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# A survey on Stark's conjectures and a result of Dasgupta-Darmon-Pollack

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**Abstract**

This is a survey on Stark's conjectures and some related topics. We present formulations of Stark's conjectures, Rubin's integral refinement for the abelian Stark conjecture, the Brumer-Stark conjecture, and a $p$-adic analogue of the rank 1 abelian Stark conjecture, which is called the Gross-Stark conjecture. In addition, we describe the recent result [DDP] by Dasgupta-Darmon-Pollack concerning the Gross-Stark conjecture.

## Introduction.

In this paper we will survey some results on Stark's conjecture and its $p$-adic analogue. Since Stark's conjecture is a generalization of the class number formula, we start by recalling this formula. Let $k$ be a number field. The Dedekind zeta function is defined by

$$\zeta_k(s) := \prod_{\mathfrak{p} \subset \mathcal{O}_k} \left(1 - N\mathfrak{p}^{-s}\right)^{-1} \qquad (\mathrm{Re}(s) > 1).$$

Here $\mathfrak{p}$ run over all prime ideals of $k$. We see that it can be extended meromorphically to the whole complex plane $\mathbb{C}$ and is holomorphic at $s = 0$. Then the class number formula states that

$$\zeta_k(s) = \frac{-h_k R_k}{e_k} s^{r_k - 1} + O(s^{r_k}) \quad (s \to 0).$$

Here $r_k, h_k, R_k, e_k$ are the number of infinite places of $k$, the class number, the regulator, and the number of roots of unity in $k$, respectively. In particular, we can write

$$\frac{\text{the leading coefficient of } \zeta_k(s) \text{ in the Taylor expansion at } s = 0}{\text{the regulator}} \in \mathbb{Q}.$$

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Very roughly speaking, the Stark conjecture is its generalization, from Dedekind zeta functions to Artin $L$ functions. In §1, we provide precise statements and some results of Stark's conjectures, the Brumer-Stark conjecture, and Rubin's integral refinement for the rank 1 abelian Stark conjecture. We also deal with a $p$-adic analogue of the Stark conjecture which is called the Gross-Stark conjecture in §2,§3. In particular, we present recent results [DDP] by Dasgupta-Darmon-Pollack. They proved the Gross-Stark conjecture assuming that Leopoldt's conjecture holds true and that some technical conditions are satisfied. By "cohomological interpretation", we can reduce the Gross-Stark conjecture to the construction of a suitable cocycle. To construct such a cocycle, some techniques of Ribet [Ri] and Wiles [Wi] are used.

Before stating Stark's conjectures, we recall the definition of the regulator $R_k$ for the sake of comparison to the "Stark regulator". Let $\mathcal{O}_k$ be the ring of integers of $k$, $\mu_k$ the group of roots of unity in $k$, and $\{\infty_1, \ldots, \infty_{r_k}\}$ the set of all infinite places of $k$. Consider the logarithmic embedding of units

$$\lambda\colon \mathcal{O}_k^\times \to \mathbb{R}^{r_k}, \qquad \lambda(x) := (\log|x|_{\infty_j})_{1\le j\le r_k}.$$

Then Dirichlet's unit theorem states that its image is a free $\mathbb{Z}$-module of rank $r_k - 1$, and so is $\mathcal{O}_k^\times/\mu_k$. Taking generators $\varepsilon_i \bmod \mu_k \in \mathcal{O}_k^\times/\mu_k$ $(1 \le i \le r_k - 1)$, we define the regulator $R_k$ of $k$ by

$$\begin{aligned} R_k &:= |\det(\log|\varepsilon_i|_{\infty_j})_{1\le i,j\le r_k-1}| \\ &= \pm \begin{vmatrix} \log|\varepsilon_1|_{\infty_1} & \log|\varepsilon_1|_{\infty_2} & \cdots & \log|\varepsilon_1|_{\infty_{r_k-1}} \\ \log|\varepsilon_2|_{\infty_1} & \log|\varepsilon_2|_{\infty_2} & \cdots & \log|\varepsilon_2|_{\infty_{r_k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \log|\varepsilon_{r_k-1}|_{\infty_1} & \log|\varepsilon_{r_k-1}|_{\infty_2} & \cdots & \log|\varepsilon_{r_k-1}|_{\infty_{r_k-1}} \end{vmatrix}. \end{aligned}$$

Note that the definition of $R_k$ does not depend on the choice of $\varepsilon_i$ or the numbering of $\infty_j$.

## § 1. Stark's conjectures.

Unless otherwise noted, we use the following notations in this paper.

- $K/k$ is a finite Galois extension of number fields with $G := \mathrm{Gal}(K/k)$.
- For a place $p$ of $\mathbb{Q}$, we denote by $S_p$ the set of all places of $k$ lying above $p$. In particular, $S_\infty$ is the set of all infinite places. For any set $T$ of places of $k$, we put

  $$T_K := \{w \text{ of } K \mid \exists v \in T \text{ such that } w|v\}.$$

  We fix a finite set $S$ of places of $k$ satisfying

  $$S_\infty \subset S.$$

- For a $\mathbb{Z}$-module $M$ and an extension $\mathbb{Z} \subset R$ of rings, we put

$$RM := R \otimes_{\mathbb{Z}} M.$$

## § 1.1. The non-abelian Stark conjecture.

For any $\mathbb{C}$-valued character $\chi$ of $G$, we will define the following symbols after introducing an isomorphism $f$ of $\mathbb{Q}[G]$-modules.

- $C_S(\chi) \in \mathbb{C}^\times$: the leading coefficient of the Artin $L$-function $L_S(s, \chi)$ in the Taylor expansion at $s = 0$ (§1.1.1).
- $R_S(\chi, f) \in \mathbb{C}^\times$: the Stark regulator associated to the group of $S$-units of $K$ (§1.1.2).
- $A_S(\chi, f) := R_S(\chi, f)/C_S(\chi) \in \mathbb{C}^\times$.

Then Stark's conjecture in the general case is formulated as follows.

**Conjecture 1.1.** *For any $\mathbb{C}$-valued character $\chi$ of $G$, we have*

$$A_S(\chi, f)^\gamma = A_S(\chi^\gamma, f) \quad (\forall \gamma \in \mathrm{Aut}(\mathbb{C})). \tag{1.1}$$

*Here we put $\chi^\gamma := \gamma \circ \chi$.*

*Remark.* Conjecture (1.1) implies

$$\frac{\text{the leading coefficient of the Artin } L\text{-function } L_S(s, \chi) \text{ at } s = 0}{\text{the Stark regulator}} \in \mathbb{Q}(\chi),$$

where $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(\sigma) \mid \sigma \in F)$.

**1.1.1. The leading coefficient $C_S(\chi)$ of the Artin $L$ function $L(s, \chi)$ at $s = 0$.** Let $V$ be the representation space of $\chi$. For each place $v$ of $k$, we choose a place $w$ of $K$ lying above $v$, and write its decomposition group, inertia group, Frobenius automorphism as $G_v, I_v, \mathrm{Frob}_v$ respectively. Then the $S$-truncated Artin $L$ function $L_S(s, \chi)$ is defined by

$$L_S(s, \chi) := \prod_{\mathfrak{p} \notin S} \det(1 - \mathrm{Frob}_{\mathfrak{p}} N\mathfrak{p}^{-s}|_{V^{I_{\mathfrak{p}}}})^{-1} \quad (\mathrm{Re}(s) > 1).$$

It can be continued meromorphically to the whole complex plane and is holomorphic at $s = 0$. We denote its leading coefficient and its order of 0 at $s = 0$ by $C_S(\chi)$, $r_S(\chi)$ respectively. That is, we can write

$$L_S(s, \chi) = C_S(\chi) s^{r_S(\chi)} + O(s^{r_S(\chi)+1}) \quad (s \to 0,\ C_S(\chi) \neq 0).$$

By using the functional equation for Artin $L$-functions, we can show the following formula, which we will use later: When $\dim_{\mathbb{C}} V = 1$, we have

$$r_S(\chi) = \begin{cases} |S| - 1 & (\chi = 1_G), \\ |\{v \in S \mid \chi(G_v) = \{1\}\}| & (\chi \neq 1_G). \end{cases}$$

**1.1.2. The Stark regulator $R_S(\chi, f)$.** Put

$$Y := Y_{S,K} := \oplus_{w \in S_K} \mathbb{Z}w.$$

Then $G$ acts on $Y$ in the natural way. We denote by $X = X_{S,K}$ the kernel of $\deg\colon Y \to \mathbb{Z}$ ($\sum_w n_w w \mapsto \sum_w n_w$), i.e.,

$$X := X_{S,K} := \left\{ \sum_w n_w w \in Y \mid \sum_w n_w = 0 \right\}.$$

Then we can show that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) = r_S(\chi)$$

with $V^*$ the contragradient representation of $V$. On the other hand, we denote the group of $S$-units in $K$ by $U := U_{K,S}$. That is, we can write

$$U := U_{K,S} := \{x \in K^\times \mid |x|_w = 1, \ \forall w \notin S_K\}.$$

Consider the logarithmic embedding

$$\lambda := \lambda_{K,S} \colon U \to \mathbb{R}X$$

which is defined by

$$\lambda(u) := \sum_{w \in S_K} \log |u|_w w.$$

Then Dirichlet's unit theorem states that $\lambda$ induces the $\mathbb{C}[G]$-isomorphism

$$\mathrm{id}_{\mathbb{C}} \otimes \lambda \colon \mathbb{C}U \cong \mathbb{C}X.$$

In particular we see that $\mathbb{Q}[G]$-modules $\mathbb{Q}U, \mathbb{Q}X$ has the same character. Therefore there exists a (non-canonical) $\mathbb{Q}[G]$-module isomorphism

$$f \colon \mathbb{Q}X \cong \mathbb{Q}U.$$

It follows from, for example, [Se, Proposition 32, §12.1]. We consider the automorphism

$$\begin{aligned} (\lambda \circ f)_V \colon \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) &\to \mathrm{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X), \\ \phi &\mapsto (\mathrm{id}_{\mathbb{C}} \otimes \lambda) \circ (\mathrm{id}_{\mathbb{C}} \otimes f) \circ \phi. \end{aligned}$$

Now we define the Stark regulator $R_S(\chi)$ by

$$R_S(\chi) = R_S(\chi, f) := \det(\lambda \circ f)_V.$$

**1.1.3. Some results for the non-abelian Stark conjecture.**

- The truth of Conjecture (1.1) does not depend on the choice of $S$ and $f$. For the proof, see [Da, §3.6 and Proposition 3.7.2].
- If Conjecture (1.1) holds true for $k = \mathbb{Q}$, then it also holds true for any $k$ as well. For the proof, see [Da, Proposition 3.7.3].
- If Conjecture (1.1) holds true for any abelian extension $K/k$, then it also holds true for any Galois extension $K/k$ as well. For the proof, see [Da, Proposition 3.7.3].
- If $r_S(\chi) = 0$, then $R_S(\chi) = 1$. Therefore Stark's conjecture (1.1) with $r_S(\chi) = 0$ is equivalent to
$$L_S(0, \chi)^\gamma = L_S(0, \chi^\gamma) \quad (\forall \gamma \in \mathrm{Aut}(\mathbb{C})),$$
which follows from a result of Siegel in [Si].
- When $\chi = 1_G$, Stark's conjecture (1.1) follows from the class number formula. For the proof, see [Da, Proposition 3.7.4].
- When $\mathbb{Q}(\chi) = \mathbb{Q}$ then Stark's conjecture (1.1) holds true. For the proof, see §9 in [Da] or Yamamoto's article (Japanese) in [SS2012].

## § 1.2. The rank 1 abelian Stark conjecture.

In this subsection we introduce a refinement of Conjecture (1.1) under the following additional assumption.

1. $K/k$ is abelian.
2. $S$ contains all ramified places in $K/k$ and all infinite places.
3. $S$ contains a distinguished place $v$ which splits completely in $K/k$. We fix a place $w$ of $K$ lying above $v$.
4. $|S| \geq 2$.

Let $e_K$ be the number of roots of unity in $K$, and $\widehat{G}$ the group of irreducible characters of $G$. We define the $S$-truncated partial zeta function $\zeta_S(s, \sigma)$ associated to $\sigma \in G$ by

$$\zeta_S(s, \sigma) := \sum_{(\mathfrak{a}, S)=1,\ (\mathfrak{a}, K/k)=\sigma} N\mathfrak{a}^{-s}.$$

Here $\mathfrak{a}$ runs over all integral ideals prime to any prime ideal in $S$ whose image under the Artin symbol $(*, K/k)$ is equal to $\sigma$. Note that we have

$$\zeta_S(s,\sigma) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \overline{\chi}(\sigma) L_S(s,\chi),$$
$$L_S(s,\chi) = \sum_{\sigma \in G} \chi(\sigma)\zeta_S(s,\sigma).$$

Therefore we can show that the assumptions 3,4 implies $\zeta_S(0,\sigma) = 0$ $(\forall \sigma \in G)$ by using formulas on $r_S(\chi)$ in §1.1.1.

**Conjecture 1.2** ($\mathrm{St}(K/k, S, v, w)$). *Under the assumptions 1,2,3,4, there exists an element $\varepsilon = \varepsilon(K/k, S, v, w) \in K^\times$ satisfying*

- *If $|S| > 2$, then $\varepsilon$ is a $\{v\}$-unit.*
- *If $|S| = 2$, put $S =: \{v, v'\}$. Then $\varepsilon$ is an $S$-unit and $|\varepsilon|_{w'}$ stays constant when $w'|v'$.*
- $\log |\varepsilon^\sigma|_w = -e_K \zeta'_S(0,\sigma) \quad (\forall \sigma \in G)$.
- *$K(\varepsilon^{1/e_K})/k$ is an abelian extension.*

*Note that such an element $\varepsilon$ is unique up to roots of unity, if it exists. We call Conjecture* $\mathrm{St}(K/k, S, v, w)$ *the rank 1 abelian Stark conjecture, and the element $\varepsilon$ a Stark unit.*

### 1.2.1. Some results for the rank 1 abelian Stark conjecture.

1. If $r_S(\chi) > 1$ for any $\chi \in \widehat{G}$, then $\zeta'_S(0,\sigma) = 0$ for any $\sigma \in G$. In this case, Conjecture $\mathrm{St}(K/k, S, v, w)$ is trivial with $\varepsilon = 1$.
2. The truth of Conjecture $\mathrm{St}(K/k, S, v, w)$ does not depend on the choice of $v, w$ (for the proof, see [Da, Remark 4.3.3, Proposition 4.3.4]). So we may write Conjecture $\mathrm{St}(K/k, S, v)$ or Conjecture $\mathrm{St}(K/k, S)$.
3. Under the assumptions 1,2,3,4, Conjecture $\mathrm{St}(K/k, S)$ implies Conjecture (1.1) for all $\chi$ with $r_S(\chi) = 1$. For the proof, see §4 in [Da] or the author's article (Japanese) in [SS2012].
4. Conjecture $\mathrm{St}(K/k, S)$ is related to Hilbert's 12th problem. Assume that $G$ is cyclic and $v$ is the only place in $S$ which splits completely in $K/k$. In this case, Conjecture $\mathrm{St}(K/k, S, v, w)$ implies $K = k(\varepsilon)$. Additionally assume that $v$ is real. We may regard that $k, K$ are subfields of $\mathbb{R}$, that is, $k \subset K \subset K_w = \mathbb{R}$. Then the Stark unit $\varepsilon$ is given by $\varepsilon = \exp(-2\zeta'_S(0, \mathrm{id}))$ [Da, Remark 4.3.3].

