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On constructions of theta functions on GSp_4 and its mod p nonvanishing

By

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§ 1. Introduction

This article is a survey of [10], which is a joint work with Ming-Lun Hsieh.

In this article, we consider arithmetic properties of certain theta lifts by Yoshida and Harris-Soudry-Taylor. In order to clarify our motivation to study such theta lifts, let us review known results briefly.

In [17] and [18], Yoshida studied a theta correspondence from automorphic forms on the orthogonal group SO_4 to Siegel modular forms of degree 2 and gave an example of nonvanishing Siegel modular forms, which are known as Yoshida lifts. The nonvanishing of Yoshida lifts in general setting are proved in [2] and [3] by representation theoretic method. In this article, we also study another theta correspondence from automorphic forms on the orthogonal groups $\mathrm{O}_{3,1}$ to Siegel modular forms of degree 2, which is first studied by Harris, Soudry and Taylor in [8]. Harris, Soudry and Taylor also proved nonvanishing of such theta lifts, which we call HST lifts, by representation theoretic method.

In this article, we show the nonvanishing of explicit Yoshida and HST lifts. To state our result precisely, we introduce some notations.

We begin with our result on Yoshida lifts. Let N^- be a square-free product of an odd number of primes. Let (N_1^+, N_2^+) be a pair of positive integers prime to N^-. We put $(N_1, N_2) = (N^-N_1^+, N^-N_2^+)$. For $i = 1, 2$, let f_i be an elliptic newform of level $\Gamma_0(N_i)$ and weight $2k_i + 2$, where k_i is a nonnegative integer such that $k_1 \geq k_2$. Let D be a definite quaternion algebra of absolute discriminant N^-. For each $i \in \{1, 2\}$, let $\mathcal{W}_{k_i}(\mathbf{C}) := \det^{-k_i} \otimes \mathrm{Sym}^{2k_i}(\mathbf{C}^{\oplus 2})$ be the algebraic representation of $\mathrm{GL}_2(\mathbf{C})$ of highest

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weight $(k_i, -k_i)$. By the Jacquet-Langlands-Shimizu correspondence, there exist vector-valued new forms $\mathbf{f}_i : D^\times \backslash D^\times_{\mathbf{A}} \to \mathcal{W}_{k_i}(\mathbf{C})$ on $D^\times_{\mathbf{A}}$ unique up to scalar such that $\mathbf{f}_i$ shares the same Hecke eigenvalues with f_i at all $p \nmid N^-$. We will make an appropriate choice of test function ϕ in the space of Bruhat-Schwartz functions on $D_{\mathbf{A}} \oplus D_{\mathbf{A}}$. Then the datum $(\mathbf{f}_1, \mathbf{f}_2, \phi)$ gives rise to the Yoshida lift $\theta^*_{f_1,f_2}$ associated to f_1 and f_2 via the usual theta lifting construction. This Yoshida lift $\theta^*_{f_1,f_2}$ is a genus two Siegel modular form of weight $(k_1 - k_2 + 2, k_1 + k_2 + 2)$ and level $\Gamma_0(N)$ with $N = \mathrm{lcm}(N_1, N_2)$ and is precisely the one considered in [17] and [3] if N_1 and N_2 are square-free.

Let ℓ be a rational prime and fix a place λ of $\bar{\mathbf{Q}}$ above ℓ. Our first result is a sufficient condition for the nonvanishing of $\theta^*_{f_1,f_2}$ modulo λ. For each prime factor p of $\gcd(N_1, N_2)$, we denote by $\epsilon_p(f_1), \epsilon_p(f_2) \in \{\pm 1\}$ the Atkin-Lehner eigenvalues at p on f_1 and f_2 respectively.

The following is the first main theorem in this article:

Theorem 1.1. *Suppose that*

1. $\ell > 2k_1$ *and* $\ell \nmid N$,
2. *the residual Galois representations* $\bar{\rho}_{f_i,\ell} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\bar{\mathbf{F}}_\ell)$ *attached to* f_i *are absolutely irreducible,*
3. $\epsilon_p(f_1) = \epsilon_p(f_2)$ *for every prime* p *with* $\mathrm{ord}_p(N_1) = \mathrm{ord}_p(N_2) > 0$.

Then the Yoshida lift $\theta^*_{f_1,f_2}$ *has* λ*-integral Fourier expansion, and there are infinitely many Fourier coefficients which are nonzero modulo* λ.

By choosing ℓ and λ so that the conditions 1 and 2 in Theorem 1.1 are satisfied, we obtain the following immediate consequence:

Corollary 1.2. *Suppose that* $\epsilon_p(f_1) = \epsilon_p(f_2)$ *for every prime* p *with* $\mathrm{ord}_p(N_1) = \mathrm{ord}_p(N_2)$. *Then the Yoshida lift* $\theta^*_{f_1,f_2}$ *is nonzero.*

Remark.

