotient of the tensor product $\mathscr{M}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{M}_n$ (for the definition, see (2.1) in the next section). Hence we have a canonical surjection

$$(\mathscr{M}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{M}_n)(\operatorname{Spec} F) \twoheadrightarrow T(\mathscr{M}_1,\dots,\mathscr{M}_n)(\operatorname{Spec} F).$$

Although the tensor product $\overset{M}{\otimes}$ gives a structure of a symmetric monoidal category in the abelian category of Mackey functors, it is not known that whether this product T satisfies the associativity and then gives a monoidal structure or not. However, this product coincides with the K-group of homotopy invariant Nisnevich sheaves with transfers on the category of smooth schemes over F ([6]). In particular, we obtain an isomorphism

$$T(G_1,\dots,G_n)(\operatorname{Spec} F) \simeq K(F; G_1,\dots,G_n),$$

for semi-abelian varieties $G_1,\dots,G_n$ over F, where $K(F;G_1,\dots,G_n)$ is Somekawa's K-group [13] which was limited on considering only semi-abelian varieties. For semi-abelian varieties $G_1,\dots,G_n$ over a *finite field* F, B. Kahn showed in [5] that

$$K(F; G_1,\dots,G_n) = (G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(\operatorname{Spec} F) = 0 \tag{1.1}$$

if $n > 1$. Because of the isomorphism ([13], Thm. 1.4)

$$K(F; \overbrace{\mathbb{G}_m,\dots,\mathbb{G}_m}^{n}) \xrightarrow{\simeq} K_n^M(F),$$

where $K_n^M(F)$ is the Milnor K-group of the field F, these results generalize the classical fact that $K_n^M(F) = 0$ if F is a finite field and $n > 1$. For algebraic groups G_1, G_2 which may contain unipotent part, the product $(G_1 \overset{M}{\otimes} G_2)(\operatorname{Spec} F)$ may not be trivial. In this note, we shall show the following theorem.

Theorem 1.1 (Thm. 2.3). *Let $G_1, \ldots, G_n$ be smooth commutative and connected algebraic groups over a finite field F. Then*

$$T(G_1, \ldots, G_n)(\operatorname{Spec} F), \quad \textit{and} \quad (G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(\operatorname{Spec} F)$$

are finite.

As an application of (1.1), the class field theory of a product of projective smooth curves over a finite field, a special case of the higher dimensional class field theory of S. Bloch, K. Kato and S. Saito (e.g., [7]) is deduced from the classical (unramified) class field theory (= class field theory of curves over a finite field) and Lang's theorem: the reciprocity map on a normal variety over a finite field has dense image. In Section 3, we will pursue related results on the (ramified) class field theory of a product of open (=non-proper) curves as a byproduct of the above theorem. In particular, we obtain a finiteness of the relative Chow group $\mathrm{CH}_0(X, D)$ for a product of smooth curves $X = X_1 \times \cdots \times X_n$ over a finite field and an effective Weil divisor D on the product $\overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_n$ of the smooth compactification $\overline{X}_i$ of X_i with support $|D| \subset \overline{X} \smallsetminus X$ (Thm. 3.1).

Throughout this note, we mean by an *algebraic group* a smooth connected and commutative group scheme over a field.

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§ 2. Mackey functors and reciprocity functors

Throughout this section, F is a perfect field.

Definition 2.1. A Mackey functor M over F is a contravariant functor from the category of étale schemes over F to the category of abelian groups equipped with a covariant structure for finite morphisms such that $M(x_1 \sqcup x_2) = M(x_1) \oplus M(x_2)$ and if

$$\begin{array}{ccc} x' & \xrightarrow{g'} & x \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ y' & \xrightarrow{g} & y \end{array}$$

is a Cartesian diagram with g and g' are finite, then the induced diagram

$$\begin{array}{ccc} M(x') & \xrightarrow{g'_*} & M(x) \\ \uparrow{\scriptstyle f'^*} & & \uparrow{\scriptstyle f^*} \\ M(y') & \xrightarrow{g_*} & M(y) \end{array}$$

commutes.