5. If $|G| = 2$ then Conjecture $\mathrm{St}(K/k, S)$ holds true. There is a partial result by Sands when $G$ has exponent 2. For detail, see §7 in [Da].

6. When $k = \mathbb{Q}$ or an imaginary quadratic field, Conjecture $\mathrm{St}(K/k, S)$ holds true. For the proof, see [Ta1], [St], or [SS2012, the author's article (Japanese) in the case of $k = \mathbb{Q}$, $v = \infty$, Onodera's article (Japanese) in the case of imaginary quadratic fields]. There is a sketch of a proof in the case of $k = \mathbb{Q}$ below.

7. When $k$ is a real quadratic field, Shintani independently formulated a conjecture which is almost equivalent to Conjecture $\mathrm{St}(K/k, S)$ in [Shin].

For example let $k = \mathbb{Q}$, $K := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ ($\zeta_m := \exp(\frac{2\pi i}{m})$), $S := \{p|m\} \cup \{\infty\}$, $v := \infty$. Define an element $\sigma_{\pm a} \in G = \mathrm{Gal}(K/\mathbb{Q})$ by

$$\sigma_{\pm a} \colon \zeta_m + \zeta_m^{-1} \mapsto \zeta_m^a + \zeta_m^{-a}$$

for $a \in \mathbb{Z}$ with $0 < a < m$, $(a, m) = 1$. Then we can write

$$\zeta_S(s, \sigma_{\pm a}) = \zeta(s, m, a) + \zeta(s, m, m - a)$$

by using the Hurwitz zeta function

$$\zeta(s, m, a) := \sum_{n=0}^{\infty} (a + nm)^{-s}.$$

Recall Lerch's formula

$$\zeta'(0, m, a) = \log(\Gamma(\frac{a}{m})(2\pi)^{-\frac{1}{2}} m^{\frac{2a-m}{2m}})$$

and Euler's formulas

$$\frac{\pi}{\sin(z\pi)} = \Gamma(z)\Gamma(1 - z),$$
$$\sin(z) = \frac{\exp(zi) - \exp(-zi)}{2i}.$$

Then we see that

$$\exp(-2\zeta_S'(0, \sigma_{\pm a})) = \zeta_m^a + \zeta_m^{-a} - 2 = -\zeta_m^a(1 - \zeta_m^{-a})^2.$$

Therefore we see that $\mathrm{St}(\mathbb{Q}(\zeta_m + \zeta_m^{-1})/\mathbb{Q}, S, \infty)$ holds true with $\varepsilon = \zeta_m + \zeta_m^{-1} - 2$ and that the Stark unit in this case is essentially equal to a cyclotomic unit. On the other hand, Stark units in the case of $k = \mathbb{Q}$, $v = p < \infty$ are essentially equal to (products of) Gauss sums. For example, let $K/k := \mathbb{Q}(\zeta_{p-1})/\mathbb{Q}$, $S := \{l|p(p-1)\} \cup \{\infty\}$ ($p \neq 2, 3$).

Then $v := p$ splits completely in $\mathbb{Q}(\zeta_{p-1})/\mathbb{Q}$. Fix a prime ideal $w := \mathfrak{P}$ of $\mathbb{Q}(\zeta_{p-1})$ lying above $p$. Define an element $\sigma_a \in G = \mathrm{Gal}(\mathbb{Q}(\zeta_{(p-1)})/\mathbb{Q})$ by

$$\sigma_a \colon \zeta_{p-1} \mapsto \zeta_{p-1}^a$$

for $a \in \mathbb{Z}$ with $0 < a < p-1$, $(a, p-1) = 1$. Since $\zeta_S(s, \sigma_a) = (1 - p^{-s})\zeta_{S-\{p\}}(s, \sigma_a)$, $(1-p^{-s})|_{s=0} = 0$, $\frac{d}{ds}(1-p^{-s})|_{s=0} = \log p$, $\zeta_{S-\{p\}}(0, \sigma_a) = \zeta(0, p-1, a) = \frac{1}{2} - \frac{a}{p-1}$, we have

$$\zeta_S'(0, \sigma_a) = \log p \cdot \left(\frac{1}{2} - \frac{a}{p-1}\right).$$

There exists a unique homomorphism $\chi = \chi_{\mathfrak{P}} \colon (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}[\zeta_{p-1}]^\times$ satisfying

$$\chi(x) \bmod \mathfrak{P} = x \text{ (as an element in } \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}[\zeta_{p-1}]/\mathfrak{P})$$

for all $x \in (\mathbb{Z}/p\mathbb{Z})^\times$. We put

$$G(\chi) := \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(x)\zeta_p^x.$$

Then we can show that

$$\varepsilon := \frac{G(\chi)^{p-1}}{p^{\frac{p-1}{2}}} \in \mathbb{Q}(\zeta_{p-1})$$

satisfies the condition of $\mathrm{St}(\mathbb{Q}(\zeta_{p-1})/\mathbb{Q}, S, p)$.

### § 1.3. Rubin's integral refinement for the abelian Stark conjecture.

Assume that $K/k$ is abelian and put $\widehat{G}$ to be the group of all irreducible characters of $G$. For $\chi \in \widehat{G}$, we define

$$e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \overline{\chi}(\sigma)\sigma \in \mathbb{C}[G].$$

Note that $e_\chi$ is the idempotent associated to $\chi$. Then related to the statement (1.1) of Stark's conjecture, we have the following equivalence.

$$\begin{aligned} &A_S(\chi, f)^\gamma = A_S(\chi^\gamma, f) \quad (\forall \gamma \in \mathrm{Aut}(\mathbb{C}),\ \forall \chi \in \widehat{G}) \\ \Leftrightarrow &\sum_{\chi \in \widehat{G}} A_S(\chi, f)\chi(\sigma) \in \mathbb{Q} \quad (\forall \sigma \in G) \\ \Leftrightarrow &\sum_{\chi \in \widehat{G}} A_S(\chi, f)e_{\overline{\chi}} \in \mathbb{Q}[G]. \end{aligned}$$

Therefore, it is natural to consider the "$G$-equivariant $L$-function"

$$\sum_{\chi \in \widehat{G}} L_S(s, \chi)e_{\overline{\chi}}$$

and study the ratio

$$\frac{\text{the leading coefficient of the } G\text{-equivariant } L\text{-function at } s=0}{\text{a "}G\text{-equivariant regulator"}}.$$

In this subsection we present Rubin's conjecture on such ratios, which is a refinement of Stark's conjecture (1.1) in the case of abelian extensions $K/k$, and is a generalization of Conjecture $\mathrm{St}(K/k, S)$ to the higher order case.

**1.3.1. Rubin's integral refinement.** Let $K/k$ be a finite abelian extension of number fields with $G = \mathrm{Gal}(K/k)$. (We can formulate Rubin's conjecture in the case of global function fields similarly.) We denote by $\widehat{G}$, $\mu_K$ the group of irreducible characters of $G$, the group of roots of unity in $K$ respectively. Different from previous sections,

we take two finite and non-empty sets $S, T$ of places of $k$.

For $r = 0, 1, 2, \ldots$, we consider the following assumption.

**Definition 1.3** (Assumption $(H_r)$)**.**

1. $S$ contains all infinite places of $k$ and all ramified places in $K/k$.
2. $T \cap S = \emptyset$.
3. $\{\zeta \in \mu_K \mid \zeta \equiv 1 \bmod T_K\} = \{1\}$.
4. $S$ contains more than or equal to $r$ places which split completely in $K/k$.
5. $|S| \geq r + 1$.

Here we define $x \equiv 1 \bmod T_K$ by $x \equiv 1 \bmod w$ $(\forall w \in T_K)$.

*Remark.* Assumption $(H_r)$-4,5 implies $r_S(\chi) = \mathrm{ord}_{s=0} L_S(s, \chi) \geq r$ $(\forall \chi \in \widehat{G})$.

**Definition 1.4.** Let $e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \overline{\chi}(\sigma)\sigma \in \mathbb{C}[G]$. We put

$$\Theta_S(s) := \sum_{\chi \in \widehat{G}} L_S(s, \chi) e_{\overline{\chi}},$$

$$\Theta_{S,T}(s) := \left( \prod_{\mathfrak{p} \in T} (1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} N\mathfrak{p}^{1-s}) \right) \Theta_S(s).$$

Then $\Theta_S(s)$ (resp. $\Theta_{S,T}(s)$) is a $\mathbb{C}[G]$-valued meromorphic (resp. holomorphic) function. Under Assumption $(H_r)$, we put

$$\Theta_S^{(r)}(0) := \lim_{s \to 0} \frac{\Theta_S(s)}{s^r},$$

$$\Theta_{S,T}^{(r)}(0) := \lim_{s \to 0} \frac{\Theta_{S,T}(s)}{s^r}.$$

*Remark.* We can write

$$\Theta_{S,T}^{(r)}(0) = \delta_T \Theta_S^{(r)}(0)$$

with $\delta_T := \prod_{\mathfrak{p}\in T}(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} N\mathfrak{p}) \in \mathbb{Q}[G]^\times$.

As usual we denote the ring of $S$-integers of $K$ by

$$\mathcal{O}_S := \mathcal{O}_{K,S} := \{x \in K \mid |x|_w \leq 1 \ (\forall w \notin S_K)\}.$$

We put

$$U_S := U_{K,S} := \mathcal{O}_{K,S}^\times,$$

and

$$U_{S,T} := U_{K,S,T} := \{x \in U_{K,S} \mid x \equiv 1 \bmod T_K\}.$$

Note that $[U_S : U_{S,T}] < \infty$.

*Remark.* Assumption $(H_r)$-3 implies $U_{S,T}$ is torsion-free.

**Definition 1.5.** Under Assumption $(H_r)$, we take $r$ places $v_1, \dots, v_r \in S$ which split completely in $K/k$. Choose a place $w_i$ of $K$ dividing $v_i$ for each $i$. We define the $G$-equivariant regulator map

$$R_W \in \mathrm{Hom}_{\mathbb{Z}[G]}\left(\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}, \mathbb{C}[G]\right)$$

with respect to $W := (w_1, w_2, \dots, w_r)$ by

$$R_W\,(u_1 \wedge \cdots \wedge u_r) := \det\left(\left(-\sum_{\sigma\in G} \log |u_i^{\sigma^{-1}}|_{w_j} \sigma\right)_{1\leq i,j\leq r}\right).$$

We put

$$e_{S,r} := \sum_{\chi\in\widehat{G},\ r_S(\chi)=r} e_\chi.$$

We see that $e_{S,r} \in \mathbb{Q}[G]$ since for all $\gamma \in \mathrm{Aut}(\mathbb{C})$, we have $r_S(\chi) = r_S(\chi^\gamma)$. It is clear that Assumption $(H_r)$ implies

$$\Theta_{S,T}^{(r)}(0) \in e_{S,r}\mathbb{C}[G].$$

Moreover we can show that Dirichlet's unit theorem gives the $\mathbb{C}[G]$-isomorphism

$$\mathrm{id}_{\mathbb{C}} \otimes R_W \colon e_{S,r}\left(\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\right) \cong e_{S,r}\mathbb{C}[G].$$

Therefore we can define

$$\varepsilon_{S,T,r} := (\mathrm{id}_{\mathbb{C}} \otimes R_W)^{-1}\left(\Theta_{S,T}^{(r)}(0)\right) \in e_{S,r}\left(\mathbb{C}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\right).$$

*Remark.* We can write the relation between Stark's regulator $R_S(\chi, f)$ and Rubin's $G$-equivariant regulator map $R_W$ as follows: We can take $f\colon \mathbb{Q}X_S \cong \mathbb{Q}U_S$ and $\varepsilon_i \in e_{S,r}\mathbb{Q}U_S$ so that

$$(\mathrm{id}_{\mathbb{C}} \otimes R_W)(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r) = \sum_{\chi\in\widehat{G},\ r_S(\chi)=r} R_S(\chi, f)e_{\overline{\chi}}.$$

We note that this relation "corresponds" to

$$\begin{aligned}\Theta_S^{(r)}(0) &= \sum_{\chi\in\widehat{G}}\left(\lim_{s\to 0}\frac{L_S(s,\chi)}{s^r}\right)e_{\overline{\chi}}\\ &= \sum_{\chi\in\widehat{G},\ r_S(\chi)=r} \text{"the leading coefficient of } L_S(s,\chi)\text{"}\ e_{\overline{\chi}}.\end{aligned}$$

Associated to $\Phi := (\phi_1, \dots, \phi_{r-1}) \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}[G])^{r-1}$, we define

$$\begin{aligned}&\widetilde{\Phi} \in \mathrm{Hom}_{\mathbb{Q}[G]}\left(\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}, \mathbb{Q}U_{S,T}\right),\\ &\widetilde{\Phi}(u_1 \wedge \cdots \wedge u_r) := \sum_{k=1}^{r}(-1)^k \det\left((\phi_i(u_j))_{j\neq k}\right)u_k.\end{aligned}$$

Here $(\phi_i(u_j))_{j\neq k}$ is a matrix of the size $(r-1)\times(r-1)$ where $i$ runs over the range $1 \le i \le r-1$, $j$ runs over the range $1 \le j \le r$, $j \neq k$.

**Definition 1.6.**

$$\Lambda_{S,T,r} := \left\{\varepsilon \in e_{S,r}\left(\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\right) \mid \widetilde{\Phi}(\varepsilon) \in U_{S,T}\ (\forall\Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}[G])^{r-1})\right\}.$$

We now state Rubin's integral refinement for the abelian Stark conjecture.

**Conjecture 1.7.** *Under Assumption $(H_r)$, we have*

$$\varepsilon_{S,T,r} \in \Lambda_{S,T,r}. \tag{1.2}$$

*Remark.* For any abelian extension $K/k$ of number fields, the following are equivalent.

1. Conjecture (1.1) holds true for any $\chi$ with $r_S(\chi) = r$.

2. Under Assumption $(H_r)$, we have $\varepsilon_{S,T,r} \in e_{S,r}\left(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\right) = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{S,T,r}$.

Therefore the above conjecture is called Rubin's integral refinement for the abelian Stark conjecture.

#### 1.3.2. Some results for Rubin's integral refinement.

1. The truth of Rubin's conjecture (1.2) does not depend on the choice of places $W = (w_1, \ldots, w_r)$ [BPSS, Remark 2, §2.1 in Popescu's article].

2. When $K/k$ is a quadratic extension of number fields, Rubin's conjecture (1.2) holds true [Ru, Theorem 2.5].

3. Under Assumption $(H_r)$-1,4,5, Rubin's conjecture (1.2) implies Stark's conjecture (1.1) for all $\chi$ with $r_S(\chi) = r$. For the proof, see [Ru] or the author's article in [SS2012] (Japanese).

4. When $r = 0$, Rubin's conjecture (1.2) only states $\Theta_{S,T}(0) \in \mathbb{Z}[G]$. It follows from a result of Deligne-Ribet.

5. When $r = 1$, the following are equivalent [Ru, Proposition 2.5].

   (a) Rubin's conjecture (1.2) with $r = 1$ holds true for all $T$.