1. In [3], by a different method, the nonvanishing of the theta lift as a representation space is proved if N_1 and N_2 are square-free.
2. If $k_2 = 0$, a nonvanishing modulo λ result for the (scalar-valued) Yoshida lifts is discussed in [12], assuming Artin's conjecture on primitive roots. (See [12, Conjecture 6.6] for the Artin's conjecture.)

Next, we introduce our result on HST lifts. Let $\mathfrak{N}$ be an ideal of the ring of integers of an imaginary quadratic field E of the discriminant D. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(E_{\mathbf{A}})$ of conductor $\mathfrak{N}$. Let $f : \mathrm{GL}_2(E_{\mathbf{A}}) \to \mathrm{Sym}^{2k-2}(\mathbf{C}^{\oplus 2})(k \geq 2)$ be a normalized cusp form in π of central character ω_f. Let χ be a Hecke character of $\mathbf{A}^\times$ such that $\omega_f = \chi \circ \mathrm{N}_{E/\mathbf{Q}}$. We denote the set of places of $\mathbf{Q}$ by $\Sigma_{\mathbf{Q}}$. Let $\delta : \Sigma_{\mathbf{Q}} \to \{\pm 1\}$ be a function such that $\delta(v) = 1$ for all but finitely many $v \in \Sigma_{\mathbf{Q}}$, $\delta(v) = 1$ whenever $\pi_v \neq \pi_v^c$ and $\prod_{v \in \Sigma_{\mathbf{Q}}} \delta(v) = 1$ if $\pi = \pi^c$, where c is the complex conjugate. The datum (f, χ, δ) gives rise to an automorphic form on $\mathrm{GO}_{3,1}(\mathbf{A})$. We assume that $\delta(\infty) = -1$. Then, together with some Bruhat-Schwartz function ϕ, we obtain a holomorphic Siegel cusp form $\theta_{f,\chi,\delta}$ of genus two which we call HST lift. This HST lift has weight $(k, 2)$ and level $\Gamma_0(N_D)$ where $N_D = \mathrm{lcm}(N, D)$ for $N\mathbf{Z} = \mathfrak{N} \cap \mathbf{Z}$.

Let $T \subset \Sigma_{\mathbf{Q}}$ be a subset of places v of $\mathbf{Q}$ such that $\pi_v = \pi_v^c$. We define Σ_{ram} to be the set of places v of $\mathbf{Q}$ such that either v divides $\mathrm{N}_{E/\mathbf{Q}}(\mathfrak{N})$, $v = \infty$ or v is a ramified place of $E/\mathbf{Q}$.

Then, we have the second main theorem in this article as follows:

Theorem 1.3. *Suppose that*

1. $\chi_v(-1) = 1$ *for any* $v \in T \cap \Sigma_{\mathrm{ram}}$,
2. $\delta(\infty) = -1$ *and* $\delta(v) = 1$ *for every finite place of* $\mathbf{Q}$.

Then, for infinitely many quadratic characters η *of* $E_{\mathbf{A}}^\times$, *the HST lift* $\theta_{f\otimes\eta,\chi,\delta}$ *is nonzero.*

Remark. The nonvanishing of HST lifts as a representation space without any quadratic twists is discussed in [16].

The proof of these results is based on an explicit calculation of the Bessel coefficients of theta lifts, which are certain period integrals of Siegel cusp forms of genus two. We see that Bessel coefficients are roughly central values of L-functions of cusp forms we start with. For the case of Yoshida lifts, we deduce Theorem 1.1 from a nonvanishing modulo λ result of central values with anticyclotomic twists in [5]. For the case of HST lifts, we use a nonvanishing result due to [6], following the argument in [8].

Remark. As in the case of Yoshida lifts, we see that the HST lift $\theta_{f,\chi,\delta}$ has λ-integral Fourier expansion after we divide $\theta_{f,\chi,\delta}$ by the λ-optimal period of f. However, in lack of the nonvanishing modulo λ result for the algebraic part of central value $L(\frac{1}{2}, \pi \otimes \phi)$ of L-function of π with anticyclotomic twists ϕ, we do not have a nonvanishing modulo λ result for Fourier coefficients of $\theta_{f,\chi,\delta}$. Nonetheless, by [14], we see that there exist infinitely many twists ϕ such that the algebraic part of $L(\frac{1}{2}, \pi \otimes \phi)$ is nonzero modulo λ if the class number of E is 1. Hence, we hope to improve this result to allow anticycltomic twists in the future.

Indeed, the motivation of this study is to construct elements of Selmer groups by congruences among automorphic forms. The first example is Ribet's proof of the converse of Herbrand's theorem in [15] via congruences between Eisenstein series and cusp forms. We expect that congruences between Hecke eigensystems of Siegel cusp forms which are given in Theorem 1.1, 1.3 and another non-theta lift Siegel cusp forms provide evidences of Bloch-Kato conjecture for the Rankin-Selberg L-functions of degree 4. This is a natural generalization of [9] in which Hida gives anticyclotomic analogue of Ribet's theorem by studying congruences between CM forms and cusp forms. For instance, see [1] and [4] for more details in this direction.

