A Lefschetz trace formula for $p^{n}$-torsion etale cohomology : a resume (Algebraic Number Theory and Related Topics 2012)

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A Lefschetz trace formula for $p^n$-torsion étale cohomology: a resume

By

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This is a resume of our results ([5]) on a Lefschetz trace formula on varieties defined over a finite field $\mathbb{F}_q$ of characteristic $p$. It is a $p^n$-torsion version of a conjecture of Deligne which was originally formulated with $\ell$-adic étale cohomology ($\ell \neq p$) and has been proved by Fujiwara in full generality ([2]).

We introduce some notations to state our results. We fix an algebraic closure $k$ of $\mathbb{F}_q$. For an object $\mathcal{X}_0$ (e.g. scheme, sheaf on a scheme, morphism of schemes) over $\mathbb{F}_q$, $\mathcal{X}$ denotes the base change of $\mathcal{X}_0$ by the injection $\mathbb{F}_q \hookrightarrow k$. Let $S$ be a scheme. For a morphism of $S$-schemes $b: V \to U \times_S U$, we put $b_1 = \text{pr}_1 \circ b$ and $b_2 = \text{pr}_2 \circ b$, where $\text{pr}_1$ (resp. $\text{pr}_2$) is the first (resp. second) projection of $U \times_S U$. The $S$-scheme $\text{Fix}(b) = V \times_{U \times_S U} U$ is defined by the following cartesian diagram

$$
\begin{array}{ccc}
\text{Fix}(b) & \longrightarrow & U \\
\downarrow & & \downarrow \Delta_{U/S} \\
V & \stackrel{b}{\longrightarrow} & U \times_S U,
\end{array}
$$

where $\Delta_{U/S}$ is the diagonal morphism. Remark that if $U$ and $V$ are smooth over $S$, $db_1: b_1^* \Omega_{U/S} \to \Omega_{V/S}$ is zero and $b_2$ is étale, then $\text{Fix} b$ is étale over $S$ ([7, Cor. 17.13.6]). For an $S$-endomorphism $f: U \to U$, we put $\text{Fix} f = \text{Fix}(f \times_S \text{id}_U)$. Let $U_0$ and $V_0$ be $\mathbb{F}_q$-schemes and $b_0: V_0 \to U_0 \times_{\mathbb{F}_q} U_0$ an $\mathbb{F}_q$-morphism of schemes. We put $b^{(m)} = (\text{Fr}_U^m \circ b_1, b_2)$, where $\text{Fr}_U$ is the relative $q$-th power Frobenius morphism of $U$ i.e. the base change of the absolute $q$-th power Frobenius morphism of $U_0$ by $\mathbb{F}_q \to k$.

First, we state a $p$-torsion version of Fujiwara’s trace formula.
**Theorem 1** ([5, Corollary 3.2], [5, Theorem 6.1]). Let $U_0$ and $V_0$ be separated $\mathbb{F}_q$-schemes of finite type and $\mathcal{F}_0$ a constructible étale $\mathbb{Z}/p$-sheaf on $U_0$.

(1) Let $f_0: U_0 \to U_0$ be an automorphism of finite order and $m \geq 1$ an integer. Let $u_0: (\text{Fr}_{U_0}^m \circ f_0)^* \mathcal{F}_0 \to \mathcal{F}_0$ be an isomorphism of sheaves whose order is the same as that of $f_0$. Then we have the following equality

$$\sum_i (-1)^i \text{Tr}(u \circ (\text{Fr}_{U_0}^m \circ f)^* | H^i_c(U, \mathcal{F})) = \sum_{P \in \text{Fix}(\text{Fr}_{U_0}^m \circ f)} \text{Tr}(u_P | \mathcal{F}_P).$$

(2) Let $b_0: V_0 \to U_0 \times_{\mathbb{F}_q} U_0$ be a morphism of $\mathbb{F}_q$-schemes. We assume that $\mathcal{F}_0$ is smooth. Further we assume that there exist proper smooth $\mathbb{F}_q$-schemes $X_0$ and $Y_0$, and an $\mathbb{F}_q$-morphism $a_0: Y_0 \to X_0 \times_{\mathbb{F}_q} X_0$ such that

(a) $U_0$ (resp. $V_0$) is an open $\mathbb{F}_q$-subscheme of $X_0$ (resp. $Y_0$), the diagram

$$\begin{array}{ccc}
V_0 & \xrightarrow{b_0} & U_0 \times_{\mathbb{F}_q} U_0 \\
\downarrow & & \downarrow \\
Y_0 & \xrightarrow{a_0} & X_0 \times_{\mathbb{F}_q} X_0
\end{array}$$

is cartesian,

(b) $b_1$ is proper, $a_2$ is étale, $a$ is a closed immersion,

(c) $X \setminus U$ is a Cartier divisor, and

(d) there exists a smooth constructible étale $\mathbb{Z}/p$-sheaf $\mathcal{G}_0$ on $X_0$ such that $\mathcal{G}_0|_{U_0} = \mathcal{F}_0$.