   (b) Conjecture $\mathrm{St}(K/k, S)$ holds true.

6. In the case of $r = 1$ and $v_1 < \infty$, Rubin's conjecture (1.2) is equivalent to the Brumer-Stark conjecture and there are some partial results. We will formulate the Brumer-Stark conjecture in the next subsection. There is a survey on this topic by Miura in [SS2012] (Japanese).

### § 1.4. The Brumer-Stark conjecture.

In this subsection, we study Conjecture $\mathrm{St}(K/k, S, v, w)$ in the case of finite places $v, w$. We write $v = \mathfrak{p}$, $w = \mathfrak{P}$. We assume that $|S| > 2$ and put $R := S - \{\mathfrak{p}\}$. Since $\mathfrak{p}$ splits completely, we see $(\mathfrak{p}, K/k) = \mathrm{id}_K$. Therefore we have

$$\begin{aligned}
\zeta_S(s,\sigma) &= \sum_{(\mathfrak{a},S)=1,\ (\mathfrak{a},K/k)=\sigma} N\mathfrak{a}^{-s} \\
&= \sum_{(\mathfrak{a},R)=1,\ (\mathfrak{a},K/k)=\sigma} N\mathfrak{a}^{-s} - \sum_{(\mathfrak{a},R)=1,\ \mathfrak{p}|\mathfrak{a},\ (\mathfrak{a},K/k)=\sigma} N\mathfrak{a}^{-s} \\
&= (1 - N\mathfrak{p}^{-s})\zeta_R(s,\sigma).
\end{aligned}$$

So we get

$$\zeta'_S(0,\sigma) = \zeta_R(0,\sigma) \cdot \log N\mathfrak{p}.$$

Hence Conjecture $\mathrm{St}(K/k, S, \mathfrak{p}, \mathfrak{P})$ implies that

$$\theta\mathfrak{P} \text{ is a principal ideal } (= (\varepsilon))$$

for

$$\theta := \sum_{\sigma\in G} e_K \zeta_R(0,\sigma)\sigma^{-1}.$$

We note that we have $e_K\zeta_R(0,\sigma) \in \mathbb{Z}$ with $e_K := |\mu_K|$ by a result of Deligne-Ribet and that the action of $\sum_{\sigma\in G} n_\sigma \sigma \in \mathbb{Z}[G]$ is defined by $(\sum_{\sigma\in G} n_\sigma\sigma)\mathfrak{a} := \prod_{\sigma\in G}(\sigma(\mathfrak{a}))^{n_\sigma}$. Moreover, we can show that the truth of Conjecture $\mathrm{St}(K/k, S, \mathfrak{p}, \mathfrak{P})$ for all $\mathfrak{p}, \mathfrak{P}$ is equivalent to the truth of the following conjecture, which is called the Brumer-Stark conjecture.

**Conjecture 1.8.** *Let the notation be as in §1.3.1. Additionally let $e_K := |\mu_K|$, $\pi$ the natural map $K^\times \to \mathbb{Q}K^\times$, and*

$$K^\times_{S,0} := \{x \in K^\times \mid \pi(x) \in e_{S,0}\mathbb{Q}K^\times\}.$$

*Then under Assumption $(H_0)$-1, for any fractional ideal $I \subset K$, there exists $\alpha_I \in K^\times_{S,0}$ satisfying*

$$e_K\Theta_S(0)I = (\alpha_I),$$
$$K(\alpha_I^{1/e_K})/k \text{ is abelian.}$$

*Remark.* There exists a Strong version of the Brumer-Stark conjecture. Let $A_{K,S,T}$ be the $(S,T)$-modified ideal class group. That is

$$A_{K,S,T} := \frac{\{\text{ fractional ideals } \mathfrak{a} \text{ of } \mathcal{O}_{K,S} \mid (\mathfrak{a}, T_K) = 1\}}{\{\text{ principal ideals } (x) \text{ of } \mathcal{O}_{K,S} \mid x \equiv 1 \bmod T_K\}}.$$

Then the Strong Brumer-Stark "conjecture" states that

Under the assumption $(H_0)$, we have $\Theta_{S,T}(0) \in (\mathbb{Z}[G] \cap e_{S,0}\mathbb{Q}[G]) \cdot \mathrm{Fitt}_{\mathbb{Z}[G]}(A_{K,S,T})$.

Here we denote the Fitting ideal by Fitt. Noting that

$$\mathrm{Fitt}_{\mathbb{Z}[G]}(A_{S,T}) \subset \mathrm{Ann}_{\mathbb{Z}[G]}(A_{S,T}),$$

we can show that the Strong Brumer-Stark "conjecture" implies the Brumer-Stark conjecture. There are some partial results and some counterexamples for the Strong version.

## § 2. The Gross-Stark conjecture.

In this section, we present a formulation of the Gross-Stark conjecture, which is a $p$-adic analogue of the rank 1 abelian Stark conjecture. Hereafter we denote $H/F$ instead of $K/k$ as in [DDP]. Without loss of generality, we may assume that

- $F$ is a totally real field of degree $n$, $H$ is a totally complex field, and $H/F$ is a cyclic extension of conductor $\mathfrak{n}$.
- The character $\chi\colon G := \mathrm{Gal}(H/F) \to \overline{\mathbb{Q}}^{\times}$ is injective.

Take a rational prime $p$ and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$ for simplicity. Let $E := \mathbb{Q}_p(\chi(\sigma) \mid \sigma \in G)$ and $\omega$ the Teichmüller character. We denote the set of primes of $F$ dividing $p$ by $S_p$ and assume that

$$S_p \subset S.$$

Then the $p$-adic $L$-function $L_{S,p}(s, \chi\omega)$ is characterized by the following.

$$\text{There exists a unique } E\text{-valued continuous function } L_{S,p}(s,\chi\omega) \text{ satisfying}$$
$$L_{S,p}(n, \chi\omega) = L_S(n, \chi\omega^n)\ (\forall n \leq 0).$$

We can also show that $L_{S,p}(s, \chi\omega)$ is holomorphic at $s = 0$. Moreover Gross conjectured ([Gr, Conjecture 2.12]

$$\mathrm{ord}_{s=0} L_{S,p}(s, \chi\omega) = r_S(\chi).$$

We consider the simplest case. Namely assume that

$$S := S_\infty \cup S_p \cup \{v|\mathfrak{n}\}.$$

Then the above conjecture states

$$\mathrm{ord}_{s=0} L_{S,p}(s, \chi\omega) = |\{\mathfrak{q} \in S_p \mid \chi(\mathfrak{q}) = 1\}|.$$

*Remark.* Put $R := S - \{\mathfrak{q} \in S_p \mid \chi(\mathfrak{q}) = 1\}$. Then we can write

$$L_S(s, \chi) = \left( \prod_{\mathfrak{q} \in S_p,\ \chi(\mathfrak{q})=1} (1 - N\mathfrak{q}^{-s}) \right) L_R(s, \chi).$$

Therefore, $L_S(s, \chi)$ has a "trivial zero" of order $r_S(\chi) = |\{\mathfrak{q} \in S_p \mid \chi(\mathfrak{q}) = 1\}|$ at $s = 0$. However, it is non-trivial whether $L_{S,p}(s, \chi\omega)$ has a zero of the same order.

The following condition is essential for the Gross-Stark conjecture: We assume that $r_S(\chi) = 1$, i.e., there exists a unique prime ideal $\mathfrak{p} \in S_p$ satisfying $\chi(\mathfrak{p}) = 1$.

In particular, we see that $\mathfrak{p}$ splits completely in $H/F$ and that $\mathrm{ord}_{s=0} L_{S,p}(s, \chi\omega) \geq 1$. In this case, Dirichlet's unit theorem states that we have

$$\dim_E U_\chi = r_S(\chi) = 1$$

with

$$U_\chi := (\mathcal{O}_{H,S}^\times \otimes_{\mathbb{Z}} E)^{\chi^{-1}} = \{u \in \mathcal{O}_{H,S}^\times \otimes_{\mathbb{Z}} E \mid \sigma u = \chi^{-1}(\sigma)u \ (\forall \sigma \in G)\}.$$

Take a non-zero element $u_\chi$ of $U_\chi$ and a prime ideal $\mathfrak{P}$ of $H$ dividing $\mathfrak{p}$. We define $E$-linear maps $\mathrm{ord}_{\mathfrak{P}}$, $\mathbf{L}_{\mathfrak{P}}$ by

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}} &\colon U_\chi \to E, \ \varepsilon \otimes \alpha \mapsto \alpha \cdot \mathrm{ord}_{\mathfrak{P}} \varepsilon, \\ \mathbf{L}_{\mathfrak{P}} &\colon U_\chi \to E, \ \varepsilon \otimes \alpha \mapsto \alpha \cdot \log_p(N_{H_{\mathfrak{P}}/\mathbb{Q}_p} \varepsilon). \end{aligned}$$

Then the $\mathcal{L}$-invariant $\mathcal{L}(\chi)$ is defined by

$$\mathcal{L}(\chi) := -\frac{\mathbf{L}_{\mathfrak{P}}(u_\chi)}{\mathrm{ord}_{\mathfrak{P}}(u_\chi)}.$$

*Remark.* We can show that the value $\mathcal{L}(\chi) \in E$ does not depend on the choice of $u_\chi$, $\mathfrak{P}$. For example, write $\mathfrak{P}^{h_K} = (\pi)$ with $\pi \in H$, $h_K$ the class number of $K$. Then we see that $U_\chi \ni \sum_{\sigma \in G} \chi(\sigma) \otimes \pi^\sigma \neq 0$. Therefore there exists $t \in E^\times$ satisfying

$$u_\chi = t \sum_{\sigma \in G} \chi(\sigma) \otimes \pi^\sigma.$$

It is clear that

$$\begin{aligned} \mathbf{L}_{\mathfrak{P}}(u_\chi) &= t \sum_{\sigma \in G} \chi(\sigma) \log_p(N_{H_{\mathfrak{P}}/\mathbb{Q}_p} \pi^\sigma), \\ \mathrm{ord}_{\mathfrak{P}}(u_\chi) &= t h_K. \end{aligned}$$

Hence we get

$$\mathcal{L}(\chi) = \frac{\sum_{\sigma \in G} \chi(\sigma) \log_p(N_{H_{\mathfrak{P}}/\mathbb{Q}_p} \pi^\sigma)}{h_K},$$

which does not depend on the choice of $u_\chi$. By a similar argument, we can prove that it does not depend on the choice of $\mathfrak{P}$, either.

We now state the Gross-Stark conjecture.

**Conjecture 2.1** ([Gr], Conjecture 3.13)**.** *Let $F$ be a totally real field, $H$ a totally complex field with $H/F$ a cyclic extension of conductor $\mathfrak{n}$. Assume that the character $\chi\colon G = \mathrm{Gal}(H/F) \to \overline{\mathbb{Q}}^\times$ is injective, $S = S_\infty \cup S_p \cup \{v | \mathfrak{n}\}$, and $r_S(\chi) = 1$. Then we have*

$$L'_{S,p}(0, \chi\omega) = \mathcal{L}(\chi) L_R(0, \chi).$$

## § 3. A result of Dasgupta-Darmon-Pollack.

To describe the main result of [DDP], we need the following notations.

**Definition 3.1.**

$$\mathcal{L}_{\mathrm{an}}(s,\chi) := \frac{-L_{S,p}(1-s,\chi\omega)}{L_R(0,\chi)},$$
$$\mathcal{L}_{\mathrm{an}}(\chi) := \frac{L'_{S,p}(0,\chi\omega)}{L_R(0,\chi)} = \mathcal{L}'_{\mathrm{an}}(1,\chi).$$

Now here is the main Theorem.

**Theorem 3.2** ([DDP], Theorem 2). *Assume the following.*

$$\text{(3.1)}\qquad \begin{aligned} &\textit{If } |S_p|>1, \textit{ assume that Leopoldt's conjecture is true for } F.\\ &\textit{If } |S_p|=1, \textit{ assume that Leopoldt's conjecture is true for } F,\\ &\quad \textit{and that } \mathrm{ord}_{s=1}(\mathcal{L}_{\mathrm{an}}(s,\chi)+\mathcal{L}_{\mathrm{an}}(s,\chi^{-1})) = \mathrm{ord}_{s=1}\mathcal{L}_{\mathrm{an}}(s,\chi^{-1}).\end{aligned}$$

*Then the Gross-Stark conjecture holds true.*

### § 3.1. Cohomological interpretation.

We reformulate the Gross-Stark conjecture in terms of Galois cohomology in this subsection. To do this, we will use the following notations.

- We put $G_H := \mathrm{Gal}(\overline{F}/H) \subset G_F := \mathrm{Gal}(\overline{F}/F)$, $G_{F_v} := \mathrm{Gal}(\overline{F_v}/F_v)$ (for each place $v$ of F). Then we may regard $G = \mathrm{Gal}(H/F) = G_F/G_H$, $G_{F_v} \subset G_F$.
- We put $\mu_n := \{\zeta \in \overline{F} \mid \zeta^n = 1\}$, $\epsilon_{\mathrm{cyc}} \colon G_F \to \mathbb{Z}_p^\times$ to be the cyclotomic character. That is, we have
$$\sigma(\zeta) = \zeta^{\epsilon_{\mathrm{cyc}}(\sigma)}$$
for all $\zeta \in \mu_{p^n}$, $n \in \mathbb{N}$.
- We denote by $E(\chi^{-1})$, $E(1)$, $E(1)(\chi)$ the representation spaces over $E$ of characters $\chi^{-1}$, $\epsilon_{\mathrm{cyc}}$, $\chi\epsilon_{\mathrm{cyc}}$ respectively. As $E$-vector spaces, $E(\chi^{-1}) = E(1) = E(1)(\chi) = E$. For any $E$-vector space $V$, we put $V(\chi^{-1}) := V \otimes E(\chi^{-1})$, etc.
- We define elements $\kappa_{\mathrm{nr}} \in \mathrm{Hom}(G_{F_v}, E)$, $\kappa_{\mathrm{cyc}} \in \mathrm{Hom}(G_F, E)$ as follows.
$$\kappa_{\mathrm{nr}} \text{ is unramified and } \kappa_{\mathrm{nr}}(\mathrm{Frob}_v) := 1,$$
$$\kappa_{\mathrm{cyc}} := \log_p \circ \epsilon_{\mathrm{cyc}}.$$
When $\chi|_{G_{F_v}} = 1$, we use the same symbols $\kappa_{\mathrm{nr}}, \kappa_{\mathrm{cyc}}$ for the corresponding elements in $H^1(G_{F_v}, E(\chi^{-1})) = \mathrm{Hom}_{\mathrm{conti}}(G_{F_v}, E)$.