This article is organized as follows. In Section 2, we recall definitions of theta correspondence and Bessel coefficients. In Section 3, we introduce explicit description of Bessel coefficients of Yoshida lifts (Theorem 3.1). Such a formula for HST lifts is introduced in Section 4 (Theorem 4.1). Since our main results Theorem 1.1 and Theorem 1.3 are immediate consequences from Theorem 3.1 and Theorem 4.1 respectively, we only give Bessel coefficients formulas. More detail can be found in our forthcoming paper [10].

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§ 2. Generality

§ 2.1. Notations

We denote the set of homogeneous polynomials of degree n with indeterminate X, Y by $\mathbf{C}[X,Y]_n$. We define a bilinear form $\langle\ ,\ \rangle_n$ on $\mathbf{C}[X,Y]_n \times \mathbf{C}[X,Y]_n$ by

$$\langle X^iY^{n-i}, X^jY^{n-j}\rangle_n = \begin{cases} (-1)^i\binom{n}{i}^{-1} & i+j=n, \\ 0 & i+j\neq n. \end{cases}$$

We regard $\mathrm{Sym}^n(\mathbf{C}^{\oplus 2}) := \mathbf{C}[X,Y]_n$ as a representation space of $\mathrm{GL}_2(\mathbf{C})$ by the right translation.

Let $\lambda = (a, b)$ be a pair of integers such that $a \geq b \geq 0$. We put $\mathcal{L}_\lambda = \mathbf{C}[\mathbf{U}, \mathbf{V}]_{a-b}$. We regard $\mathcal{L}_\lambda$ as a representation space of a representation ρ_λ of $\mathrm{GL}_2(\mathbf{C})$ which is defined by

$$\rho_\lambda(g)P(\mathbf{U}, \mathbf{V}) = (\det g)^a P((\mathbf{U}, \mathbf{V})g),$$

where $g \in \mathrm{GL}_2(\mathbf{C})$ and $P(\mathbf{U}, \mathbf{V}) \in \mathcal{L}_\lambda$.

§ 2.2. Theta correspondence

Let (V, n) be a 4-dimensional quadratic space over the rational number field $\mathbf{Q}$ and we define a bilinear form $(\ ,\ ) : V \times V \to \mathbf{Q}$ by $(x, y) = \mathrm{n}(x+y) - \mathrm{n}(x) - \mathrm{n}(y)$. Denote by $\mathrm{GO}(V)$ the orthogonal similitude group with the similitude morphism $\nu : \mathrm{GO}(V) \to \mathbb{G}_m$. We put $\mathbf{X} = V \oplus V$. For a place v of $\mathbf{Q}$, we denote by $|\cdot|_v$ the normalized absolute value on $\mathbf{Q}_v$. We put $V_v = V \otimes_{\mathbf{Q}} \mathbf{Q}_v$ and $\mathbf{X}_v = \mathbf{X} \otimes_{\mathbf{Q}} \mathbf{Q}_v$. Let $\mathcal{S}(\mathbf{X}_v)$ be the space of $\mathbf{C}$-valued Bruhat-Schwartz functions on $\mathbf{X}_v$. For each $x = (x_1, x_2) \in \mathbf{X}_v = V_v \oplus V_v$, we put

$$S_x = \begin{pmatrix} \mathrm{n}(x_1) & \frac{1}{2}(x_1, x_2) \\ \frac{1}{2}(x_1, x_2) & \mathrm{n}(x_2) \end{pmatrix}.$$

Let $\chi_{V_v} : \mathbf{Q}_v^\times \to \mathbf{C}^\times$ be the quadratic character attached to V_v.

We denote the Weil representation of $\mathrm{Sp}_4(\mathbf{Q}_v)$ by ω_{V_v} . Let $\psi : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}^\times$ be the additive character such that $\psi(x) = e^{2\pi\sqrt{-1}x}$ for $x \in \mathbf{A}_\infty = \mathbf{R}$. We denote by ψ_v the composition of the natural embedding $\mathbf{Q}_v \to \mathbf{A}$ and ψ. We define $\gamma(\frac{1}{2}\psi_v \circ V_v)$ to be the Weil index of V_v. We put $\gamma_{V_v} = \gamma(\frac{1}{2}\psi_v \circ V_v)^2$. The representaion ω_{V_v} has a Schrödinger model $\mathrm{Sp}_4(\mathbf{Q}_v) \to \mathrm{Aut}_{\mathbf{C}}\mathcal{S}(\mathbf{X}_v)$, which is characterized by the following formulas (see [13, Section 5], [11, Section 4.2]):

$$\begin{aligned}
\omega_{V_v}(\begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{-1} \end{pmatrix})\phi(x) =& \chi_{V_v}(\mathrm{det} a)|\mathrm{det} a|_v^2 \phi(xa), \\
\omega_{V_v}(\begin{pmatrix} 1_2 & b \\ & 1_2 \end{pmatrix})\phi(x) =& \psi_v(\mathrm{Tr}(S_x b))\phi(x), \\
\omega_{V_v}(\begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix})\phi(x) =& \gamma_{V_v}\hat{\phi}(x),
\end{aligned}$$

where $\hat{\phi}$ is the Fourier transform of ϕ with respect to the self-dual Haar measure $d\mu$ on $V_v \oplus V_v$:

$$\hat{\phi}(x) = \int_{\mathbf{X}_v} \phi(y)\psi((x, y))d\mu(y).$$

We define

$$R(\mathrm{GO}(V_v)\times \mathrm{GSp}_4(\mathbf{Q}_v)) = \{(h,g)\in \mathrm{GO}(V_v)\times \mathrm{GSp}_4(\mathbf{Q}_v); \nu(h)=\nu(g)\}.$$