We call a morphism $x \to \operatorname{Spec} F$ a *finite point* if $x = \operatorname{Spec} E$ for some finite field extension of F. For Mackey functors $M_1, \dots, M_n$, the product $M_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} M_n$ called the *Mackey product* is defined as follows. For any finite point $x \to \operatorname{Spec} F$,

$$(2.1) \qquad (M_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} M_n)(x) := \left(\bigoplus_{y \to x\text{: finite}} M_1(y) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} M_n(y) \right) \Big/ R(x)$$

where $y \to x$ runs through all finite points over x, and $R(x)$ is the subgroup generated by elements of the following form: For any morphism $j : y' \to y$ of finite points over x, and if $a'_{i_0} \in M_{i_0}(y')$ and $a_i \in M_i(y)$ for $i \neq i_0$, then

$$(2.2) \qquad j^*(a_1) \otimes \cdots \otimes a'_{i_0} \otimes \cdots \otimes j^*(a_n) - a_1 \otimes \cdots \otimes j_*(a'_{i_0}) \otimes \cdots \otimes a_n \in R(x),$$

where j^* and j_* are the pull-back and the push-forward along j respectively. We write $\{a_1, \dots, a_n\}_{y/x}$ for the image of $a_1 \otimes \cdots \otimes a_n \in M_1(y) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} M_n(y)$ in the product $(M_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} M_n)(x)$. Using this symbol, the above relation (2.2) defining $R(x)$ above gives the following equation which is often called the *projection formula*:

$$(2.3) \qquad \{j^*(a_1), \dots, a'_{i_0}, \dots, j^*(a_n)\}_{y'/x} = \{a_1, \dots, j_*(a'_{i_0}), \dots, a_n\}_{y/x}.$$

The Mackey product (2.1) satisfies the following properties:

(M1) The Mackey product $\overset{M}{\otimes}$ gives a tensor product in the abelian category of the Mackey functors. Its unit is the constant Mackey functor $\mathbb{Z}$. In particular, the product commutes with the direct sum $\oplus$ and satisfies the associativity: $M_1 \overset{M}{\otimes} M_2 \overset{M}{\otimes} M_3 \xrightarrow{\simeq} (M_1 \overset{M}{\otimes} M_2) \overset{M}{\otimes} M_3; \{a_1, a_2, a_3\}_{y/x} \mapsto \{\{a_1, a_2\}_{y/y}, a_3\}_{y/x}$.

(M2) The product $- \overset{M}{\otimes} M$ is right exact for any Mackey functor M.

(M3) For any finite point $j : x' \to x$, the push-forward $j_* : (M_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} M_n)(x') \to (M_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} M_n)(x)$ along j is given by $j_*(\{a_1, \dots, a_n\}_{y'/x'}) = \{a_1, \dots, a_n\}_{y'/x}$ on symbols.

Lemma 2.2. *Let G be a unipotent (smooth and commutative) algebraic group over F and A a semi-abelian variety over F. If F is a perfect field of characteristic $p > 0$, we have $G \overset{M}{\otimes} A = 0$.*

Proof. The unipotent group G has a composition series:

$$0 = G^r \subset \cdots \subset G^1 \subset G,$$

each G^i/G^{i+1} being isomorphic to $\mathbb{G}_a$. By the right exactness (M2), it is enough to show $(\mathbb{G}_a \overset{M}{\otimes} A) = 0$. By (M3) above, the assertion is reduced to showing $\{a, b\}_{x/x} = 0$ for any $a \in \mathbb{G}_a(x), b \in A(x)$. There exists a finite point $j : x' \to x$ such that $j^*(b) = pb'$ for some $b' \in A(x')$. Since the trace map (= the push-forward map on $\mathbb{G}_a$) $j_* = \mathrm{Tr}_{x'/x} : \mathbb{G}_a(x') \to \mathbb{G}_a(x)$ is surjective, we obtain

$$\begin{aligned}\{a, b\}_{x/x} &= \{\mathrm{Tr}_{x'/x}(a'), b\}_{x/x} \quad \text{for some } a' \in \mathbb{G}_a(x')\\ &= \{a', j^*b\}_{x'/x} \quad \text{by the projection formula (2.3)}\\ &= \{a', pb'\}_{x'/x}\\ &= 0.\end{aligned}$$