Then, for any integer $m \geq 1$ and any $u_0 \in \text{Hom}(b_{01}^{(m)*} \mathcal{F}_0, b_{02}^* \mathcal{F}_0)$, we have the following equality

$$\sum_i (-1)^i \text{Tr}(u_! | H^i_c(U, \mathcal{F})) = \sum_{P \in \text{Fix}(b^{(m)})} \text{Tr}(u_P | \mathcal{F}_P),$$

where $u_!$ is the composition

$$H^i_c(U, \mathcal{F}) \xrightarrow{b^{(m)*}_1} H^i_c(V, b_1^{(m)*} \mathcal{F}) \xrightarrow{u} H^i_c(V, b_2^* \mathcal{F}) \xrightarrow{b_2} H^i_c(U, \mathcal{F}).$$

Remark that $\text{Fix}(\text{Fr}_{U_0}^m \circ f)$ is finite over $k$ by Zink's lemma [6, Lemma 2.3] and $\text{Fix}(b^{(m)})$ is finite étale over $k$ since $U$ and $V$ are smooth over $k$, the differential of $b_1^{(m)}$ is zero and $b_2$ is étale.

Theorem 1 (1) is proved by using the Lefschetz trace formula for the Frobenius correspondence ([8, Fonct. L mod. $\ell$ Théorème 4.1]) and Deligne-Lusztig’s method ([1,
Section 3). We sketch the proof of Theorem 1 (2). This is a generalization of the proof of [8, Fonct. L mod. ℓⁿ, Théorème 4.1]. We put $\mathcal{G}' = \mathcal{I}_0(\mathcal{G}_0 \otimes \mathcal{O}_X)$, where $\mathcal{I}_0$ is the ideal sheaf of definition of $X_0 \setminus U_0$. Since $X \setminus U$ is a Cartier divisor, $\mathcal{G}'$ is a locally free $\mathcal{O}_X$-module of finite rank and sits in the exact sequence

$$0 \to j_1 \mathcal{F} \to \mathcal{G}' \xrightarrow{1-\Phi} \mathcal{G}' \to 0,$$

where $\Phi: \mathcal{G}' \rightarrow \mathcal{G}'$ is the morphism induced by the $p$-th power map on $\mathcal{O}_X$. Then we can reduce the calculation of the trace of the endomorphism of the cohomology group of $\mathcal{F}$ to that of $\mathcal{G}'$. By applying the following trace formula to the trace, we obtain Theorem 1 (2).

**Theorem 2** (Woods Hole formula, [5, Theorem 4.1]). Let $S$ be the spectrum of an artinian local ring, $X$ and $Y$ proper smooth schemes over $S$, and $a: Y \hookrightarrow X \times S$ a closed immersion over $S$. We assume that $a_2$ is étale and the homomorphism $da_1: a_1^* \Omega_{YS} \to \Omega_{YS}$ is zero. Then, for any perfect complex $\mathscr{K}$ of $\mathcal{O}_X$-modules and any $u \in \text{Hom}(a_1^* \mathscr{K}, a_2^* \mathscr{K})$, we have

$$\sum_i (-1)^i \text{Tr}(u_\ast | H^i(X, \mathscr{K})) = \sum_{\beta \in \pi_0(\text{Fix}(a))} \text{Tr}_{\beta/S}(\text{Tr}(u_{\beta} | \mathscr{K}_{\beta})),$$

where $u_\ast$ is the composition

$$H^i(X, \mathscr{K}) \xrightarrow{a_1^*} H^i(Y, a_1^* \mathscr{K}) \xrightarrow{u} H^i(Y, a_2^* \mathscr{K}) \xrightarrow{a_2*} H^i(X, \mathscr{K}),$$

$\pi_0(\text{Fix}(a))$ is the set of connected components of $\text{Fix}(a)$, $\mathscr{K}_\beta$ (resp. $u_\beta$) is the pull-back of $\mathscr{K}$ (resp. $u$) by the immersion $i_\beta: \beta \hookrightarrow Y$ and $\text{Tr}_{\beta/S}$ is the trace map $\Gamma(\beta, \mathcal{O}_\beta) \to \Gamma(S, \mathcal{O}_S)$.

Remark that $\text{Fix}(a)$ is finite étale over $S$. Theorem 2 is a generalization of [9, Exp. III, Corollaire 6.12], and can be proved by using the Lefschetz-Verdier trace formula ([9, Exp. III, Théorème 6.10]) and properties of residue symbols in [3, Ch. III, §9].

Secondly, we state a $p^n$-torsion version of Fujiwara’s trace formula. At present, this requires more assumptions than Theorem 1.

For a perfect field $K$ of characteristic $p$, we denote by $W_n(K)$ the ring of Witt vectors of $K$ of length $n$. We write $\sigma_0$ for the Frobenius automorphism of $W_n(K)$. For a scheme $S$, we write $\mathcal{O}_S$ for the structure sheaf of $S$. If $S$ is of characteristic $p$, denote by $\Phi_{\mathcal{O}_S}$ the $p$-th power map on $\mathcal{O}_S$.