Moreover, we will use the following well-known results. For each place $v$ of $F$, we consider the perfect pairing

$$\langle\ ,\ \rangle_v\colon H^1(G_{F_v}, E(\chi^{-1})) \times H^1(G_{F_v}, E(1)(\chi)) \to E,$$

which is defined by "Tate's local duality" in §3.1.1. Then the global reciprocity law of class field theory gives the following relation.

$$\forall\kappa \in H^1(G_F, E(\chi^{-1})),\ \forall u \in H^1(G_F, E(1)(\chi)),\ \langle\kappa, u\rangle := \sum_v \langle \mathrm{res}_v\kappa, \mathrm{res}_v u\rangle_v = 0.$$

Here we put $\mathrm{res}_v f := f|_{G_{F_v}}$. As we will see in §3.1.2, we can embed

$$\delta\colon U_\chi \hookrightarrow H^1(G_F, E(1)(\chi))$$

by using "Kummer theory". In [DDP], the subspace

$$H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \subset H^1(G_F, E(\chi^{-1})),$$

which is characterized by (3.4), is constructed. This subspace "corresponds" to

$$\delta(U_\chi) \subset H^1(G_F, E(1)(\chi))$$

in the following sense:

- $\forall v \neq \mathfrak{p}$, $\mathrm{res}_v(H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1}))) \perp \mathrm{res}_v(\delta(U_\chi))$ w.r.t. $\langle\ ,\ \rangle_v$. That is,

$$\langle \mathrm{res}_v\kappa, \mathrm{res}_v u\rangle_v = 0 \quad (\forall v \neq \mathfrak{p},\ \forall\kappa \in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})),\ \forall u \in \delta(U_\chi)).$$

Moreover, it satisfies the following properties:

- $\mathrm{res}_\mathfrak{p}\colon H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \hookrightarrow E\cdot\kappa_{\mathrm{nr}} \oplus E\cdot\kappa_{\mathrm{cyc}}$. That is,

  $\mathrm{res}_\mathfrak{p}$ on $H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1}))$ is injective and
  $\mathrm{res}_\mathfrak{p}(H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1}))) \subset E\cdot\kappa_{\mathrm{nr}} \oplus E\cdot\kappa_{\mathrm{cyc}}$.

- $\dim_E H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) = 1$.

Therefore the Gross-Stark conjecture is equivalent to the following conjecture.

**Conjecture 3.3.** *There exists a non-trivial element* $\kappa \in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1}))$ *satisfying the following.*

*Write* $\mathrm{res}_\mathfrak{p}\kappa = x\cdot\kappa_{\mathrm{nr}} + y\cdot\kappa_{\mathrm{cyc}}$ *with* $x, y \in E$*. Then* $\mathcal{L}_{\mathrm{an}}(\chi) = -x/y$*.*

We can see the equivalence of two conjectures as follows: Put $\mathrm{res}_{\mathfrak{p}}\kappa = x\cdot\kappa_{\mathrm{nr}}+y\cdot\kappa_{\mathrm{cyc}}$ for $0 \neq \kappa \in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1}))$. Then we have

$$\begin{aligned} 0 = \langle\delta(u_\chi),\kappa\rangle &= \sum_v \langle \mathrm{res}_v(\delta(u_\chi)), \mathrm{res}_v(\kappa)\rangle_v \\ &= \langle \mathrm{res}_{\mathfrak{p}}(\delta(u_\chi)), \mathrm{res}_{\mathfrak{p}}(\kappa)\rangle_{\mathfrak{p}} \\ &= x\langle \mathrm{res}_{\mathfrak{p}}(\delta(u_\chi)), \kappa_{\mathrm{nr}}\rangle_{\mathfrak{p}} + y\langle \mathrm{res}_{\mathfrak{p}}(\delta(u_\chi)), \kappa_{\mathrm{cyc}}\rangle_{\mathfrak{p}}. \end{aligned}$$

On the other hand, the reciprocity law of local class field theory states that

$$\begin{aligned} \langle\delta(u),\kappa_{\mathrm{nr}}\rangle_{\mathfrak{p}} &= -\mathrm{ord}_{\mathfrak{P}}(u), \\ \langle\delta(u),\kappa_{\mathrm{cyc}}\rangle_{\mathfrak{p}} &= \mathbf{L}_{\mathfrak{P}}(u). \end{aligned}$$

Combining these, we get the desired result.

**3.1.1. Tate's local duality.** Let $v$ be a finite place of $F$, $V$ a finite-dimensional representation of $G_{F_v}$ over $E$. We have the following perfect pairing, which is called "Tate's local duality."

$$H^1(G_{F_v}, V) \times H^1(G_{F_v}, \mathrm{Hom}(V, E(1))) \to H^2(G_{F_v}, E(1)) = E.$$

Putting $V = E(\chi^{-1})$, we get the desired $E$-linear pairing

$$\langle\ ,\ \rangle_v \colon H^1(G_{F_v}, E(\chi^{-1})) \times H^1(G_{F_v}, E(1)(\chi)) \to E.$$

We define the unramified part of $H^1(G_{F_v}, V)$ by

$$H^1_{\mathrm{nr}}(G_{F_v}, V) := \mathrm{Ker}[H^1(G_{F_v}, V) \to H^1(I_v, V)].$$

Here we denote the inertia group of $G_{F_v}$ by $I_v$. Then we have

$$\begin{aligned} &H^1_{\mathrm{nr}}(G_{F_v}, E(\chi^{-1})) \cong H^1(G_{F_v}/I_v, (E(\chi^{-1}))^{I_v}), \\ &H^1_{\mathrm{nr}}(G_{F_v}, E(1)(\chi)) \\ &\ = \{u \in H^1(G_{F_v}, E(1)(\chi)) \mid \langle\kappa, u\rangle_v = 0\ \forall\kappa \in H^1_{\mathrm{nr}}(G_{F_v}, E(\chi^{-1}))\}. \end{aligned}$$

We can calculate the dimension of each space [DDP, Lemma 1.3 and §1.2]:

$$\begin{aligned} &\dim_E H^1(G_{F_v}, E(\chi^{-1})) = \dim_E H^1(G_{F_v}, E(1)(\chi)) \\ &= \begin{cases} [F_v : \mathbb{Q}_p] & \text{if } v|p, \chi|_{G_{F_v}} \neq 1, \\ [F_v : \mathbb{Q}_p] + 1 & \text{if } v|p, \chi|_{G_{F_v}} = 1, \\ 1 & \text{if } v \nmid p\infty,\ \chi|_{G_{F_v}} = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$H^1_{\mathrm{nr}}(G_{F_v}, E(\chi^{-1})) = \begin{cases} E \cdot \kappa_{\mathrm{nr}} & \text{if } \chi|_{G_{F_v}} = 1, \\ \{0\} & \text{otherwise,} \end{cases}$$

$$H^1_{\mathrm{nr}}(G_{F_v}, E(1)(\chi)) \cong \begin{cases} \mathcal{O}_{F_v}^{\times} \widehat{\otimes} E & \text{if } \chi|_{G_{F_v}} = 1, \\ H^1(G_{F_v}, E(1)(\chi)) & \text{otherwise.} \end{cases}$$

Here we write the completed tensor product by $* \widehat{\otimes} E := (\lim_{\leftarrow} * \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} E$. In particular, we have

$$H^1_{\mathrm{nr}}(G_{F_v}, E(\chi^{-1})) = H^1(G_{F_v}, E(\chi^{-1})) \quad \text{if } v \nmid p, \tag{3.2}$$

$$H^1_{\mathrm{nr}}(G_{F_v}, E(1)(\chi)) = H^1(G_{F_v}, E(1)(\chi)) \quad \text{if } \chi|_{G_{F_v}} \neq 1. \tag{3.3}$$

**3.1.2. Kummer Theory.** The connecting homomorphism of “Kummer theory” gives an isomorphism

$$H^{\times}/(H^{\times})^n \cong H^1(G_H, \mu_n).$$

Therefore we get $H^{\times} \widehat{\otimes} E \cong H^1(G_H, E(1))$. Moreover we have an isomorphism

$$\delta\colon (H^{\times} \widehat{\otimes} E)^{\chi^{-1}} \cong H^1(G_F, E(1)(\chi))$$

by

$$(H^{\times} \widehat{\otimes} E)^{\chi^{-1}} \cong H^1(G_H, E(1))^{\chi^{-1}} \cong H^1(G_H, E(1)(\chi))^G \cong H^1(G_F, E(1)(\chi)).$$

Here we used the exact sequence

$$H^1(G, E(1)(\chi)^{G_H}) \to H^1(G_F, E(1)(\chi)) \overset{\mathrm{res}}{\to} H^1(G_H, E(1)(\chi))^G \to H^2(G, E(1)(\chi)^{G_H})$$

and $E(1)(\chi)^{G_H} = \{0\}$. For the local field, we similarly get

$$H_w^{\times} \widehat{\otimes} E \cong H^1(G_{H_w}, E(1)),$$

$$\begin{array}{ccc} (H_w^{\times} \widehat{\otimes} E)^{\chi^{-1}} & \cong & H^1(G_{F_v}, E(1)(\chi)) \\ \cup & & \cup \\ (\mathcal{O}_{H_w}^{\times} \widehat{\otimes} E)^{\chi^{-1}} & \cong & H^1_{\mathrm{nr}}(G_{F_v}, E(1)(\chi)). \end{array}$$

Therefore if we put $H^1_{\mathfrak{p}}(G_F, E(1)(\chi)) := \delta(U_\chi)$, then we can write

$$\begin{aligned} &H^1_{\mathfrak{p}}(G_F, E(1)(\chi)) \\ &= \{u \in H^1(G_F, E(1)(\chi)) \mid \mathrm{res}_v(u) \in H^1_{\mathrm{nr}}(G_{F_v}, E(1)(\chi)) \ (\forall v \notin S)\}. \end{aligned}$$

Moreover we see that by (3.3)

$$\begin{aligned} &H^1_{\mathfrak{p}}(G_F, E(1)(\chi)) \\ &= \{u \in H^1(G_F, E(1)(\chi)) \mid \mathrm{res}_v(u) \in H^1_{\mathrm{nr}}(G_{F_v}, E(1)(\chi)) \ (\forall v \neq \mathfrak{p})\}. \end{aligned}$$

(For detail, see [DDP, Proposition 1.4].) Now we put

$$
\begin{aligned}
&H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \\
&:= \mathrm{res}_{\mathfrak{p}}^{-1}(E\cdot\kappa_{\mathrm{nr}} \oplus E\cdot\kappa_{\mathrm{cyc}}) \cap \left(\bigcap_{v\neq\mathfrak{p}} \mathrm{res}_v^{-1}(H^1_{\mathrm{nr}}(G_{F_v}, E(\chi^{-1})))\right).
\end{aligned} \tag{3.4}
$$

Then we have

$$\langle \mathrm{res}_v(\kappa), \mathrm{res}_v(\delta(u))\rangle_v = 0 \text{ for } v\neq\mathfrak{p},\ \kappa\in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})),\ u\in U_\chi.$$

Note that by (3.2) we can write

$$
\begin{aligned}
&H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \\
&= \mathrm{res}_{\mathfrak{p}}^{-1}(E\cdot\kappa_{\mathrm{nr}} \oplus E\cdot\kappa_{\mathrm{cyc}}) \cap \left(\bigcap_{v\in S_p-\{\mathfrak{p}\}} \mathrm{res}_v^{-1}(H^1_{\mathrm{nr}}(G_{F_v}, E(\chi^{-1})))\right).
\end{aligned} \tag{3.5}
$$

Furthermore, by using Poitou-Tate exact sequence, we can show the following properties [DDP, Lemma 1.5]

$$
\begin{aligned}
&\dim_E H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) = 1, \\
&\mathrm{res}_{\mathfrak{p}} \colon H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \hookrightarrow E\cdot\kappa_{\mathrm{nr}} \oplus E\cdot\kappa_{\mathrm{cyc}}.
\end{aligned}
$$

## § 3.2. A very rough sketch of the proof of the main theorem.

We reduced the problem to the construction of a cocycle in the previous subsection. Before we go into details we shall give a sketch of the construction in [DDP]. We put

$$
\begin{aligned}
\mathfrak{n}_R &:= \mathrm{lcm}(\mathfrak{n}, \prod_{\mathfrak{p}\neq\mathfrak{q}\in S_p} \mathfrak{q}), \\
\mathfrak{n}_S &:= \mathrm{lcm}(\mathfrak{n}, \prod_{\mathfrak{q}\in S_p} \mathfrak{q}),
\end{aligned}
$$

and denote the character modulo $\mathfrak{n}_R$ (resp. $\mathfrak{n}_S$) associated to $\chi$ by $\chi_R$ (resp. $\chi_S$).

1. We denote the Eisenstein series of weight $k$ associated to characters $\eta, \psi$ by $E_k(\eta,\psi)$. $E_k(\eta,\psi)$ is characterized by its $\mathfrak{m}$-th Fourier coefficients $c(\mathfrak{m}, E_k(\eta,\psi))$, that is

$$c(\mathfrak{m}, E_k(\eta,\psi)) = \sum_{\mathfrak{r}\mid\mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})N\mathfrak{r}^{k-1}$$

   for all non-zero integral ideals $\mathfrak{m}$ of $\mathcal{O}_F$. Here $\mathfrak{r}$ runs over all integral ideals dividing $\mathfrak{m}$. Consider the product of Eisenstein series

$$P_k := E_1(1,\chi_R)\cdot\frac{2^n}{L(2-k,\omega^{1-k})}E_{k-1}(1,\omega^{1-k}).$$

   Then we see that $P_k$ is a Hilbert modular form of weight $k$, level $\mathfrak{n}_S$, character $\chi\omega^{1-k}$.

2. By well-known results for (a family of) Eisenstein series, we see that the family $\{P_k\}_k$ becomes a $\Lambda$-adic Hilbert modular form. Since we have to apply Wiles' results on ordinary $\Lambda$-adic cusp forms, we shall modify the family $\{P_k\}_k$ as in §3.3 in order to get a family of ordinary cusp forms. First we take the ordinary part $eP_k$ by Hida's ordinary operator $e$. By a general theory of Hilbert modular forms, we can uniquely write $eP_k$ as

$$eP_k = \text{“an ordinary cusp form”} + \sum_{j\in J} a_k(\eta_j,\psi_j)E_k(\eta_j,\psi_j),$$

where $J$ is a finite set of indices, $a_k(\eta_j,\psi_j)$ are constants, $E_k(\eta_j,\psi_j)$ are ordinary Eisenstein series. We can remove some of $\{a_k(\eta_j,\psi_j)E_k(\eta_j,\psi_j)\}_{j\in J}$ by multiplying $eP_k$ by $T_j-\alpha_j$ with $T_j$ a suitable Hecke operator, $\alpha_j$ the eigenvalue for $T_j$, $E_k(\eta_j,\psi_j)$ (for detail, see [DDP, Lemma 2.9] or Lemma 3.9 in this paper). Here each operator $T_j$ has to satisfy a certain condition (that is, $(T_j-\alpha_j)E_1(1,\chi_S)\neq 0$) in order to guarantee that the weight 1 specialization of the ordinary $\Lambda$-adic cusp form obtained finally is non-zero (that is, $\nu_1(\mathcal{F})\neq 0$ in the next step). When we can not remove $a_k(\eta_j,\psi_j)E_k(\eta_j,\psi_j)$ by Hecke operators for this reason, we compute $a_k(\eta_j,\psi_j)$ explicitly by comparing constant terms ([DDP, Propositions 2.6, 2.7] or Proposition 3.7 in this paper) and subtract $a_k(\eta_j,\psi_j)E_k(\eta_j,\psi_j)$ from $eP_k$. By a combination of these methods, we get a family of ordinary cusp forms $\{\mathcal{F}_k\}_k$ ([DDP, Corollary 2.10] or Corollary 3.10 in this paper). We see that this family $\{\mathcal{F}_k\}_k$ becomes an ordinary $\Lambda$-adic cusp form, which is denoted by $\mathcal{F}$.