The assertion follows from this. □

A *point* x is a morphism $x = \operatorname{Spec} k \to \operatorname{Spec} F$ where k is a finitely generated field extension over F. Let $\mathrm{Reg}^{\leq 1}$ be the category of regular schemes over F which are separated and of finite type over some point and Ab is the category of abelian groups. A *reciprocity functor* is a Nisnevich sheaf with transfers $\mathscr{M} : (\mathrm{Reg}^{\leq 1}\mathrm{Cor})^{\mathrm{op}} \to \mathrm{Ab}$ satisfying several axioms ([3], Def. 1.5.1), where $\mathrm{Reg}^{\leq 1}\mathrm{Cor}$ is the category with objects the objects of $\mathrm{Reg}^{\leq 1}$ and morphisms are given by correspondences. In particular, $\mathscr{M}$ has the transfer $\mathrm{Tr}_{X/Y} := f_* : \mathscr{M}(X) \to \mathscr{M}(Y)$ for a finite flat morphism $f : X \to Y$. As an example, an algebraic group G over F is a reciprocity functor which is given by $X \mapsto G(X)$. For a finite flat map $X \to Y$, The transfer map $\mathrm{Tr}_{X/Y} : G(X) \to G(Y)$ equals the usual trace if $G = \mathbb{G}_a$ and the norm map if $G = \mathbb{G}_m$. Note that any reciprocity functor gives a Mackey functor by restricting to the category of finite points over F. The "product" $T(\mathscr{M}_1, \ldots, \mathscr{M}_n)$ for reciprocity functors $\mathscr{M}_1, \ldots, \mathscr{M}_n$ is a reciprocity functor and satisfies some functorial properties. In particular, there are functorial isomorphisms

$$T(\mathscr{M}_1, \ldots, \mathscr{M}_i, \ldots, \mathscr{M}_j, \ldots, \mathscr{M}_n) \simeq T(\mathscr{M}_1, \ldots, \mathscr{M}_j, \ldots, \mathscr{M}_i, \ldots, \mathscr{M}_n)$$

and

$$T(\mathscr{M}_1, \ldots, \mathscr{M}_i \oplus \mathscr{M}_i', \ldots, \mathscr{M}_n) \simeq T(\mathscr{M}_1, \ldots, \mathscr{M}_i, \ldots, \mathscr{M}_n) \oplus T(\mathscr{M}_1, \ldots, \mathscr{M}_i', \ldots, \mathscr{M}_n).$$

There is a surjective map as Nisnevich sheaves $T(\mathscr{M}_1, \mathscr{M}_2, \mathscr{M}_3) \twoheadrightarrow T(T(\mathscr{M}_1, \mathscr{M}_2), \mathscr{M}_3)$. However, it is not known whether this map becomes an isomorphism. By the very construction, for any finite point x over F, the product $T(\mathscr{M}_1, \dots, \mathscr{M}_n)(x)$ evaluated at x is a quotient of the Mackey product $(\mathscr{M}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{M}_n)(x)$. More precisely, the product $T(\mathscr{M}_1, \dots, \mathscr{M}_n)$ is defined to be the Nisnevich sheafification $\mathscr{L}^{\infty}_{\mathrm{Nis}}$ of a quotient $\mathscr{L}^{\infty}$ of the product $\mathscr{M}_1 \otimes \cdots \otimes \mathscr{M}_n$ whose underlying Mackey functor is the Mackey product $\mathscr{M}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{M}_n$. However, an isomorphism $\mathscr{L}^{\infty}(x) \xrightarrow{\simeq} \mathscr{L}^{\infty}_{\mathrm{Nis}}(x)$ exists since any Nisnevich covering of $\operatorname{Spec} F$ refines a trivial covering.

Theorem 2.3. *Let $G_1, \dots, G_n$ be algebraic groups over a finite field F. Then the group $(G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(x)$ is finite for any finite point x over F. In particular, the product $T(G_1, \dots, G_n)(x)$ is also finite.*