**Theorem 3** ([5, Theorem 7.1]). Let $U_0$ and $V_0$ be smooth $\mathbb{F}_q$-schemes, $b_0: V_0 \to U_0 \times_{\mathbb{F}_q} U_0$ an $\mathbb{F}_q$-morphism, and $\mathcal{F}_0$ a locally free constructible étale $\mathbb{Z}/p^n$-sheaf on $U_0$. We assume that there exist proper smooth $W_n(\mathbb{F}_q)$-schemes $\mathcal{X}_0$ and $\mathcal{Y}_0$, a Cartier divisor $\mathcal{D}_0$ of $\mathcal{X}_0$ which is flat over $W_n(\mathbb{F}_q)$, a $W_n(\mathbb{F}_q)$-morphism $\tilde{a}_0: \mathcal{Y}_0 \to \mathcal{X}_0 \times_{W_n(\mathbb{F}_q)} \mathcal{X}_0$, and a morphism $\Phi_{\mathcal{O}_{\mathcal{X}_0}}: \mathcal{O}_{\mathcal{X}_0} \to \mathcal{O}_{\mathcal{X}_0}$ such that

$$1-\Phi_{\mathcal{O}_{\mathcal{X}_0}}: \mathcal{O}_{\mathcal{X}_0} \to \mathcal{O}_{\mathcal{X}_0}.$$
(a) when we put $X_0 = \mathcal{X}_0 \times W_n(F_q)$, $Y_0 = \mathcal{Y}_0 \times W_n(F_q)$ and define $a_0$ such that the diagram

$$
\begin{array}{c}
\mathcal{Y}_0 \xrightarrow{\bar{a}_0} \mathcal{X}_0 \times W_n(F_q) \\
\uparrow \quad \uparrow \\
Y_0 \xrightarrow{a_0} X_0 \times W_n(F_q)
\end{array}
$$

is cartesian, then $(U_0, V_0, b_0, X_0, Y_0, a_0)$ satisfies the condition (a) in Theorem 1 (2),

(b) $b_1$ is proper, $\tilde{a}_2$ is étale, $\tilde{a}$ is a closed immersion,

(c) $\mathcal{D}_0$ is a lift of $X_0 \setminus U_0$ to $W_n(F_q)$,

(d) the diagrams

$$
\begin{array}{c}
W_n(F_q) \xrightarrow{\sigma_0} W_n(F_q) \\
\downarrow \quad \downarrow \\
\mathcal{O}_{\mathcal{X}_0} \xrightarrow{\Phi_{\mathcal{O}_{\mathcal{X}_0}}} \mathcal{O}_{\mathcal{X}_0}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\mathcal{O}_{\mathcal{X}_0} \xrightarrow{\Phi_{\mathcal{O}_{\mathcal{X}_0}}} \mathcal{O}_{\mathcal{X}_0} \\
\downarrow \quad \downarrow \\
\mathcal{O}_{X_0} \xrightarrow{\Phi_{\mathcal{O}_{X_0}}} \mathcal{O}_{X_0}
\end{array}
$$

commute,

(e) the inclusion $\Phi_{\mathcal{O}_{\mathcal{X}_0}}(\mathcal{I}_0) \subset \mathcal{I}_0$ holds, where $\mathcal{I}_0$ is the defining ideal of $\mathcal{D}_0$,

(f) there exists a locally free constructible étale $\mathbb{Z}/p^n$-sheaf $\mathcal{G}_0$ on $X_0$ such that $\mathcal{G}_0|_{U_0} = \mathcal{F}_0$ and

(g) $H^i_c(U, \mathcal{F})$ (resp. $H^i(\mathcal{D}, \mathcal{G} \otimes_{\mathbb{Z}/p^n} \mathcal{I})$) is free over $\mathbb{Z}/p^n$ (resp. $W_n(k)$) for any $i$.

Then there exists an integer $M$ such that, for any integer $m \geq M$ and any homomorphism $u_0 \in \text{Hom}(b_0^{(m)*}\mathcal{G}_0, b_0^{(m)}\mathcal{G}_0)$, we have the following equality

$$
\sum_{i} (-1)^i \text{Tr}(u_! | H^i_c(U, \mathcal{F})) = \sum_{P \in \text{Fix}(b^{(m)})} \text{Tr}(u_P | F_P).
$$

We note that the integer $M$ in Theorem 3 depends on the sheaf $\mathcal{F}_0$. We need the assumption on existence of $\mathcal{X}_0$, $\mathcal{Y}_0$, $\mathcal{D}_0$, $\bar{a}_0$ and $\Phi_{\mathcal{O}_{\mathcal{X}_0}}$ in order to use the same argument used in the proof of Theorem 1 (2), and the assumption (g) in order to compute the trace in the category of $\mathbb{Z}/p^n$-modules, not in that of perfect complexes of $\mathbb{Z}/p^n$-modules. If $X_0$ is a curve, then the assumption (e) automatically holds ([4, Lemma 1.1.2]).

The proof of Theorem 3 is similar to that of Theorem 1 (2).

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