3. Let $\Lambda$ be the Iwasawa algebra, $\nu_k$ the weight $k$ specialization $\Lambda\to\mathcal{O}_E$, $\Lambda_{(k)}$ the localization of $\Lambda$ at $\operatorname{Ker}\nu_k$. Then under the assumption (3.1), we see that Fourier coefficients of $\mathcal{F}$ are in $\Lambda_{(1)}$ and its weight 1 specialization is equal to $E_1(1,\chi_S)$ up to multiplication by a non-zero constant. More precisely, in Proposition 3.13, we write $\mathcal{F}$ in the form of

$$\mathcal{F} = \text{“a }\Lambda\text{-adic Hecke operator”} \cdot \left(u\mathcal{E}(1,\chi)+v\mathcal{E}(\chi,1)+\omega\mathcal{P}^0\right).$$

Here $\mathcal{E}(1,\chi),\mathcal{E}(\chi,1)$ are $\Lambda$-adic Eisenstein series whose Fourier coefficients are in $\Lambda$, $\mathcal{P}^0$ is the ordinary $\Lambda$-adic Hilbert modular form with respect to $\{eP_k\}_k$, and $u,v,w$ are functions expressed in terms of $\mathcal{L}_{\mathrm{an}}(s,\chi)$, $\mathcal{L}_{\mathrm{an}}(s,\chi^{-1})$ as in Corollary 3.10. We need the additional condition of (3.1) in the case of $|S_p|=1$ in order to ensure that $u,v,w$ are elements in $\Lambda_{(1)}$. Moreover Leopoldt's conjecture implies that $\nu_1(\mathcal{F}) = t\cdot E_1(1,\chi_S)$ with $t\in E^\times$. We note that $\nu_1(\mathcal{F})$ is an eigenform, but $\mathcal{F}$ itself is not. Therefore we may guess $\mathcal{F}$ is "approximately" an eigenform near $k=1$. In fact, by a direct computation, we see that $\mathcal{F} \bmod (\operatorname{Ker}\nu_1)^2$ is a simultaneous eigenform of all Hecke operators whose eigen values are contained in $\Lambda_{(1)}/(\operatorname{Ker}\nu_1)^2$. To explain more precisely, let $T_{\mathfrak{l}}$ be the $\mathfrak{l}$-th Hecke operator, $c(\mathfrak{m},\mathcal{F}) \in \Lambda_{(1)}$ the $\mathfrak{m}$-th Fourier coefficient of $\mathcal{F}$. Then we will see that

$$c(\mathfrak{m}, T_{\mathfrak{l}}\mathcal{F}) \equiv \alpha_{\mathfrak{l}} c(\mathfrak{m},\mathcal{F}) \bmod (\operatorname{Ker}\nu_1)^2$$

with $\alpha_{\mathfrak{l}} \in \Lambda_{(1)}$. (Strictly speaking, Fourier coefficients of $\mathcal{H} := u\mathcal{E}(1,\chi) + v\mathcal{E}(\chi,1) + \omega\mathcal{P}^0$ are computed in §3.5, instead of those of $\mathcal{F}$.) Therefore we get a $\Lambda$-algebra homomorphism

$$\phi_{1+\varepsilon} : \mathbf{T} \to \Lambda_{(1)}/(\operatorname{Ker}\nu_1)^2, \quad T_{\mathfrak{l}} \mapsto \alpha_{\mathfrak{l}} \bmod (\operatorname{Ker}\nu_1)^2.$$

Here we denote by $\mathbf{T}$ the ordinary $\Lambda$-adic Hecke algebra acting on the space of ordinary $\Lambda$-adic cusp forms.

4. Define

$$\phi_1 \colon \mathbf{T} \to \Lambda_{(1)}/\operatorname{Ker}\nu_1 \cong E,$$
$$\phi_1(T) := \phi_{1+\varepsilon}(T) \bmod \operatorname{Ker}\nu_1.$$

We denote the localization of $\mathbf{T}$ at $\phi_1$ by $\mathbf{T}_{(1)}$, the total ring of fractions of $\mathbf{T}_{(1)}$ by $\mathcal{F}_{\mathbf{T}_{(1)}}$. Wiles constructed a "big" Galois representation (§3.4)

$$\rho_{(1)} \colon G_F \to \mathrm{GL}(\mathcal{F}_{\mathbf{T}_{(1)}})$$

which is characterized by

$$\operatorname{Tr}\rho_{(1)}(\mathrm{Frob}_{\mathfrak{l}}) = \text{the } \mathfrak{l}\text{-th Hecke operator } T_{\mathfrak{l}}$$

for almost all primes $\mathfrak{l}$. Taking a suitable basis, we write $\rho_{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we can show that $\phi_{1+\varepsilon} \circ a, \phi_{1+\varepsilon} \circ d$ become characters $\psi_1, \psi_2 \colon G_F \to (\Lambda_{(1)}/(\operatorname{Ker}\nu_1)^2)^\times$ which are defined in Definition 3.16. Therefore $\phi_{1+\varepsilon} \circ (b/d)$ becomes a cocycle

$$K \colon G_F \to \Lambda_{(1)}/(\operatorname{Ker}\nu_1)^2.$$

5. We may regard any element $f \in \Lambda_{(1)}$ as a meromorphic function $f(s)$ on $\mathbb{Z}_p$ which is analytic at $s = 1$. Then we can identify

$$\begin{array}{ccc} \Lambda_{(1)}/(\operatorname{Ker}\nu_1)^2 & \cong E[\varepsilon]/(\varepsilon^2) = & E \oplus E\cdot\varepsilon, \\ f & \longleftrightarrow & f(1) + f'(1)\varepsilon, \end{array}$$

where $E[\varepsilon]$ is the polynomial ring in one variable $\varepsilon$. After multiplication by a constant, we may assume that

$$\{0\} \neq K(G_F) \subset E\cdot\varepsilon.$$

Then we can define a function

$$\kappa\colon G_F \to E$$

by

$$K(\sigma) = \kappa(\sigma) \cdot \varepsilon \quad (\sigma \in G_F),$$

which again becomes a cocycle. In §3.6, we will check this $\kappa$ satisfies the desired conditions as in Conjecture 3.3.

*Remark.* As stated in [DDP, *Construction of a cusp form*, p445], some techniques in [Ri] and [Wi] are used to construct the cocycle $\kappa$. For example, Wiles [Wi, proof of Theorem 4.1, p508] suggested the following strategy: Let $\mathbf{T}$ be the $\Lambda$-adic Hecke algebra acting on the space of ordinary $\Lambda$-adic cusp forms, $\mathcal{F}$ an ordinary $\Lambda$-adic cusp form, and $\mathfrak{b}$ an ideal of $\Lambda$. Assume that $\mathcal{F}$ is a Hecke eigenform mod$\mathfrak{b}$, that is, there exist elements $\alpha_{\mathfrak{l}} \in \Lambda$ satisfying $c(\mathfrak{m}, T_{\mathfrak{l}}\mathcal{F}) \equiv \alpha_{\mathfrak{l}} c(\mathfrak{m}, \mathcal{F}) \bmod \mathfrak{b}$ for all $\mathfrak{l}, \mathfrak{m}$. Then we get a homomorphism $\eta_{\mathcal{F}}\colon \mathbf{T} \to \Lambda/\mathfrak{a}$ given by $T_{\mathfrak{l}} \mapsto \alpha_{\mathfrak{l}}$, where $\mathfrak{a} := \{\lambda \in \Lambda \mid \lambda c(\mathcal{O}_F, \mathcal{F}) \in \mathfrak{b}\}$. In [Wi], such an ordinary $\Lambda$-adic cusp form $\mathcal{F}'$ was constructed by modifying a product of Eisenstein series. This "strategy" and a similar modification of a product of Eisenstein series are used also in [DDP]. However, taking $\mathfrak{a} = (\operatorname{Ker} \nu_1)^2$ seems to be one of their new ideas, in order to relate the homomorphism $\eta_{\mathcal{F}}$ ($= \phi_{1+\varepsilon}$ in the above sketch) to the first derivatives of $p$-adic $L$-function. Moreover we need an explicit formula [DDP, (94)] ((3.13) in this paper) for Fourier coefficients in order to investigate the cocycle $\kappa$. By this formula, we easily see that $\mathcal{H}$ is (and hence $\mathcal{F}$ is also) a Hecke eigenform mod$(\operatorname{Ker} \nu_1)^2$ with eigenvalues given by [DDP, Proposition 3.6] (Proposition 3.17 in this paper), and we can write down the map $\phi_{1+\varepsilon}$ explicitly as in [DDP, Theorem 3.7] (Theorem 3.18 in this paper). We need this expression of $\phi_{1+\varepsilon}$ to check that the cocycle $\kappa$, which is defined by using $\phi_{1+\varepsilon}$ as in (3.15), satisfies the desired formulas.

### § 3.3. Λ-adic Hilbert modular forms.

We will use the following notations. $F$ is a totally real field of degree $n$. The narrow ideal class group of $F$ is denoted by $\mathrm{Cl}^+(F)$. For each class $\lambda \in \mathrm{Cl}^+(F)$, we fix a representative $\mathfrak{t}_\lambda \in \lambda$. We put $M_k(\mathfrak{n}, \psi)$ (resp. $S_k(\mathfrak{n}, \psi)$) to be the space of Hilbert modular forms (resp. Hilbert cusp forms) over $F$, of weight $k$, level $\mathfrak{n}$, character $\psi$. For $f \in M_k(\mathfrak{n}, \psi)$, we denote the normalized Fourier coefficient by $c_\lambda(0, f)$ (resp. $c(\mathfrak{m}, f)$) at $\lambda \in \mathrm{Cl}^+(F)$ (resp. at a non-zero integral ideal $\mathfrak{m} \subset \mathcal{O}_F$). Let $\Lambda \cong \mathcal{O}_E[[T]]$ be the Iwasawa algebra equipped with the weight $k$ specialization $\nu_k\colon \Lambda \to \mathcal{O}_E$ for $k \in \mathbb{Z}_p$. We denote the fraction field of $\Lambda$ by $\mathcal{F}_\Lambda$, the localization of $\Lambda$ at $\operatorname{Ker} \nu_k$ by $\Lambda_{(k)}$. Then $\nu_k$ is extended to the homomorphism $\nu_k\colon \Lambda_{(k)} \to E$.

**Definition 3.4.** A family $\mathcal{F} = \{c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}) \mid \mathfrak{m}, \lambda\}$ of formal Fourier coefficients $c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}) \in \Lambda$ is called a $\Lambda$-adic form (resp. a $\Lambda$-adic cusp form) of level $\mathfrak{n}$, character $\chi$ if it satisfies

$$\text{for almost all } k \geq 2, \text{ there exist } f_k \in M_k(\mathfrak{n}', \chi\omega^{1-k}) \text{ (resp. } S_k(\mathfrak{n}', \chi\omega^{1-k})) \text{ satisfying}$$
$$\nu_k(c(\mathfrak{m}, \mathcal{F})) = c(\mathfrak{m}, f_k),\ \nu_k(c_\lambda(0, \mathcal{F})) = c_\lambda(0, f_k)\ (\forall \lambda, \mathfrak{m}).$$

Here we put $\mathfrak{n}' := \operatorname{lcm}(\mathfrak{n}, \prod_{\mathfrak{q} \in S_p} \mathfrak{q})$. For such an $\mathcal{F}$, we write $\mathcal{F}_k := \nu_k(\mathcal{F}) := f_k$. We denote the space of all $\Lambda$-adic forms (resp. $\Lambda$-adic cusp forms) of level $\mathfrak{n}$, character $\chi$ by $\mathcal{M}(\mathfrak{n}, \chi)$ (resp. $\mathcal{S}(\mathfrak{n}, \chi)$). Actually we also call an element in $\mathcal{M}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ (resp. $\mathcal{S}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$) a $\Lambda$-adic form (resp. a $\Lambda$-adic cusp form).

We now recall some properties of the Eisenstein series $E_k(\eta, \psi)$, which is one of the main tools. It is constructed explicitly and is characterized by the following property. For details we refer to [Shim1].

**Proposition 3.5** ([Shim1, Proposition 3.4])**.** *Let $\eta, \psi$ be characters of the narrow ideal class groups modulo $\mathfrak{m}_\eta, \mathfrak{m}_\psi$ with associated signs $q, r \in (\mathbb{Z}/2\mathbb{Z})^n$ respectively. Assume that $q + r \equiv (k, k, \ldots, k) \bmod 2\mathbb{Z}^n$ with $k \in \mathbb{N}$. Then there exists an element $E_k(\eta, \psi) \in M_k(\mathfrak{m}_\eta \mathfrak{m}_\psi, \eta\psi)$ satisfying*

$$c(\mathfrak{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{r} \mid \mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r}) \psi(\mathfrak{r}) N\mathfrak{r}^{k-1}. \tag{3.6}$$

*Here $\mathfrak{r}$ runs over all integral ideals dividing $\mathfrak{m}$.*

*Proof.* We only give a sketch of the construction of the desired modular form in [Shim1, §3]. We denote the upper half plane by $\mathfrak{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, the set of all embeddings $F \hookrightarrow \mathbb{R}$ by $\{\tau_1, \ldots, \tau_n\}$. Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of $F$, $a_0, b_0 \in F$, $k \in \mathbb{N}$, and $U$ a subgroup of $\mathcal{O}_F^\times$ of finite index. We assume that $U$ is "sufficiently small", that is, $N(u)^k = 1$, $ua_0 - a_0 \in \mathfrak{a}$, $ub_0 - b_0 \in \mathfrak{b}$ for all $u \in U$. Then we define for $z = (z_1, \ldots, z_n) \in \mathfrak{H}^n$, $s \in \mathbb{C}$

$$E_{k,U}(z, s; a_0, b_0; \mathfrak{a}, \mathfrak{b}) := D_F^{\frac{1}{2}} N(\mathfrak{b})(-2\pi i)^{-kn} \sum_{(a,b)U} \prod_{i=1}^n (a^{\tau_i} z_i + b^{\tau_i})^{-k} |a^{\tau_i} z_i + b^{\tau_i}|^{-2s}.$$