Proof. The case of $n = 1$, there is nothing to show. So we assume $n > 1$. First we show that there is a surjection $(U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_n)(x) \twoheadrightarrow (G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(x)$ by induction on n. We consider the case $n = 2$. The algebraic group G_i has a decomposition

$$0 \to U_i \to G_i \to A_i \to 0$$

with unipotent part U_i and a semi-abelian variety A_i over F. From the right exactness (M2), we obtain the following exact sequences

$$\begin{array}{ccccccc}
U_1 \overset{M}{\otimes} U_2 & & & & A_1 \overset{M}{\otimes} U_2 & & \\
\downarrow & & & & \downarrow & & \\
U_1 \overset{M}{\otimes} G_2 & \longrightarrow & G_1 \overset{M}{\otimes} G_2 & \longrightarrow & A_1 \overset{M}{\otimes} G_2 & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & \\
U_1 \overset{M}{\otimes} A_2 & & & & A_1 \overset{M}{\otimes} A_2 & & \\
\downarrow & & & & \downarrow & & \\
0 & & & & 0 & & .
\end{array} \tag{2.4}$$

By Kahn's theorem (1.1), we have $A_1 \overset{M}{\otimes} A_2 = 0$. Lemma 2.2 implies $U_1 \overset{M}{\otimes} A_2 = A_1 \overset{M}{\otimes} U_2 = 0$. Hence we obtain $U_1 \overset{M}{\otimes} U_2 \twoheadrightarrow G_1 \overset{M}{\otimes} G_2$. For $n > 2$, consider a decomposition

$$0 \to U_n \to G_n \to A_n \to 0$$

as above. The right exactness (M2) again gives an exact sequence,

$$U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} U_n \to U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} G_n \to U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} A_n \to 0.$$

Lemma 2.2 induces $U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} U_n \twoheadrightarrow U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \overset{M}{\otimes} G_n$. By induction hypothesis, we have $U_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} U_{n-1} \twoheadrightarrow G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_{n-1}$ and the claim follows from this. Thus it is enough to show the assertion for unipotent groups $G_i = U_i$. Take a composition series of the unipotent group $G := G_i$:

$$0 = G^r \subset \cdots \subset G^1 \subset G,$$

each $G^i/G^{i+1} \simeq \mathbb{G}_a$. By the right exactness (M2), we may assume $G_i = \mathbb{G}_a$ for all i and $x = \operatorname{Spec} F$ without loss of generality. We show the finiteness of $(\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)$ by induction on n. We assume that $(\mathbb{G}_a)^{\overset{M}{\otimes}(n-1)}(x)$ is finite. The group $(\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)$ has a structure of an F-vector space given by $a\{a_1, \dots, a_n\}_{x'/x} := \{j^*(a)a_1, \dots, a_n\}_{x'/x}$, for any $a \in F$ and a symbol $\{a_1, \dots, a_n\}_{x'/x}$ on a finite point $j : x' \to x = \operatorname{Spec} F$. Consider a subspace $I(x)$ of $(\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)$ generated by the elements of the form

$$\{1_{x'}, a_2, \dots, a_n\}_{x'/x} - \{a_2 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}_{x'/x}$$

for any finite point $x' \to x$, where $1_{x'} \in \mathbb{G}_a(x')$ is the unit. By identifying the canonical isomorphism $(\mathbb{G}_a)^{\overset{M}{\otimes} n} \simeq \mathbb{G}_a \overset{M}{\otimes} (\mathbb{G}_a)^{\overset{M}{\otimes}(n-1)}$ by (M1) and $j^*(1_x) = 1_{x'}$, we have

$$\begin{aligned} \{1_{x'}, a_2, \dots, a_n\}_{x'/x} &= \{j^*(1_x), \{a_2, \dots, a_n\}_{x'/x'}\}_{x'/x} \\ &= \{1_x, j_*\{a_2, \dots, a_n\}_{x'/x'}\}_{x/x} \quad \text{by the projection formula} \\ &= \{1_x, \{a_2, \dots, a_n\}_{x'/x}\}_{x/x} \quad \text{by (M3).} \end{aligned}$$

By the induction hypothesis, the set of elements of this form is finite. On the other hand, the projection formula implies

$$\begin{aligned} \{a_2 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}_{x'/x} &= \{a_2 \cdots a_n, j^*(1_x), \dots, j^*(1_x)\}_{x'/x} \\ &= \{j_*(a_2 \cdots a_n), 1_x, \dots, 1_x\}_{x/x}. \end{aligned}$$