The Weil representation $\omega_v : R(\mathrm{GO}(V_v)\times \mathrm{GSp}_4(\mathbf{Q}_v)) \to \mathrm{Aut}_{\mathbf{C}}\mathcal{S}(\mathbf{X}_{\mathrm{v}})$ is given by

$$\omega_v(h,g)\phi(x) = |\nu(h)|_v^{-2}(\omega_{V_v}(g_1)\phi)(h^{-1}x) \quad (g_1 = \begin{pmatrix} 1_2 & \\ & \nu(g)^{-1}1_2 \end{pmatrix} g).$$

Let $\mathcal{S}(\mathbf{X}_{\mathbf{A}}) = \otimes_v \mathcal{S}(\mathbf{X}_v)$. We put $\omega_V := \otimes_v \omega_{V_v} : \mathrm{Sp}_4(\mathbf{A}) \to \mathrm{Aut}_{\mathbf{C}}\mathcal{S}(\mathbf{X}_{\mathbf{A}})$ and $\omega := \otimes_v \omega_v : R(\mathrm{GO}(V)_{\mathbf{A}}\times \mathrm{GSp}_4(\mathbf{A})) \to \mathrm{Aut}_{\mathbf{C}}\mathcal{S}(\mathbf{X}_{\mathbf{A}})$.

We denote by $\mathcal{W}$ (resp. $\mathcal{L}$) an irreducible representaion of $\mathrm{O}(V_\infty)$ (resp. U_2) over $\mathbf{C}$. Let $\langle\ ,\ \rangle : \mathcal{W}\times\mathcal{W}\to\mathbf{C}$ be an $\mathrm{O}(V_\infty)$-equivariant pairing. For each vector-valued Bruhat-Schwartz function $\varphi \in \mathcal{S}(\mathbf{X}_{\mathbf{A}})\otimes\mathcal{W}\otimes\mathcal{L}$, define the theta kernel $\theta(-,-;\varphi) : R(\mathrm{GO}(V)_{\mathbf{A}}\times \mathrm{GSp}_4(\mathbf{A})) \to \mathcal{W}\otimes\mathcal{L}$ by

$$\theta(g,h;\varphi) = \sum_{x\in\mathbf{X}} \omega(g,h)\varphi(x).$$

We define $\mathrm{GSp}_4(\mathbf{A})^+$ to be the image of the second projection of $R(\mathrm{GO}(V)_{\mathbf{A}}\times \mathrm{GSp}_4(\mathbf{A}))$. For an automorphic form $\mathbf{f}$ on $\mathrm{GO}(V)_{\mathbf{A}}$ and a Bruhat-Schwartz function φ, we define a theta lift $\theta(-;\mathbf{f},\varphi) : \mathrm{GSp}_4(\mathbf{A})^+ \to \mathcal{L}$ by

$$\theta(g;\mathbf{f},\varphi) = \int_{\mathrm{O}(V)\backslash \mathrm{O}(V)_{\mathbf{A}}} \langle \theta(g,hh';\varphi), \mathbf{f}(hh')\rangle dh, \quad (\nu(h')=\nu(g)).$$

We extend $\theta(-;\mathbf{f},\varphi)$ to a function $\mathrm{GSp}_4(\mathbf{A})\to\mathcal{L}$ so that $\theta(-;\mathbf{f},\varphi)$ gives an automorphic form on $\mathrm{GSp}_4(\mathbf{A})$.

§ 2.3. Bessel coefficients

We introduce Bessel coefficients as defined in [7, Section 1.1]. Let $S\in \mathrm{M}_2(\mathbf{Q})$ such that $S = {}^{\mathrm{t}}S$. Define an $\mathbf{Q}$-algebraic group T_S by

$$T_S = \{g\in \mathrm{GL}_2 \mid {}^{\mathrm{t}}gSg = \det g S\}. \tag{2.1}$$

We shall consider T_S as a subgroup of GSp_4 by the embedding

$$g \mapsto \begin{pmatrix} g & \\ & \det g\,{}^{\mathrm{t}}g^{-1} \end{pmatrix}.$$

Define the subgroup R of GSp_4 by

$$R = T_S U = T_S \ltimes U,$$

where we define

$$U = \{u(X) := \begin{pmatrix} 1_2 & X \\ & 1_2 \end{pmatrix} | X = {}^{\mathrm{t}}X\} \subset \mathrm{GSp}_4.$$