Here $D_F$ is the discriminant of $F$ and the element $a^{\tau_i} z_i + b^{\tau_i} \in \mathbb{C}$ is well-defined since $\mathbb{R}, \mathfrak{H} \subset \mathbb{C}$. The sum over $(a, b)U$ runs over all cosets $(a, b)U \in \{(a, b) \in F \times F - \{(0, 0)\} \mid a - a_0 \in \mathfrak{a},\ b - b_0 \in \mathfrak{b}\}/U$, where the action of $u \in U$ is defined by $(a, b)u := (au, bu)$. By assumption on $U$, this definition does not depend on the choice of representatives $(a, b)$. The series is convergent for $\operatorname{Re}(2s + k) > 2$ and continued meromorphically to the whole complex $s$-plane. Moreover, we can show that

$$E_{k,U}(z; a_0, b_0; \mathfrak{a}, \mathfrak{b}) := E_{k,U}(z, 0; a_0, b_0; \mathfrak{a}, \mathfrak{b})$$

is a Hilbert modular form of weight $k$ in the traditional sense (except when $n = 1$, $k = 2$). We note that when $k > 2$, we can define the function $E_{k,U}(z; a_0, b_0; \mathfrak{a}, \mathfrak{b})$ without introducing another variable $s$ and without analytic continuation. Taking a sufficiently small $U$, we put

$$G(a, b, \mathfrak{x}, \mathfrak{y}) := \Gamma(k)^n N(\mathfrak{m}_\psi)^{-1} \sum_{t \in \mathfrak{m}_\psi^{-1}\mathfrak{d}^{-1}\mathfrak{y}^{-1}/\mathfrak{d}^{-1}\mathfrak{y}^{-1}} \mathbf{e}_F(-tb) E_{k,U}(z; a, t; \mathfrak{x}, \mathfrak{d}^{-1}\mathfrak{y}^{-1}),$$

$$H(\mathfrak{x}, \mathfrak{y}) := \sum_{a \in \mathfrak{x}/\mathfrak{x}\mathfrak{m}_\eta,\ b \in \mathfrak{y}/\mathfrak{y}\mathfrak{m}_\psi} \mathrm{sgn}(a^q)\eta(a\mathfrak{x}^{-1})\mathrm{sgn}(b^r)\psi(b\mathfrak{y}^{-1})N(\mathfrak{y})^{1-k} G(a, b; \mathfrak{x}\mathfrak{m}_\eta, \mathfrak{y}),$$

$$K(\mathfrak{t}) := [\mathcal{O}_F^\times : U]^{-1} \sum_{\mathfrak{x}} H(\mathfrak{x}, \mathfrak{t}\mathfrak{x}^{-1})$$

for fractional ideals $\mathfrak{x}, \mathfrak{y}, \mathfrak{t}$, elements $a \in F$, $b \in \mathfrak{y}$. Here $\mathfrak{d}$ is the different of $F$, $\mathbf{e}_F(x) := e^{2\pi i \cdot \mathrm{tr}_{F/\mathbb{Q}}(x)}$, $\mathrm{sgn}(x^{(m_1,\ldots,m_n)}) := \prod_{i=1}^n \mathrm{sgn}((x^{\tau_i})^{m_i})$ ($x \in F$, $m_1, \ldots, m_n \in \mathbb{Z}$), and $\mathfrak{x}$ runs over a complete set of representatives for the ideal class group of $F$. Note that $K(\mathfrak{t})$ does not depend on the choice of $U$ when $U$ is sufficiently small. Then we can show that

$$E_k(\eta, \psi) := (g_\lambda)_{\lambda \in \mathrm{Cl}^+(F)} \text{ with } g_\lambda = N(\mathfrak{t}_\lambda)^{\frac{k}{2}} K(\mathfrak{t}_\lambda)$$

satisfies the desired properties. We note that (3.6) is equivalent to [Shim1, (3.21)] as mentioned in [Shim1, p24, l18]. □

As commented in [DDP, Remark 2.2], the explicit formula for the constant terms of $E_k(\eta, \psi)$ seem to be well-known to the experts (e.g., in [Ka]). Since notations differ, these are recalculated in [DDP, Propositions 2.1, 2.3, 2.4, 2.5]. For example, we have

$$\begin{aligned} &c_\lambda(0, E_k(\eta, \psi)) \\ &= \begin{cases} 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L(1-k, \psi\eta^{-1}) & \text{if } k > 1,\ \mathfrak{m}_\eta = (1), \\ 0 & \text{if } k > 1,\ \mathfrak{m}_\eta \neq (1), \\ 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L(0, \psi\eta^{-1}) & \text{if } k = 1,\ \mathfrak{m}_\eta = (1),\ \mathfrak{m}_\psi \neq (1), \\ 2^{-n}\psi^{-1}(\mathfrak{t}_\lambda)L(0, \eta\psi^{-1}) & \text{if } k = 1,\ \mathfrak{m}_\eta \neq (1),\ \mathfrak{m}_\psi = (1), \\ 2^{-n}\eta^{-1}(\mathfrak{t}_\lambda)L(0, \psi\eta^{-1}) + 2^{-n}\psi^{-1}(\mathfrak{t}_\lambda)L(0, \eta\psi^{-1}) & \text{if } k = 1,\ \mathfrak{m}_\eta = (1),\ \mathfrak{m}_\psi = (1), \\ 0 & \text{if } k = 1,\ \mathfrak{m}_\eta \neq (1),\ \mathfrak{m}_\psi \neq (1). \end{cases} \end{aligned}$$

**Definition 3.6.** Additionally assume $L(1-k, \psi) \neq 0$. Then we define the normalized Eisenstein series $G_k(1, \psi)$ by

$$G_k(1, \psi) := \frac{2^n}{L(1-k, \psi)} E_k(1, \psi).$$

Let $\chi$ be a primitive character with conductor $\mathfrak{n}$, as in the setting of the Gross-Stark conjecture. We put

$$\mathfrak{n}_S := \operatorname{lcm}\left(\mathfrak{n}, \prod_{\mathfrak{q}\in S_p} \mathfrak{q}\right) = \mathfrak{n}\cdot \prod_{\mathfrak{q}\in S_p,\ \mathfrak{q}\nmid\mathfrak{n}} \mathfrak{q},$$

$$\mathfrak{n}_R := \operatorname{lcm}\left(\mathfrak{n}, \prod_{\mathfrak{p}\neq\mathfrak{q}\in S_p} \mathfrak{q}\right) = \mathfrak{n}\cdot \prod_{\mathfrak{p}\neq\mathfrak{q}\in S_p,\ \mathfrak{q}\nmid\mathfrak{n}} \mathfrak{q},$$

and denote the character modulo $\mathfrak{n}_S$ (resp. $\mathfrak{n}_R$) associated to $\chi$ by $\chi_S$ (resp. $\chi_R$). We consider the following product of Eisenstein series.

$$P_k := E_1(1,\chi_R)\cdot G_{k-1}(1,\omega^{1-k}) \in M_k(\mathfrak{n}_S, \chi\omega^{1-k}).$$

Here we consider $\omega^{1-k}$ (resp. $\chi\omega^{1-k}$) is a character modulo $\prod_{\mathfrak{q}\in S_p}\mathfrak{q}$ (resp. $\mathfrak{n}_S$) for any $k$ (even if $k=1$). We denote by $E_k(\mathfrak{n},\psi)$ the Eisenstein series part of $M_k(\mathfrak{n},\psi)$, which is the $\mathbb{C}$-subspace of $M_k(\mathfrak{n},\psi)$ spanned by Eisenstein series in $M_k(\mathfrak{n},\psi)$. Then we have the following decomposition which is stable under the action of Hecke operators:

$$M_k(\mathfrak{n},\psi) = S_k(\mathfrak{n},\psi)\oplus E_k(\mathfrak{n},\psi).$$

For details we refer the reader to [Shim2, §7,§8]. Therefore we can uniquely write $P_k$ as

$$P_k = \text{ “A cusp form” } + \sum_{(\eta,\psi)\in J} a_k(\eta,\psi)E_k(\eta,\psi)$$

with constants $a_k(\eta,\psi)\in\mathbb{C}$. Here we put

$$J := \{(\eta,\psi)\mid \eta,\psi \text{ characters of modulus } \mathfrak{m}_\eta,\mathfrak{m}_\psi \text{ with } \mathfrak{m}_\eta\mathfrak{m}_\psi = \mathfrak{n}_S,\ \eta\psi = \chi\omega^{1-k}\}.$$

We will remove the Eisenstein series part of $P_k$ (strictly speaking, the Eisenstein series part of the ordinary part $P_k^o$ of $P_k$) in this expression to get a family of ordinary cusp forms $\mathcal{F} = \{F_k\}_k$ satisfying $F_1\neq 0$ as follows. We compute the eigenvalue $\alpha$ of $E_k(\eta,\psi)$ with respect to a suitable Hecke operator $T$ and multiply $P_k$ by $T-\alpha$ for each pair $(\eta,\psi)\in J$. It can be done when $\eta\neq 1$, $|S_p|>1$ or $\eta\neq 1,\chi$, $|S_p|=1$ by Lemma 3.9. When $\eta=1$, $|S_p|>1$ or $\eta=1,\chi$, $|S_p|=1$, we have to compute $a_k(\eta,\psi)$ explicitly as in Proposition 3.7 and subtract $a_k(\eta,\psi)E_k(\eta,\psi)$ from $P_k$.

First of all, we get the following proposition [DDP, Proposition 2.6, 2.7] by comparing their constant terms.

**Proposition 3.7.** *For $k\geq 2$, we have*

$$a_k(1,\chi\omega^{1-k}) = -\mathcal{L}_{\mathrm{an}}(k,\chi)^{-1}.$$

*If $k > 2$, $|S_p| = 1$ then we have*

$$a_k(\chi, \omega^{1-k}) = -\mathcal{L}_{\mathrm{an}}(k, \chi^{-1})^{-1} \cdot \langle N\mathfrak{n} \rangle^{k-1}.$$

*Here we put $\langle z \rangle := z/\omega(z)$ for $z \in \mathbb{Z}_p^\times$ as usual.*

We denote the $\mathfrak{l}$-th Hecke operators by $T_\mathfrak{l}, U_\mathfrak{l}$ for prime ideals $\mathfrak{l}$ of $\mathcal{O}_F$. Then for any pair $(\eta, \psi) \in J$, we can easily see that

$$\begin{aligned} T_\mathfrak{l} E_k(\eta, \psi) &= (\eta(\mathfrak{l}) + \psi(\mathfrak{l}) N\mathfrak{l}^{k-1}) E_k(\eta, \psi) \quad (\mathfrak{l} \nmid \mathfrak{n}_S), \\ U_\mathfrak{l} E_k(\eta, \psi) &= (\eta(\mathfrak{l}) + \psi(\mathfrak{l}) N\mathfrak{l}^{k-1}) E_k(\eta, \psi) \quad (\mathfrak{l} \mid \mathfrak{n}_S) \\ &= \begin{cases} \eta(\mathfrak{l}) E_k(\eta, \psi) & (\mathfrak{l} \mid \mathfrak{m}_\psi), \\ \psi(\mathfrak{l}) N\mathfrak{l}^{k-1} E_k(\eta, \psi) & (\mathfrak{l} \mid \mathfrak{m}_\eta). \end{cases} \end{aligned}$$

Therefore we get the following Proposition and Lemma [DDP, Proposition 2.8, Lemma 2.9].

**Proposition 3.8.** *For simplicity, we fix an embedding $\overline{\mathbb{Q}_p} \subset \mathbb{C}$. For any subring $A$ of $\overline{\mathbb{Q}_p}$, we denote by $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; A)$ the space of modular forms whose Fourier coefficients are in $A$. Let $e$ be the ordinary operator defined by*

$$e := \lim_{r \to \infty} \left( \prod_{\mathfrak{q} \in S_p} U_\mathfrak{q} \right)^{r!}$$

*on $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_L)$ with $L$ a finite extension of $\mathbb{Q}_p$. It can be extended to an operator on $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; L)$ linearly. Then we have for $k \geq 2$*

$$eE_k(\eta, \psi) = \begin{cases} E_k(\eta, \psi) & \text{if } (p, \mathfrak{m}_\eta) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Therefore we can write*

$$P_k^o := eP_k = \text{“An ordinary cusp form”} + \sum_{(\eta, \psi) \in J^o} a_k(\eta, \psi) E_k(\eta, \psi)$$

*with $J^o := \{(\eta, \psi) \in J \mid (p, \mathfrak{m}_\eta) = 1\}$.*

**Lemma 3.9** ([DDP, Lemma 2.9]). *For $(\eta, \psi) \in J^o$ with $\eta \neq 1, \chi$, there exists a prime ideal $\mathfrak{l} = \mathfrak{l}_{\eta,\psi} \nmid \mathfrak{n}_S$ satisfying*

$$\begin{aligned} &T_{\eta,\psi,k} E_k(\eta, \psi) = 0, \\ &T_{\eta,\psi,k} E_1(1, \chi_S) \neq 0 \\ &\text{with } T_{\eta,\psi,k} := T_\mathfrak{l} - \eta(\mathfrak{l}) - \psi(\mathfrak{l}) N\mathfrak{l}^{k-1}. \end{aligned}$$

*Additionally assume that* $|S_p| > 1$. *Then for* $\mathfrak{q} \in S_p - \{\mathfrak{p}\}$, *we have*

$$\begin{aligned} &T_{\chi,\omega^{1-k},k} E_k(\chi, \omega^{1-k}) = 0, \\ &T_{\chi,\omega^{1-k},k} E_1(1, \chi_S) \neq 0 \\ &\textit{with } T_{\chi,\omega^{1-k},k} := U_{\mathfrak{q}} - \chi(\mathfrak{q}). \end{aligned}$$

Summarizing the above, we get the following result.

**Corollary 3.10** ([DDP, Corollary 2.10]). *Put*

$$u_k := \begin{cases} \dfrac{1}{1+\mathcal{L}_{\mathrm{an}}(k,\chi)} & \textit{if } |S_p| > 1, \\ \dfrac{\mathcal{L}_{\mathrm{an}}(k,\chi)^{-1}}{\mathcal{L}_{\mathrm{an}}(k,\chi)^{-1} + \mathcal{L}_{\mathrm{an}}(k,\chi^{-1})^{-1}\langle N\mathfrak{n}\rangle^{k-1} + 1} & \textit{if } |S_p| = 1, \end{cases}$$

$$v_k := \begin{cases} 0 & \textit{if } |S_p| > 1, \\ \dfrac{\mathcal{L}_{\mathrm{an}}(k,\chi^{-1})^{-1}\langle N\mathfrak{n}\rangle^{k-1}}{\mathcal{L}_{\mathrm{an}}(k,\chi)^{-1} + \mathcal{L}_{\mathrm{an}}(k,\chi^{-1})^{-1}\langle N\mathfrak{n}\rangle^{k-1} + 1} & \textit{if } |S_p| = 1, \end{cases}$$

$$w_k := \begin{cases} \dfrac{\mathcal{L}_{\mathrm{an}}(k,\chi)}{1+\mathcal{L}_{\mathrm{an}}(k,\chi)} & \textit{if } |S_p| > 1, \\ \dfrac{1}{\mathcal{L}_{\mathrm{an}}(k,\chi)^{-1} + \mathcal{L}_{\mathrm{an}}(k,\chi^{-1})^{-1}\langle N\mathfrak{n}\rangle^{k-1} + 1} & \textit{if } |S_p| = 1, \end{cases}$$

$$H_k := u_k E_k(1, \chi\omega^{1-k}) + v_k E_k(\chi, \omega^{1-k}) + w_k P_k^o,$$

$$F_k := \begin{cases} \left(\prod_{(\eta,\psi)\in J^o,\ \eta\neq 1} T_{\eta,\psi,k}\right) H_k & \textit{if } |S_p| > 1, \\ \left(\prod_{(\eta,\psi)\in J^o,\ \eta\neq 1,\chi} T_{\eta,\psi,k}\right) H_k & \textit{if } |S_p| = 1. \end{cases}$$

*Then* $F_k \in S_k(\mathfrak{n}_S, \chi\omega^{1-k})$.