Thus the subspace $I(x)$ is finite. Define $Q(x) := (\mathbb{G}_a)^{\overset{M}{\otimes} n}(x)/I(x)$ the quotient space. We denote by $\overline{\{a_1, \dots, a_n\}}_{x'/x}$ the image of $\{a_1, \dots, a_n\}_{x'/x}$ in $Q(x)$. Now we consider a subspace $S(x)$ of $Q(x)$ generated by symbols of the form $\overline{\{a_1, \dots, a_n\}}_{x/x}$. It is easy to see that the subspace $S(x)$ is finite. We show that any symbol $\overline{\{a_1, \dots, a_n\}}_{x'/x}$ in $Q(x)$ for a finite point $j : x' \to x$ is in $S(x)$. In fact,

$$\begin{aligned} \overline{\{a_1, \dots, a_n\}}_{x'/x} &= j_*(\overline{\{a_1, \dots, a_n\}}_{x'/x'}) \quad \text{by (M3)} \\ &= j_*(a_1\overline{\{1_{x'}, a_2, \dots, a_n\}}_{x'/x'}) \quad \text{because of } \overline{\{a_1, \dots, a_n\}}_{x'/x'} \in Q(x') \\ &= j_*(a_1\overline{\{a_2 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}}_{x'/x'}) \\ &= \overline{\{a_1 \cdots a_n, 1_{x'}, \dots, 1_{x'}\}}_{x'/x} \quad \text{by (M3)} \\ &= \overline{\{j_*(a_1 \cdots a_n), 1_x, \dots, 1_x\}}_{x/x} \quad \text{by the projection formula.} \end{aligned}$$

Thus we obtain $Q(x) = S(x)$ and the assertion follows from it. □

§ 3. Applications

Let X be a smooth (and connected) variety over a finite field F. Assume that there is a smooth compactification $\overline{X}$ of X, that is, a projective smooth variety which contains X as an open subscheme. Let D be an effective Weil divisor on $\overline{X}$ with support $|D| \subset \overline{X} \smallsetminus X$. We define the relative Chow group of the pair (X, D) by

$$\mathrm{CH}_0(X, D) := \mathrm{Coker}\left(\mathrm{div} : \bigoplus_{\phi: C \to X} P_C(\overline{\phi}^* D) \to Z_0(X)\right),$$

where the direct sum runs over the normalization $\phi : C \to X$ of a curve in X, $Z_0(X)$ is the group of 0-cycles on X, $\overline{\phi} : \overline{C} \to \overline{X}$ is the extension of the map ϕ to the smooth compactification $\overline{C}$ of C, the map div is given by the divisor map on each curve C and

$$P_C(\overline{\phi}^* D) := \{f \in F(C)^\times \mid f \equiv 1 \bmod \overline{\phi}^* D + (\overline{C} \smallsetminus C)_{\mathrm{red}}\}$$

(*cf.* [1], Set. 8.1, see also [9], Sect. 3.4 and 3.5). Putting $X_y := X \times_{\operatorname{Spec} F} y$ and denoting by D_y the pull-back of D to $\overline{X}_y := \overline{X} \times_x y$ for any finite point $y \to x = \operatorname{Spec} F$, the assignment

$$\mathscr{C}\!H_0(X, D) : y \mapsto \mathrm{CH}_0(X_y, D_y)$$

gives a Mackey functor $\mathscr{C}\!H_0(X, D)$.

Theorem 3.1. *Let $X_1, \dots, X_n$ be smooth and geometrically connected curves over a finite field F and put $X := X_1 \times \cdots \times X_n$. For an effective Weil divisor D on $\overline{X} := \overline{X}_1 \times \cdots \times \overline{X}_n$ with support $|D| \subset \overline{X} \smallsetminus X$, the kernel of the degree map $\mathrm{CH}_0(X, D)^0 := \mathrm{Ker}(\deg : \mathrm{CH}_0(X, D) \to \mathbb{Z})$ is finite.*

Recently, P. Deligne has showed the finiteness of the group $\mathrm{CH}_0(X, D)^0$ for a smooth variety X over a finite field and an effective Cartier divisor D with support in the boundary $\overline{X} \smallsetminus X$ only assuming the existence of some normal compactification $\overline{X}$ of X as an application of his finiteness theorem for l-adic Galois representations of function fields ([1], Thm. 8.1).