Define a character $\psi_S : U \to \mathbf{C}^\times$ by $\psi_S(u(X)) = \psi(-\mathrm{Tr}(SX))$. Let F be a cuspidal automorphic form on $\mathrm{GSp}_4(\mathbf{A})$. For each character $\phi : T_S(\mathbf{Q})\backslash T_S(\mathbf{A}) \to \mathbf{C}^\times$ such that $\phi|_{\mathbf{A}^\times} = \chi^{-1}$, we define the Bessel coefficient $\mathbf{B}_{F,S,\phi} : \mathrm{GSp}_4(\mathbf{A}) \to \mathcal{L}$ of type (S, ϕ, ψ) by

$$\mathbf{B}_{F,S,\phi}(g) = \int_{\mathbf{A}^\times R_{\mathbf{Q}}\backslash R_{\mathbf{A}}} F(rg)\phi \otimes \psi_S(r)dr.$$

§ 3. Yoshida lift case

In this section, we introduce our result on Yoshida lifts (Theorem 3.1) which are theta lifts from SO_4 to GSp_4. In Section 3.1, we introduce automorphic forms on orthogonal group. Since we choose an orthogonal space to be a definite quaternion algebra, automorphic forms on orthogonal group are described by automorphic forms on a quaternion algebra. In Section 3.2, we introduce a specific choice of Bruhat-Schwartz function φ which we made in [10]. In Section 3.3, after choosing some symmetric matrix S and $g \in \mathrm{GSp}_4(\mathbf{A})$ to compute the Bessel coefficients $\mathbf{B}_{F,S,\phi}$, we write down our formula.

§ 3.1. cusp forms

Let D be a definite quaternion algebra over $\mathbf{Q}$ of discriminant N^-. We define a 4-dimensional vector space V over $\mathbf{Q}$ to be D. We define a quadratic form n on V by $\mathrm{n}(x) = xx^*$, where $*$ is the main involution of D. Let $\varrho : D^\times \times D^\times/\mathbf{Q}^\times \cong \mathrm{GSO}(V)$ be the isomorphism given by

$$\varrho(a, b)x = axb^{-1}, \quad (a, b \in D^\times, x \in D).$$

For each finite place $p \nmid N^-$, we fix an isomorphism $i_p : D_p \cong \mathrm{M}_2(\mathbf{Q}_p)$. Let $\mathbb{H}$ be the Hamilton quaternion algebra given by

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in \mathrm{M}_2(\mathbf{C}) \right\}.$$

The main involution $^* : \mathbb{H} \to \mathbb{H}$ is given by $x \mapsto {}^{\mathrm{t}}\bar{x}$. Fix an identification $i_\infty : D_\infty \cong \mathbb{H}$ such that $i_\infty(x^*) = i_\infty(x)^*$, which induces an embedding $i_\infty : D_\infty^\times \cong \mathbb{H}^\times \hookrightarrow \mathrm{GL}_2(\mathbf{C})$.

We put $i = 1, 2$. Let N_i^+ be a positive integer which is prime to N^-. We put $N_i = N_i^+ N^-$. Let π_i be an irreducible automorphic representation of $D_{\mathbf{A}}^\times$ of conductor

N_i with the trivial central character. We assume that $\pi_{i,\infty}$ is the restriction of the representation $\mathcal{W}_{k_i}(\mathbf{C}) = \det^{-k_i} \otimes \mathrm{Sym}^{2k_i}(\mathbf{C}^{\oplus 2})$ of $\mathrm{GL}_2(\mathbf{C})$ to $D_\infty^\times$, where $k_1 \geq k_2 \geq 0$. We note that $\pi_{i,\infty}$ corresponds to the discrete series representation of $\mathrm{GL}_2(\mathbf{R})$ of weight $2k_i + 2$ via the Jacquet-Langlands-Shimizu correspondence. Let $f_i \in \pi_i$ be a newform on $D_\mathbf{A}^\times$. We may assume that f_i is a function $D_\mathbf{A}^\times \to \mathcal{W}_{k_i}(\mathbf{C})$.

We put $\underline{k} = (k_1, k_2)$. We define $\mathcal{W}_{\underline{k}} := \mathcal{W}_{k_1}(\mathbf{C}) \otimes \mathcal{W}_{k_2}(\mathbf{C})$ to be an algebraic representation of $\mathrm{GSO}(V)_\infty$ via ϱ. We define an automorphic form $\mathbf{f}$ on $\mathrm{GSO}(V)_\mathbf{A}$ by $f_1 \boxtimes f_2$. Since, in this case, the theta correspondence only depends on automorphic forms on $\mathrm{GSO}(V)_\mathbf{A}$, we do not need to introduce an extension of $\mathbf{f}$ to $\mathrm{GO}(V)_\mathbf{A}$.

§ 3.2. Bruhat-Schwartz function

We put $\lambda = (k_1 + k_2 + 2, k_1 - k_2 + 2)$. We introduce a Bruhat-Schwartz function $\varphi = \otimes'_v \varphi_v \in \mathcal{S}(\mathbf{X_A}) \otimes \mathcal{W}_{\underline{k}} \otimes \mathcal{L}_\lambda$, which will be used in the explicit computation of Bessel coefficients.