Hereafter in this subsection, we see the family $\{F_k\}_k$ of Hilbert cusp forms becomes a $\Lambda$-adic cusp form. We can define the $\mathfrak{l}$-th Hecke operators $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ on spaces $\mathcal{M}(\mathfrak{n},\chi)$, $\mathcal{S}(\mathfrak{n},\chi)$ as usual. Then the ordinary parts of the spaces of $\Lambda$-adic forms is defined by

$$\begin{aligned} \mathcal{M}^o(\mathfrak{n},\chi) &:= e\mathcal{M}(\mathfrak{n},\chi), \\ \mathcal{S}^o(\mathfrak{n},\chi) &:= e\mathcal{S}(\mathfrak{n},\chi) \end{aligned}$$

with $e := \lim_{r\mapsto\infty} \left(\prod_{\mathfrak{q}\in S_p} U_{\mathfrak{q}}\right)^{r!}$. We consider the following $\Lambda$-algebras of Hecke operators:

$$\begin{aligned} \widetilde{\mathbf{T}} &\subset \mathrm{End}_\Lambda(\mathcal{M}^o(\mathfrak{n},\chi)), \\ \mathbf{T} &\subset \mathrm{End}_\Lambda(\mathcal{S}^o(\mathfrak{n},\chi)) \end{aligned}$$

generated by $T_{\mathfrak{l}}$, $U_{\mathfrak{l}}$ over $\Lambda$. The following is a well-known fact.

**Proposition 3.11.** *If $\eta\psi$ is totally odd, there exists a $\Lambda$-adic Hecke eigenform $\mathcal{E}(\eta,\psi) \in \mathcal{M}(\mathfrak{m}_\eta\mathfrak{m}_\psi, \eta\psi) \otimes_\Lambda \mathcal{F}_\Lambda$ satisfying*

$$\nu_k(\mathcal{E}(\eta,\psi)) = E_k(\eta, \psi\omega^{1-k}).$$

We define a "weight shifted $\Lambda$-adic form" as follows. For the proof, see [DDP, Proposition 3.3].

**Proposition 3.12.** *We denote by $\mathcal{M}'$ the space of all families of formal Fourier coefficients $\mathcal{F} = \{c(\mathfrak{m},\mathcal{F}), c_\lambda(0,\mathcal{F}) \mid \mathfrak{m}, \lambda\}$ ($c(\mathfrak{m},\mathcal{F}), c_\lambda(0,\mathcal{F}) \in \Lambda$) satisfying*

$$\begin{gathered}\text{for almost all } k \geq 2, \text{ there exist } \nu_k(\mathcal{F}) \in M_{k-1}(p,\omega^{1-k}) \text{ satisfying}\\ \nu_k(c(\mathfrak{m},\mathcal{F})) = c(\mathfrak{m},\nu_k(\mathcal{F})),\ \nu_k(c_\lambda(0,\mathcal{F})) = c_\lambda(0,\nu_k(\mathcal{F}))\ (\forall\lambda,\mathfrak{m}).\end{gathered}$$

*Then there exists an element $\mathcal{G} \in \mathcal{M}' \otimes_\Lambda \mathcal{F}_\Lambda$ satisfying*

$$\nu_k(\mathcal{G}) := G_{k-1}(1, \omega^{1-k}).$$

*Moreover, assuming Leopoldt's conjecture, we have*

$$\begin{aligned}&\mathcal{G} \in \mathcal{M}' \otimes_\Lambda \Lambda_{(1)},\\ &\nu_1(\mathcal{G}) = 1.\end{aligned}$$

Summing up these, we get the following proposition [DDP, Proposition 3.4, Lemma 3.5].

**Proposition 3.13.** *We denote the $p$-adic interpolations of $u_k, v_k, w_k, T_{\eta,\psi,k}$ in Lemma 3.9, Corollary 3.10 by $u, v, w, T_{\eta,\psi}$ respectively. We note that the condition $\mathrm{ord}_{s=1}(\mathcal{L}_{\mathrm{an}}(s,\chi) + \mathcal{L}_{\mathrm{an}}(s,\chi^{-1})) = \mathrm{ord}_{s=1}\mathcal{L}_{\mathrm{an}}(s,\chi^{-1})$ in the case of $|S_p| = 1$ assures that $u, v, w \in \Lambda_{(1)}$. Put*

$$\begin{aligned}\mathcal{P} &:= E_1(1,\chi_R)\mathcal{G},\\ \mathcal{P}^o &:= e\mathcal{P},\\ \mathcal{H} &:= u\mathcal{E}(1,\chi) + v\mathcal{E}(\chi,1) + w\mathcal{P}^o,\\ \mathcal{F} &:= \begin{cases}\left(\prod_{(\eta,\psi)\in J^o,\ \eta\neq 1} T_{\eta,\psi}\right)\mathcal{H} & \text{if } |S_p| > 1,\\ \left(\prod_{(\eta,\psi)\in J^o,\ \eta\neq 1,\chi} T_{\eta,\psi}\right)\mathcal{H} & \text{if } |S_p| = 1.\end{cases}\end{aligned}$$

*Then under the assumption (3.1):*

$$\begin{aligned}&\text{If } |S_p| > 1, \text{ assume that Leopoldt's conjecture is true for } F.\\ &\text{If } |S_p| = 1, \text{ assume that Leopoldt's conjecture is true for } F,\\ &\ \text{and that } \mathrm{ord}_{s=1}(\mathcal{L}_{\mathrm{an}}(s,\chi) + \mathcal{L}_{\mathrm{an}}(s,\chi^{-1})) = \mathrm{ord}_{s=1}\mathcal{L}_{\mathrm{an}}(s,\chi^{-1}),\end{aligned}$$

*we have*

$$\mathcal{P} \in \mathcal{M}(\mathfrak{n}, \chi) \otimes_\Lambda \Lambda_{(1)},$$
$$\mathcal{P}^o, \mathcal{H} \in \mathcal{M}^o(\mathfrak{n}, \chi) \otimes_\Lambda \Lambda_{(1)},$$
$$\mathcal{F} \in \mathcal{S}^o(\mathfrak{n}, \chi) \otimes_\Lambda \Lambda_{(1)}.$$

*Moreover we have for all* $k \geq 2$

$$\nu_k(\mathcal{P}) = P_k,$$
$$\nu_k(\mathcal{P}^o) = P_k^o,$$
$$\nu_k(\mathcal{H}) = H_k,$$
$$\nu_k(\mathcal{F}) = F_k,$$

*and for* $k = 1$

$$\nu_1(\mathcal{P}) = \nu_1(\mathcal{P}^o) = E_1(1, \chi_R),$$
$$\nu_1(\mathcal{H}) = E_1(1, \chi_S),$$
$$\nu_1(\mathcal{F}) = t \cdot E_1(1, \chi_S)$$

*with an element* $t \in E^\times$.

## § 3.4. Wiles' "big Galois representation".

To describe Wiles' result, we use the following notations.

- Define the homomorphism associated to $\nu_1(\mathcal{E}(1, \chi)) = E_1(1, \chi_S)$ by

$$\begin{aligned} \phi_1 \colon \widetilde{\mathbf{T}} \otimes_\Lambda \Lambda_{(1)} &\to E, \\ T &\mapsto \nu_1(c(\mathcal{O}_F, T \cdot \mathcal{E}(1, \chi)). \end{aligned}$$

  Then we have
  (3.7)
  $$\phi_1(T_\mathfrak{l}) = \nu_1(c(\mathcal{O}_F, T_\mathfrak{l} \cdot \mathcal{E}(1, \chi))) = \nu_1(c(\mathfrak{l}, \mathcal{E}(1, \chi))) = c(\mathfrak{l}, E_1(1, \chi_S)) = 1 + \chi_S(\mathfrak{l}).$$

  Actually, we may regard $\phi_1$ as

  $$\phi_1 \colon \mathbf{T} \otimes_\Lambda \Lambda_{(1)} \to E$$

  since there exists $\mathcal{F} \in \mathcal{S}^o(\mathfrak{n}, \chi) \otimes_\Lambda \Lambda_{(1)}$ satisfying $\nu_1(\mathcal{F}) = t \cdot E_1(1, \chi_S)$.

- Let $\mathbf{T}_{(1)}$ be the localization of $\mathbf{T} \otimes_\Lambda \Lambda_{(1)}$ at $\mathrm{Ker}(\phi_1)$, and $\mathcal{F}_{\mathbf{T}_{(1)}}$ the total ring of fractions of $\mathbf{T}_{(1)}$. As well known, there exists a basis of $\mathcal{S}^o(\mathfrak{n}, \chi) \otimes_\Lambda \Lambda'$ consisting of Hecke eigenforms if $\Lambda'$ is large enough. Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be the elements of such a basis which satisfy $\nu_1(\mathcal{F}_i) = E_1(1, \chi_S)$. We denote the Hecke eigenvalue of $\mathcal{F}_i$ at $T \in \mathbf{T}_{(1)}$ by $\lambda_T(\mathcal{F}_i)$, the Hecke field of $\mathcal{F}_i$ by $\mathbf{F}(\mathcal{F}_i)$. Then we can embed $\mathbf{T}_{(1)} \hookrightarrow \prod_{i=1}^r \mathbf{F}(\mathcal{F}_i)$ by $T \mapsto (\lambda_T(\mathcal{F}_i))_{i=1,\dots,r}$. Eventually, we can decompose $\mathcal{F}_{\mathbf{T}_{(1)}}$ into a product of fields $\mathcal{F}_{\mathbf{T}_{(1)}} = \mathbf{F}_1 \times \mathbf{F}_2 \times \cdots \times \mathbf{F}_t$ with $\mathbf{F}_i$ a finite extension field of $\mathcal{F}_\Lambda$. Take a factor $\mathbf{F} := \mathbf{F}_i$ of $\mathcal{F}_{\mathbf{T}_{(1)}}$. We denote by $T_\mathfrak{l}$, $U_\mathfrak{l}$ their images under the natural map $\mathbf{T} \to \mathcal{F}_{\mathbf{T}_{(1)}} \to \mathbf{F}$. The image of $\mathbf{T}_{(1)}$ under this map is denoted by $\mathbf{R}$. Then $\mathbf{R}$ is a local ring with $E$ the residue field. Let $\mathbf{m}$ be the maximal ideal of $\mathbf{R}$. Note that $\mathbf{m} = \operatorname{Ker} \phi_1 \colon \mathbf{R} \to E$.

- We define the $\Lambda$-adic cyclotomic character $\underline{\epsilon}_{\mathrm{cyc}} \colon G_F \to \Lambda^\times$ by

$$\nu_k(\underline{\epsilon}_{\mathrm{cyc}}(\mathrm{Frob}_\mathfrak{l})) = \langle N\mathfrak{l} \rangle^{k-1} \quad (\forall \mathfrak{l} \notin S_p).$$

Note that the $p$-adic cyclotomic character $\epsilon_{\mathrm{cyc}} \colon G_F \to \mathbb{Z}_p^\times$ is characterized by $\epsilon_{\mathrm{cyc}}(\mathrm{Frob}_\mathfrak{l}) = N\mathfrak{l}$ $(\forall \mathfrak{l} \notin S_p)$.

We now introduce Wiles' "big Galois representation" [DDP, Theorem 4.1].

**Theorem 3.14.** *For each $\mathbf{F}(= \mathbf{F}_i)$, there exists a continuous irreducible Galois representation*

$$\rho(= \rho_i) \colon G_F \to \mathrm{GL}_2(\mathbf{F})$$

*satisfying*

1. *If $\mathfrak{l} \notin S$, then $\rho$ is unramified at $\mathfrak{l}$ and the characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{l})$ is*

$$X^2 - T_\mathfrak{l} X + \chi \underline{\epsilon}_{\mathrm{cyc}}(\mathrm{Frob}_\mathfrak{l}).$$

2. *$\rho$ is odd.*

3. *If $\mathfrak{q} \in S_p$, then*

$$\rho|_{G_{F_\mathfrak{q}}} \cong \begin{pmatrix} \chi \underline{\epsilon}_{\mathrm{cyc}} \eta_\mathfrak{q}^{-1} & * \\ 0 & \eta_\mathfrak{q} \end{pmatrix}.$$

*Here we denote by $\eta_\mathfrak{q}$ the unramified character of $G_{F_\mathfrak{q}}$ characterized by*

$$\eta_\mathfrak{q}(\mathrm{Frob}_\mathfrak{q}) = U_\mathfrak{q}. \tag{3.8}$$

Now we prepare some properties of the representation $\rho$.

**Theorem 3.15** ([DDP, Theorem 4.2]). *Let $\rho$ be as in Theorem 3.14 and fix a complex conjugation $\delta \in G_F$. Since $\rho$ is odd, we may assume that*

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*by replacing the* $\mathbf{F}$*-basis. We define continuous functions* $a, b, c, d\colon G_F \to \mathbf{F}$ *by*

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}.$$

*Take a change of basis matrix of Theorem 3.14-3. That is, for* $\mathfrak{q} \in S_p$*, there exists* $\begin{pmatrix} A_\mathfrak{q} & B_\mathfrak{q} \\ C_\mathfrak{q} & D_\mathfrak{q} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F})$ *satisfying*

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A_\mathfrak{q} & B_\mathfrak{q} \\ C_\mathfrak{q} & D_\mathfrak{q} \end{pmatrix} = \begin{pmatrix} A_\mathfrak{q} & B_\mathfrak{q} \\ C_\mathfrak{q} & D_\mathfrak{q} \end{pmatrix} \begin{pmatrix} \chi\underline{\epsilon}_{\mathrm{cyc}}\eta_\mathfrak{q}^{-1}(\sigma) & * \\ 0 & \eta_\mathfrak{q}(\sigma) \end{pmatrix} \quad (\forall \sigma \in G_{F_\mathfrak{q}}). \tag{3.9}$$

*Then we have the following properties.*

1. *For all* $\sigma \in G_F$*, we have* $a(\sigma), d(\sigma) \in \mathbf{R}^\times$*,* $a(\sigma) \equiv 1$*,* $d(\sigma) \equiv \chi(\sigma) \bmod \mathbf{m}$*. That is,*

$$\phi_1 \circ a = 1,$$
$$\phi_1 \circ d = \chi.$$

2. $C_\mathfrak{q} \neq 0$ *(*$\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\}$*).*
3. $b|_{G_{F_\mathfrak{p}}} \neq 0$*. In particular,* $A_\mathfrak{p} \neq 0$*.*

*Proof.* By assumption on $\delta$, we see that

$$\begin{aligned} a(\sigma) &= \frac{1}{2}(\mathrm{Tr}\,\rho(\sigma) + \mathrm{Tr}\,\rho(\sigma\delta)), \\ d(\sigma) &= \frac{1}{2}(\mathrm{Tr}\,\rho(\sigma) - \mathrm{Tr}\,\rho(\sigma\delta)). \end{aligned} \tag{3.10}$$