Proof of Thm. 3.1. Let $p_i : \overline{X} = \overline{X}_1 \times \cdots \times \overline{X}_n \to \overline{X}_i$ be the projection. An irreducible component Z of the boundary

$$\overline{X} \smallsetminus X = \bigcup_{i=1}^{n} \overline{X}_1 \times \cdots \times (\overline{X}_i \smallsetminus X_i) \times \cdots \times \overline{X}_n.$$

has the form $Z = p_i^{-1}(P)$ for some $P \in \overline{X}_i \smallsetminus X_i$. Therefore, for sufficiently large divisors D_i on $\overline{X}_i$, there is a surjection $\mathrm{CH}_0(X, p_1^*D_1 + \cdots + p_n^*D_n) \twoheadrightarrow \mathrm{CH}_0(X, D)$. Thus we may assume that D is a divisor of the form $p_1^*D_1 + \cdots + p_n^*D_n$. Now we consider the map

$$\psi : (\mathscr{C}H_0(X_1, D_1) \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathscr{C}H_0(X_n, D_n))(\operatorname{Spec} F) \to \mathrm{CH}_0(X, D) \tag{3.1}$$

defined by

$$\psi(\{[P_1], \ldots, [P_n]\}_{y/x}) := (j_y)_*[P_1 \times_y \cdots \times_y P_n]$$

for a finite point $j_y : y \to x := \operatorname{Spec} F$ and the classes $[P_i]$ represented by closed points P_i of $(X_i)_y := X_i \times_x y$, where $[P_1 \times_y \cdots \times_y P_n]$ is the zero-cycle on X_y determined by P_i's and the base change $X_y \to X$ to y is also denoted by j_y.

Well-definedness of ψ: First we have to prove that ψ annihilates the element of the form

$$\{j^*[P_1], \ldots, [P'_{i_0}], \ldots, j^*[P_n]\}_{y'/x} - \{[P_1], \ldots, j_*[P'_{i_0}], \ldots, [P_n]\}_{y/x}$$

for a map $j : y' \to y$ of finite points and a closed point P'_{i_0} of $(X_{i_0})_{y'}$ and closed points P_i of $(X_i)_y$ $(i \neq i_0)$.

$$\begin{aligned}
\psi(\{j^*[P_1], \ldots, [P'_{i_0}], \ldots, j^*[P_n]\}_{y'/x}) &= (j_{y'})_*(j^*[P_1] \times_{y'} \cdots \times_{y'} [P'_{i_0}] \times_{y'} \cdots \times_{y'} j^*[P_n]) \\
&= (j_y)_* \circ (j_*)([P_1 \times_y \cdots \times_y P'_{i_0} \times_y \cdots \times_y P_n]) \\
&= (j_y)_*([P_1] \times_y \cdots \times_y j_*[P'_{i_0}] \times_y \cdots \times_y [P_n]) \\
&= \psi(\{[P_1], \ldots, j_*[P'_{i_0}], \ldots, [P_n]\}_{y/x}).
\end{aligned}$$

Next we show that ψ annihilates the element $\{[P_1], \ldots, \overset{\overset{i}{\vee}}{\operatorname{div}(f)}, \ldots, [P_n]\}_{y/x}$ for a finite point $j_y : y \to x$ and $f \in P_{(X_i)_y}((D_i)_y)$. Because of

$$\{[P_1], \ldots, \operatorname{div}(f), \ldots, [P_n]\}_{y/x} = (j_y)_*(\{[P_1], \ldots, \operatorname{div}(f), \ldots, [P_n]\}_{y/y}),$$

we may assume $y = x$ by the definition of ψ. Consider the product

$$P_1 \times_x \cdots \times_x P_{i-1} \times_x X_i \times_x P_{i+1} \times_X \cdots \times_x P_n = \bigcup_Q (X_i)_Q,$$

where the union runs over a point Q in $P_1 \times_x \cdots \times_x P_{i-1} \times_x P_{i+1} \times_x \cdots \times_x P_n$. For such a point Q, we denote by ϕ_Q the natural map $(X_i)_Q \to X$. Since the base change $j_Q : (X_i)_Q \to X_i$ is unramified, $j_Q^*(f) \in P_{(X_i)_Q}(\phi_Q^*(D))$. We obtain

$$\begin{aligned}
\psi(\{[P_1], \ldots, \overset{\overset{i}{\vee}}{\operatorname{div}(f)}, \ldots, [P_n]\}_{x/x}) &= \sum_Q j_Q^* \operatorname{div}(f) \\
&= 0.
\end{aligned}$$