Let v be a finite place of $\mathbf{Q}$. We denote by R the Eichler order of level $N^+ := \mathrm{lcm}(N_1^+, N_2^+)$. Then, we define φ_v to be the characteristic function of $R_v^{\oplus 2}$.

We define φ_∞. Let $P_{\underline{k}} : \mathbb{H}^{\oplus 2} \to \mathcal{W}_{\underline{k}} \otimes \mathcal{L}_\lambda$ be the pluri-harmonic function which is defined in [10, Section 4.2]. For $x = (x_1, x_2) \in \mathbf{X}_\infty$, we put

$$\varphi_\infty(x) = e^{-2\pi(\mathrm{n}(x_1)+\mathrm{n}(x_2))} P_{\underline{k}}(x).$$

§ 3.3. Result on Bessel coefficients

Let K be an imaginary quadratic field which satisfies the following conditions:

- every prime factor of N^+ (resp. N^-) is split (resp. inert) in K;
- 2 is unramified in K.

By the first condition on K, we can embed K into D and fix an embedding $K \hookrightarrow D$. We denote the ring of integers of K by $\mathcal{O}_K$ and we take $\vartheta \in K$ to be $\mathcal{O}_K = \mathbf{Z} + \mathbf{Z}\vartheta$. We put $S = \begin{pmatrix} \mathrm{N}_{K/\mathbf{Q}}(\vartheta) & \mathrm{Tr}(\vartheta) \\ \mathrm{Tr}(\vartheta) & 1 \end{pmatrix}$. Let $\mathfrak{X}^-$ be the set of finite order Hecke characters of $K_\mathbf{A}^\times$ which is trivial on $\mathbf{A}^\times$.

We define $\xi = \xi_\infty \times \xi_f \in \mathrm{GL}_2(\mathbf{A})$ to be

$$\xi_\infty = \begin{pmatrix} \mathrm{Im}(\vartheta) & -\mathrm{Re}(\vartheta) \\ & 1 \end{pmatrix} (\mathrm{Im}(\vartheta))^{-1}, \quad \xi_f = C1_2,$$

where C is a positive integer which is prime to N^+N^-. For each $\phi \in \mathfrak{X}^-$ whose conductor is $C\mathcal{O}_K$, we define

$$\mathbf{B}_{S,\phi}^{[0]}(f_1, f_2) = \langle \mathbf{B}_{\theta(-;\mathbf{f},\varphi),S,\phi}(\begin{pmatrix} \xi & \\ & {}^t\xi^{-1} \end{pmatrix}), (\mathbf{U}^2 + \mathbf{V}^2)^{k_2} \rangle_{2k_2}.$$

For $h \in \widehat{D}^\times$ and $i = 1, 2$, we define a toric period integral $P(f_i, \phi, h)$ to be

$$P(f_i, \phi, h) = \int_{K^\times \mathbf{A}^\times \backslash K^\times_{\mathbf{A}}} \langle (X_i Y_i)^{k_i}, f_i(th) \rangle_{2k_i} \phi(t) dt.$$

We define an element $\varsigma^{(C)} = (\varsigma_p^{(C)})_{p<\infty} \in \widehat{D}^\times$ as follows: If $p \mid N^-$, we put $\varsigma_p^{(C)} = 1$. If $p \nmid N^-$, we choose $\varsigma_p \in \mathrm{GL}_2(\mathbf{Q}_p) \cong D_p^\times$ so that

$$(\varsigma_p^{(C)})^{-1} \delta \varsigma_p^{(C)} = \begin{cases} \begin{pmatrix} 1 & -\delta\bar{\delta} p^{\mathrm{ord}_p(C)} \\ p^{-\mathrm{ord}_p(C)} & 0 \end{pmatrix} & (p: \text{ non-split in } K), \\ \begin{pmatrix} \delta & (\delta - \bar{\delta}) p^{-\mathrm{ord}_p(C)} \\ 0 & \bar{\delta} \end{pmatrix} & (p: \text{ split in } K), \end{cases}$$

where we identify $K_{\mathfrak{p}} \cong \mathbf{Q}_p$ if $p = \mathfrak{p}\bar{\mathfrak{p}}$ is split in K.