Surjectivity of ψ: We show that the map ψ is surjective. Take a cycle $[P]$ as a generator of $\mathrm{CH}_0(X,D)$ which is represented by a closed point P on X and is a finite point $j_P : P \to x = \operatorname{Spec} F$. By the definition of ψ, the push-forward map on the relative Chow group and the norm map on the Mackey product are compatible as in the following commutative diagram:

$$\begin{array}{ccc} (\mathscr{C}H_0(X_1,D_1)\overset{M}{\otimes}\cdots\overset{M}{\otimes}\mathscr{C}H_0(X_n,D_n))(P) & \xrightarrow{\psi_P} & \mathrm{CH}_0(X_P,D_P) \\ \downarrow{\scriptstyle (j_P)_*} & & \downarrow{\scriptstyle (j_P)_*} \\ (\mathscr{C}H_0(X_1,D_1)\overset{M}{\otimes}\cdots\overset{M}{\otimes}\mathscr{C}H_0(X_n,D_n))(x) & \xrightarrow{\psi} & \mathrm{CH}_0(X,D). \end{array}$$

Thus to show the surjectivity of ψ we may assume that P is an F-rational point. The point P is determined by maps $P \to X_i$. These maps give closed points P_i in X_i and $\psi(\{[P_1],\dots,[P_n]\}_{x/x}) = [P_1 \times_x \cdots \times_x P_n] = [P]$. Therefore ψ is surjective.

Finiteness of $\mathrm{CH}_0(X,D)$: By a theorem of F. K. Schmidt ([10], Sect. 8), there exists a degree 1 cycle in X. Hence, for each i, we have a decomposition $\mathscr{C}H_0(X_i,D_i) \simeq \mathbb{Z} \oplus J_{X_i,D_i}$ by the generalized Jacobian variety J_{X_i,D_i} of the pair (X_i,D_i) ([12]). According to this decomposition, we obtain

$$\mathscr{C}H_0(X_1,D_1)\overset{M}{\otimes}\cdots\overset{M}{\otimes}\mathscr{C}H_0(X_n,D_n) \simeq \mathbb{Z} \oplus \bigoplus_{r=1}^{n} \bigoplus_{1\le i_1<\cdots<i_r\le n} (J_{X_{i_1},D_{i_1}}\overset{M}{\otimes}\cdots\overset{M}{\otimes} J_{X_{i_r},D_{i_r}})$$

by (M1). The surjection ψ (3.1) induces a surjection

$$\bigoplus_{r=1}^{n} \bigoplus_{1\le i_1<\cdots<i_r\le n} (J_{X_{i_1},D_{i_1}}\overset{M}{\otimes}\cdots\overset{M}{\otimes} J_{X_{i_r},D_{i_r}})(\operatorname{Spec} F) \twoheadrightarrow \mathrm{CH}_0(X,D)^0.$$

The left is finite by Theorem 2.3 and so is $\mathrm{CH}_0(X,D)^0$. □

Let X be the product of curves over F, and D as in the above theorem (Thm. 3.1). For each normalization $\phi : C \to X$ of a curve in X, we have a divisor $D_C := \overline{\phi}^*(D) + (\overline{C} \smallsetminus C)_{\mathrm{red}}$ on $\overline{C}$. The category of étale coverings of X with ramification bounded by the collection of Weil divisors $(D_C)_{\phi:C\to X}$ forms a Galois category and gives a fundamental group $\pi_1(X,D)$ ([2], Lem. 3.3). For each such C and a point $P \in \overline{C} \smallsetminus C$ there is a canonical map $G_{C,P}^{\mathrm{ab}} := \mathrm{Gal}(F(C)_P^{\mathrm{ab}}/F(C)_P) \to \pi_1(X)^{\mathrm{ab}}$, where $F(C)_P^{\mathrm{ab}}$ is the maximal abelian extension of the completion $F(C)_P$ at P. By the very definition of the coverings, we have

$$\operatorname{Coker}\left(\bigoplus_{\phi:C\to X} \bigoplus_{P\in\overline{C}\smallsetminus C} G_{C,P}^{\mathrm{ab},m_P(D_C)} \to \pi_1(X)^{\mathrm{ab}} \right) \twoheadrightarrow \pi_1(X,D)^{\mathrm{ab}},$$

where $G_{C,P}^{\mathrm{ab},m}$ is the m-th (upper numbering) ramification subgroup of $G_{C,P}^{\mathrm{ab}}$ ([11], Chap. IV, Sect. 3) and $m_P(D_C)$ is the multiplication of the divisor D_C at P. Using the idele theoretic description of the relative Chow group

$$\mathrm{CH}_0(X,D) \simeq \mathrm{Coker}\left(\bigoplus_{\phi: C \to X} F(C)^\times \to Z_0(X) \oplus \bigoplus_{\phi: C \to X} \bigoplus_{P \in \overline{C} \smallsetminus C} F(C)_P^\times / U_{C,P}^{m_P(D_C)} \right),$$

where $U_{C,P}^m = 1 + \mathfrak{m}_{C,P}^m$ is the higher unit group, local class field theory (see e.g., [11], Chap. XV) induces a commutative diagram:

$$\begin{array}{ccc} Z_0(X)^0 & \twoheadrightarrow & \mathrm{CH}_0(X,D)^0 \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho_D} \\ \pi_1(X)^{\mathrm{ab},0} & \twoheadrightarrow & \pi_1(X,D)^{\mathrm{ab},0}. \end{array}$$

Here $Z_0(X)^0$ is the kernel of the degree map $\deg : Z_0(X) \to \mathbb{Z}$, the left vertical map ρ is the reciprocity map on X and $\pi_1(X,D)^{\mathrm{ab},0}$ is the geometric part of the abelian fundamental group (= the kernel of the canonical map $\pi_1(X,D)^{\mathrm{ab}} \to \pi_1(\mathrm{Spec}(F))^{\mathrm{ab}}$). The image of the reciprocity map $\rho : Z_0(X) \to \pi_1(X)^{\mathrm{ab}}$ is known to be dense (due to Lang [8]) and the image of ρ_D in the above diagram is finite by Theorem 3.1. Therefore, the map ρ_D is surjective and we obtain the following finiteness result.

Corollary 3.2. *Let X and D be as in Theorem 3.1. Then, the geometric part of the abelian fundamental group $\pi_1(X,D)^{\mathrm{ab},0}$ is finite.*

References

[1] H. Esnault and M. Kerz, *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Math. Vietnam. **37** (2012), no. 4, 531–562.
[2] S. Harada and T. Hiranouchi, *Smallness of fundamental groups for arithmetic schemes*, J. Number Theory **129** (2009), no. 11, 2702–2712.
[3] F. Ivorra and K. Rülling, *K-groups of reciprocity functors*, arXiv:1209.1207.
[4] B. Kahn, *The decomposable part of motivic cohomology and bijectivity of the norm residue homomorphism*, Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989), Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 79–87.
[5] ———, *Nullité de certains groupes attachés aux variétés semi-abéliennes sur un corps fini; application*, C. R. Acad. Sci. Paris Sér. I Math. **314** (1992), no. 13, 1039–1042.
[6] B. Kahn and T. Yamazaki, *Voevodsky's motives and Weil reciprocity*, arXiv:1108.2764.
[7] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), no. 2, 241–275.

[8] S. Lang, *Sur les séries L d'une variété algébrique*, Bull. Soc. Math. France **84** (1956), 385–407.
[9] H. Russell, *Albanese varieties with modulus over a perfect field*, arXiv:0902.2533.
[10] F. K. Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p*, Math. Z. **33** (1931), no. 1, 1–32.
[11] J.-P. Serre, *Corps locaux*, Deuxième édition, Actualités scientifiques et industrielles 1296, Publications de l'Institut Mathématique de l'Université de Nancago, No. VIII, Hermann, Paris, 1968.
[12] ———, *Groupes algébriques et corps de classes*, Reprint of the second edition, Actualités Scientifiques et Industrielles 1264, Publications de l'Institut Mathématique de l'Université de Nancago, No. VII, Hermann, Paris, 1984.
[13] M. Somekawa, *On Milnor K-groups attached to semi-abelian varieties*, K-Theory **4** (1990), no. 2, 105–119.