Then, we have the following explicit description of Bessel coefficients:

Theorem 3.1. *Let ϕ be an element of $\mathfrak{X}^-$ whose conductor is $C\mathcal{O}_E$. Then, we have*

$$\mathbf{B}^{[0]}_{S,\phi}(f_1, f_2) = e^{-4\pi} C^2 \mathrm{vol}(\widehat{\mathcal{O}}_C^\times, dt) e(f_1, f_2, \phi) P(f_1, \phi, \varsigma^{(C)}) P(f_2, \phi, \varsigma^{(C)}),$$

where

$$e(f_1, f_2, \phi) = \prod_{p \mid N} \prod_{\mathfrak{p} \mid p} (1 + \epsilon_p(f_1) \epsilon_p(f_2) \phi(\mathfrak{p})^{\mathrm{ord}_p(N_1 N_2^{-1})}).$$

Remark. Let π_i^{JL} be the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ which corresponds to π_i via the Jacquet-Langlands-Shimizu correspondence. Let $\pi_{i,K}^{\mathrm{JL}}$ be the base change of π_i^{JL} to K. Then, the toric integral $P(f_i, \phi, \varsigma^{(C)})$ which appears in Theorem 3.1 is roughly a square root of the central value $L(\frac{1}{2}, \pi_{i,K}^{\mathrm{JL}} \otimes \phi)$ of L-function of $\pi_{i,K}^{\mathrm{JL}}$. For the precise formula of these relation, see [5, Theorem 3.11].

§ 4. HST lift case

In this section, we describe our result on HST lifts (Theorem 4.1) which are theta lifts from $\mathrm{O}_{3,1}$ to GSp_4. We consider cusp forms on GL_2 over imaginary quadratic fields instead of automorphic forms on quaternion algebra.

§ 4.1. cusp forms

Let $E = \mathbf{Q}(\sqrt{-D})$ be an imaginary quadratic field of the discriminant D. We define a 4-dimensional $\mathbf{Q}$-vector space V to be

$$V = \{\mathrm{M}_2(E) : {}^{\mathrm{t}}x^c = x\},$$

where c is the complex conjugate. We define a quadratic form n on V by $\mathrm{n}(x) = -\det(x)$, where $x \in V$. Let $\varrho : \mathrm{GL}_2(E) \times_{E^\times} \mathbf{Q}^\times \xrightarrow{\sim} \mathrm{GSO}(V)$ be the isomorphism given by

$$(h, \alpha) \mapsto \varrho(h, \alpha)x = hx^{\mathrm{t}}h^c\alpha.$$

For $g = \varrho(h, \alpha) \in \mathrm{GSO}(V)$, we put $g^c = \varrho(h^c, \alpha)$. Let $\mathbf{t} \in \mathrm{GO}(V)$ be the element given by

$$\mathbf{t}x = {}^{\mathrm{t}}x. \tag{4.1}$$

Then we have

$$\mathrm{GO}(V) = \mathrm{GSO}(V) \rtimes \{1, \mathbf{t}\}.$$

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(E_{\mathbf{A}})$ of conductor $\mathfrak{N}$ with the central character ω_π. We assume that π_∞ is isomorphic to a principal representation $\pi(\mu, \mu^c)$, where $\mu(z) = (z/z^c)^{\frac{k-1}{2}}$ for $z \in \mathbf{C}^\times$ and $k \geq 2$. Let $f \in \pi$ be a normalized newform. We may assume that f is a function $\mathrm{GL}_2(E_{\mathbf{A}}) \to \mathcal{W}_{k-1} = \det^{1-k} \otimes \mathrm{Sym}^{2k-2}(\mathbf{C}^{\oplus 2})$. Let χ and δ be as in Section 1.

According to [8], we define an automorphic form $\tilde{\mathbf{f}}$ on $\mathrm{GO}(V)_{\mathbf{A}}$ by using the triple (f, χ, δ). We define $\mathbf{f} = f \boxtimes \chi$. Then, we consider $\mathbf{f}$ as an automorphic form on $\mathrm{GSO}(V)_{\mathbf{A}}$ via ϱ. Let T and Σ_{ram} be as in Section 1. Assume that $\delta(v) = 1$ for $v \notin \Sigma_{\mathrm{ram}}$. We denote the Whittaker function of f by W_f. We put $W(\varrho(h, \alpha)) = \chi(\alpha)W_f(h)$ for $\varrho(h, \alpha) \in \mathrm{GSO}(V)_{\mathbf{A}}$. For each subset $\mathcal{R} \subset \Sigma_{\mathrm{ram}}$, we define $\mathbf{f}_{\mathcal{R}} = 0$ if $\mathcal{R} \not\subset T$ and define $\mathbf{f}_{\mathcal{R}}$ to be the function whose associated Whittaker function is $W_{\mathcal{R}}(g) := \delta_{\mathcal{R}} \cdot W(g^c_{\mathcal{R}} g^{\mathcal{R}})$ if $\mathcal{R} \subset T$, where $\delta_{\mathcal{R}} := \prod_{v \in \mathcal{R}} \delta(v)$ and $g^c_{\mathcal{R}} g^{\mathcal{R}} := \prod_{v \in \mathcal{R}} g^c_v \prod_{v \in \Sigma_{\mathbf{Q}} \setminus \mathcal{R}} g_v$. We define an automorphic form $\tilde{\mathbf{f}}$ on $\mathrm{GO}(V)_{\mathbf{A}}$ by

$$\tilde{\mathbf{f}}(g\mathbf{t}_{\mathcal{R}}) = \mathbf{f}_{\mathcal{R}'}(g) + \mathbf{f}_{\Sigma_{\mathrm{ram}} \setminus \mathcal{R}'}(g^c) \quad (g \in \mathrm{GSO}(V)_{\mathbf{A}}, \mathcal{R} \subset \Sigma_{\mathbf{Q}}),$$

where $\mathcal{R}' = \mathcal{R} \cap \Sigma_{\mathrm{ram}}$ and $\mathbf{t}_{\mathcal{R}} = \prod_{v \in \mathcal{R}} \mathbf{t}_v$ with $\mathbf{t}_v \in \mathrm{GO}(V)(\mathbf{Q}_v)$ is the element defined in (4.1).

§ 4.2. Bruhat-Schwartz function

We introduce a Bruhat-Schwartz function $\varphi = \otimes'_v \varphi_v \in \mathcal{S}(\mathbf{X}_{\mathbf{A}}) \otimes \mathcal{W}_{k-1} \otimes \mathcal{L}_{k,2}$.

Let $N\mathbf{Z} = \mathfrak{N} \cap \mathbf{Z}$ and $N_D = \mathrm{lcm}(N, D)$. For a finite place v of $\mathbf{Q}$, we put

$$V'_v(\mathcal{O}_v) = \left\{ \begin{pmatrix} p & q \\ q^c & r \end{pmatrix} \in V_v : p \in \mathbf{Z}_v, q \in \mathcal{O}_v, r \in N_D\mathbf{Z}_v \right\},$$

where $\mathcal{O}_v$ is the ring of integers of E_v. We define φ_v to be the characteristic function of $V'(\mathcal{O}_v)^{\oplus 2}$.

We define φ_∞. Let $P_k : \mathbf{X}_\infty \to \mathcal{W}_{k-1} \otimes \mathcal{L}_{k,2}$ be the pluri-harmonic function which is defined in [10, Section 5.2]. For $x = (x_1, x_2) \in \mathbf{X}_\infty$, we put

$$\varphi_\infty(x) = e^{-\pi \mathrm{Tr}(x_1 {}^{\mathrm{t}} x_1^c + x_2 {}^{\mathrm{t}} x_2^c)} P_k(x).$$

§ 4.3. Result on Bessel coefficients

We take $\vartheta \in E$ so that $\mathcal{O}_E = \mathbf{Z} + \mathbf{Z}\vartheta$, where $\mathcal{O}_E$ is the ring of integers of E. We put $S = \begin{pmatrix} 1 & \mathrm{Tr}_{E/\mathbf{Q}}(\vartheta) \\ \mathrm{Tr}_{E/\mathbf{Q}}(\vartheta) & \mathrm{N}_{E/\mathbf{Q}}(\vartheta) \end{pmatrix}$. Let $\mathfrak{X}^-$ be the set of finite order Hecke characters ϕ of $E_{\mathbf{A}}^\times$ such that $\phi|_{\mathbf{A}^\times} = \chi^{-1}$.

We define $\xi = \xi_\infty \times \xi_f \in \mathrm{GL}_2(\mathbf{A})$ to be

$$\xi_\infty = \begin{pmatrix} \mathrm{Im}(\vartheta) & -\mathrm{Re}(\vartheta) \\ & 1 \end{pmatrix} (\mathrm{Im}(\vartheta))^{-1}, \quad \xi_f = C 1_2,$$

where C is a positive integer which is prime to ND. For each $\phi \in \mathfrak{X}^-$ such that conductor of ϕ is $C\mathcal{O}_E$, we define

$$\mathbf{B}^{[0]}_{S,\phi}(f, \chi, \delta) = \langle \mathbf{B}_{\theta(-;\tilde{\mathbf{f}},\varphi),S,\phi}(\begin{pmatrix} \xi & \\ & {}^{\mathrm{t}}\xi^{-1} \end{pmatrix}), (\mathbf{U}^2 + \mathbf{V}^2)^{\frac{k-2}{2}} \rangle_{k-2}.$$

Then, we have the following explicit description of Bessel coefficients:

Theorem 4.1. *Let ϕ be an element of $\mathfrak{X}^-$ such that conductor of ϕ is $C\mathcal{O}_E$. Then, we have*

$$\mathbf{B}^{[0]}_{S,\phi}(f, \chi, \delta) = \frac{\pi}{2^5} e^{-4\pi} (2\sqrt{-1})^{k-1} e(\pi, \phi, \delta) \Gamma_{\mathbf{C}}(\frac{k}{2})^2 L(\frac{1}{2}, \pi \otimes \phi) \prod_{v | C} \epsilon(0, \phi_v) \phi_v(C),$$

where

$$e(\pi, \phi, \delta) = (1 - \delta(\infty)) \prod_{v \in \Sigma_{\mathrm{ram}} \cap T, v < \infty} (1 + \delta(v)\phi_v(-1))(1 + \epsilon(\frac{1}{2}, \pi_v \otimes \phi_v)).$$

Remark. In [8], the authors prove that HST lifts vanish if $\delta(\infty) = 1$ by representation theoretic method. Hence, our computation is compatible with their result.

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