On the other hand, by Theorem 3.14-1, (3.7) and the Chebotarev density theorem we have

$$\phi_1(\mathrm{Tr}\,\rho(\sigma)) = (1+\chi)(\sigma).$$

Noting that $\chi(\delta) = -1$, we can write

$$\begin{aligned} \phi_1 \circ a(\sigma) &= \frac{1}{2}(\phi_1(\mathrm{Tr}\,\rho(\sigma)) + \phi_1(\mathrm{Tr}\,\rho(\sigma\delta))) = \frac{1}{2}((1+\chi)(\sigma) + (1+\chi)(\sigma\delta)) = 1, \\ \phi_1 \circ d(\sigma) &= \frac{1}{2}(\phi_1(\mathrm{Tr}\,\rho(\sigma)) - \phi_1(\mathrm{Tr}\,\rho(\sigma\delta))) = \frac{1}{2}((1+\chi)(\sigma) - (1+\chi)(\sigma\delta)) = \chi(\sigma) \end{aligned}$$

as desired. Next we see the upper left-hand entries of (3.9). Then we have

$$C_\mathfrak{q} b(\sigma) = A_\mathfrak{q}(\chi\underline{\epsilon}_{\mathrm{cyc}}\eta_\mathfrak{q}^{-1}(\sigma) - a(\sigma)). \tag{3.11}$$

Therefore $C_{\mathfrak{q}} = 0$ implies $A_{\mathfrak{q}} = 0$. Then the second statement follows from $\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} \in$ $\mathrm{GL}_2(\mathbf{F})$. Now we give a sketch of the proof of the third statement. Put

$$\mathbf{B} := \sum_{\sigma \in G_{F_{\mathfrak{p}}}} b(\sigma) \cdot \mathbf{R} \subset \mathbf{F}.$$

Actually, we see that $\mathbf{B}$ is a finitely generated $\mathbf{R}$-module. Additionally, we put

$$K \colon G_F \to \mathbf{B},$$
$$K(\sigma) := \frac{b(\sigma)}{d(\sigma)}.$$

Since we have $d(\sigma) \in \mathbf{R}^\times$ by the first statement, $\mathrm{Im}(K)$ generates $\mathbf{B}$ over $\mathbf{R}$. Put $\overline{\mathbf{B}} := \mathbf{B}/\mathbf{m}\mathbf{B}$. Then it is easy to check that the associated map

$$\overline{K} \colon G_F \to \overline{\mathbf{B}},$$
$$\overline{K}(\sigma) := K(\sigma) \bmod \mathbf{m}\mathbf{B}$$

is a cocycle $\in Z^1(G_F, \overline{\mathbf{B}}(\chi^{-1}))$ by noting that $a \equiv 1$, $d \equiv \chi \bmod \mathbf{m}$. Note that $\overline{\mathbf{B}}$ is an $E = \mathbf{R}/\mathbf{m}$ vector space generated by $\mathrm{Im}(\overline{K})$ and that $\overline{\mathbf{B}}(\chi^{-1}) := \overline{\mathbf{B}} \otimes_E E(\chi^{-1})$. Now we prove $b|_{G_{F_{\mathfrak{p}}}} \neq 0$ by contradiction. We see that $b = 0$ on $G_{F_{\mathfrak{p}}}$ implies $b = 0$ on $G_F$ by the following technique, which we will use again later.

1. $\overline{K}|_{G_{F_v}}$ satisfies the following "local triviality properties" (a),(b). In particular, by (3.5), we have

$$[\overline{K}] \in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, \overline{\mathbf{B}}(\chi^{-1})) = H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \otimes_E \overline{\mathbf{B}}.$$

   (a) For all $\mathfrak{q} \in S_p - \{\mathfrak{p}\}$, we have $[\overline{K}|_{G_{F_{\mathfrak{q}}}}] = 0 \in H^1(G_{F_{\mathfrak{q}}}, \overline{\mathbf{B}}(\chi^{-1}))$

   *Proof.* By Theorem 3.15-2 and (3.11), we can write

$$K|_{G_{F_{\mathfrak{q}}}} = \frac{A_{\mathfrak{q}}}{C_{\mathfrak{q}}} \cdot \frac{\chi \underline{\epsilon}_{\mathrm{cyc}} \eta_{\mathfrak{q}}^{-1} - a}{d}.$$

   By definition of $\underline{\epsilon}_{\mathrm{cyc}}, \eta_{\mathfrak{q}}$ and Theorem 3.15-1, we have

$$\begin{aligned} \phi_1 \circ a &= 1, \\ \phi_1 \circ d &= \chi, \\ \phi_1 \circ \underline{\epsilon}_{\mathrm{cyc}} &= 1, \\ \phi_1 \circ \eta_{\mathfrak{q}} &= 1. \end{aligned} \tag{3.12}$$

   Hence the assertion is clear. □

(b) $\overline{K}|_{G_{F_\mathfrak{p}}} = 0$.

*Proof.* By our assumption $b|_{G_{F_\mathfrak{p}}} = 0$. □

2. The fact that the class $[\overline{K}]$ is locally trivial at $\mathfrak{p}$ (by 1-(b)) implies that the class $[\overline{K}]$ is globally trivial:
$$[\overline{K}] = 0 \in H^1(G_F, \overline{\mathbf{B}}(\chi^{-1})).$$
*Proof.* It follows from the injectivity of $\mathrm{res}_\mathfrak{p} \colon H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, \overline{\mathbf{B}}(\chi^{-1})) \hookrightarrow \overline{\mathbf{B}} \cdot \kappa_{\mathrm{nr}} \oplus \overline{\mathbf{B}} \cdot \kappa_{\mathrm{cyc}}$. □

3. The fact that $[\overline{K}] = 0$ as a class implies that
$$\overline{K} = 0 \text{ as a function: } G_F \to \overline{\mathbf{B}}.$$
*Proof.* By $[\overline{K}] = 0$, we can write $\overline{K} = \theta \cdot (1 - \chi^{-1})$ with $\theta \in \overline{\mathbf{B}}$. Then we get
$$\theta = \theta \cdot (1 - \chi^{-1}(\delta))/2 = K(\delta)/2 = b(\delta)/2d(\delta) = 0.$$
Hence we get $\overline{K} = 0$. □

4. Since $\overline{\mathbf{B}}$ is generated by elements $\in \mathrm{Im}(\overline{K})$, $\overline{K} = 0$ implies $\overline{\mathbf{B}} = 0$, i.e., $\mathbf{B} = \mathbf{mB}$. Then Nakayama's Lemma states that $\mathbf{B} = 0$.

Therefore $b = 0$ if $b|_{G_{F_\mathfrak{p}}} = 0$. But it contradicts the irreducibility of $\rho$. The fact that $A_\mathfrak{p} \neq 0$ follows from (3.11), $b|_{G_{F_\mathfrak{p}}} \neq 0$ and $\begin{pmatrix} A_\mathfrak{p} & B_\mathfrak{p} \\ C_\mathfrak{p} & D_\mathfrak{p} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F})$. □

## § 3.5. The weight "$1 + \varepsilon$" specialization.

The weight 1 specialization $\nu_1 \colon \Lambda_{(1)} \to E$ induces the weight $1 + \varepsilon$ specialization
$$\nu_{1+\varepsilon} \colon \Lambda_{(1)} \to \Lambda_{(1)}/(\mathrm{Ker}\,\nu_1)^2 \cong E[\varepsilon]/(\varepsilon^2).$$
More explicitly, considering any element $f \in \Lambda_{(1)}$ as a meromorphic function $f(s)$ on $\mathbb{Z}_p$, we define
$$\nu_{1+\varepsilon}(f) := f(1) + f'(1)\varepsilon \in E[\varepsilon]/(\varepsilon^2) = E \oplus E\varepsilon.$$

**Definition 3.16.** We define two characters
$$\psi_1, \psi_2 \colon G_F \to E[\varepsilon]/(\varepsilon^2)^\times$$
by
$$\psi_1 := 1 + v(1)\kappa_{\mathrm{cyc}} \cdot \varepsilon,$$
$$\psi_2 := \chi \cdot (1 + u(1)\kappa_{\mathrm{cyc}} \cdot \varepsilon).$$
Note that these characters are lifts of $1, \chi \colon G_F \to E^\times$, respectively. Then we have

- $\psi_1$ is unramified outside of $S_p$ and satisfies

$$\psi_1(\mathrm{Frob}_{\mathfrak{l}}) = 1 + v(1)\kappa_{\mathrm{cyc}}(\mathrm{Frob}_{\mathfrak{l}})\varepsilon \quad (\forall \mathfrak{l} \notin S_p).$$

- $\psi_2$ is unramified outside of $S$ and satisfies

$$\psi_2(\mathrm{Frob}_{\mathfrak{l}}) = \chi(\mathfrak{l})\,(1 + u(1)\kappa_{\mathrm{cyc}}(\mathrm{Frob}_{\mathfrak{l}})\varepsilon) \quad (\forall \mathfrak{l} \notin S).$$

As usual, we consider $\psi_1, \psi_2$ as multiplicative functions on the set of ideals by the rule of

$$\psi_1(\mathfrak{q}) = 1 \ (\forall \mathfrak{q} \in S_p),$$
$$\psi_2(\mathfrak{l}) = 0 \ (\forall \mathfrak{l} \in S).$$

We defined the ordinary $\Lambda$-adic Hilbert modular form $\mathcal{H}$ by modifying a product of Eisenstein series. Therefore $\mathcal{H}$ is not necessarily an eigenform. Nevertheless, instead of $\mathcal{H}$, we can show that $\mathcal{H}$ mod $(\mathrm{Ker}\,\nu_1)^2$ is an eigenform. It was shown by the explicit calculation of Fourier coefficients. Namely, the following results are obtained in [DDP, Proposition 3.6].

**Proposition 3.17.** *Let $\mathcal{H}$ be as in Proposition 3.13. Consider the weight $1+\varepsilon$ specialization $\mathcal{H}_{1+\varepsilon} := \nu_{1+\varepsilon}(\mathcal{H})$. ($\mathcal{H}_{1+\varepsilon}$ is the family of formal Fourier coefficients $c(\mathfrak{m}, \mathcal{H}_{1+\varepsilon}) := \nu_{1+\varepsilon}(c(\mathfrak{m}, \mathcal{H}))$, $c(\lambda, \mathcal{H}_{1+\varepsilon}) := \nu_{1+\varepsilon}(c(\lambda, \mathcal{H}))$.) The action of a Hecke operator $T \in \widetilde{\mathbf{T}}$ is defined by*

$$c(\mathfrak{m}, T\mathcal{H}_{1+\varepsilon}) := \nu_{1+\varepsilon}(c(\mathfrak{m}, T\mathcal{H})).$$

*Then we have the following.*

- *$\mathcal{H}_{1+\varepsilon}$ is a simultaneous eigenform for the Hecke operators. Note that its eigenvalues belong to $E[\varepsilon]/(\varepsilon^2)$.*
- *$c(1, \mathcal{H}_{1+\varepsilon}) = 1$.*
- *$c(\mathfrak{l}, \mathcal{H}_{1+\varepsilon}) = \psi_1(\mathfrak{l}) + \psi_2(\mathfrak{l})$ ($\forall \mathfrak{l} \neq \mathfrak{p}$).*
- *$c(\mathfrak{p}, \mathcal{H}_{1+\varepsilon}) = 1 + w'(1)\varepsilon$.*

*Therefore we have the $\Lambda_{(1)}$-algebra homomorphism*

$$\begin{aligned} \phi_{1+\varepsilon}\colon\ \widetilde{\mathbf{T}} \otimes_\Lambda \Lambda_{(1)} &\to E[\varepsilon]/(\varepsilon^2), \\ T_{\mathfrak{l}} &\mapsto c(\mathfrak{l}, \mathcal{H}_{1+\varepsilon}) \quad (\mathfrak{l} \notin S), \\ U_{\mathfrak{l}} &\mapsto c(\mathfrak{l}, \mathcal{H}_{1+\varepsilon}) \quad (\mathfrak{l} \in S). \end{aligned}$$

*Actually,* $\phi_{1+\varepsilon}$ *factors through the quotient*

$$\phi_{1+\varepsilon}\colon \mathbf{T}\otimes_\Lambda \Lambda_{(1)} \to E[\varepsilon]/(\varepsilon^2),$$

*since we can write* $\mathcal{F} = T\mathcal{H} \in \mathcal{S}^o(\mathfrak{n},\chi)\otimes_\Lambda \Lambda_{(1)}$ *with an element* $T \in \widetilde{\mathbf{T}}$*. Note that* $\phi_{1+\varepsilon}$ *is a lift of*

$$\phi_1\colon \mathbf{T}\otimes_\Lambda \Lambda_{(1)} \to E.$$

*Proof.* For simplicity, we give the proof for the case of $|S_p| > 1$. (In the case of $|S_p| = 1$, the proof is similar but more complicated. See [DDP, Proof of Proposition 3.6]). By using that $\mathcal{H} = u\mathcal{E}(1,\chi) + w\mathcal{P}^o$, $u(1) = 1$, $w(1) = 0$, $u'(1) + w'(1) = 0$, $\nu_1(\mathcal{E}(1,\chi)) = E_1(1,\chi_S)$, $\nu_1(\mathcal{P}^o) = E_1(1,\chi_R)$, we get

$$\mathcal{H}_{1+\varepsilon} = \nu_{1+\varepsilon}(\mathcal{E}(1,\chi)) + w'(1)(E_1(1,\chi_R) - E_1(1,\chi_S)).$$

We will write down the $\mathfrak{m}$-th Fourier coefficient of this. We define an integral ideal $\mathfrak{m}_0$ and non-negative integers $\mathrm{ord}_\mathfrak{q}\,\mathfrak{m}$ by $\mathfrak{m} = \mathfrak{m}_0\prod_{\mathfrak{q}\in S_p}\mathfrak{q}^{\mathrm{ord}_\mathfrak{q}\mathfrak{m}}$, $\gcd(\mathfrak{m}_0,(p)) = 1$. Then we have

$$c(\mathfrak{m},\nu_{1+\varepsilon}(\mathcal{E}(1,\chi))) = \nu_{1+\varepsilon}(c(\mathfrak{m},\mathcal{E}(1,\chi))) = \sum_{\mathfrak{r}|\mathfrak{m}_0}\chi(\mathfrak{r})(1+\kappa_{\mathrm{cyc}}(\mathfrak{r})\varepsilon)$$

since $\nu_k(c(\mathfrak{m},\mathcal{E}(1,\chi)) = \sum_{\mathfrak{r}|\mathfrak{m}_0}\chi(\mathfrak{r})\langle N\mathfrak{r}\rangle^{k-1}$. By noting that $\chi_R(\mathfrak{r}) - \chi_S(\mathfrak{r}) = 0$ if $\mathfrak{p}\nmid\mathfrak{r}$ and that $\chi_R(\mathfrak{r}) - \chi_S(\mathfrak{r}) = \chi_R(\mathfrak{r}) = \chi_R(\mathfrak{r}/\mathfrak{p})$ if $\mathfrak{p}|\mathfrak{r}$, we can write

$$c(\mathfrak{m},E_1(1,\chi_R)) - c(\mathfrak{m},E_1(1,\chi_S)) = \sum_{\mathfrak{r}|\mathfrak{m}}(\chi_R(\mathfrak{r}) - \chi_S(\mathfrak{r})) = \mathrm{ord}_\mathfrak{p}\,\mathfrak{m}\sum_{\mathfrak{r}|\mathfrak{m}_0}\chi(\mathfrak{r}).$$

Consequently, we get

$$c(\mathfrak{m},\mathcal{H}_{1+\varepsilon}) = \left(\sum_{\mathfrak{r}|\mathfrak{m}_0}\psi_1(\mathfrak{m}_0/\mathfrak{r})\psi_2(\mathfrak{r})\right)\times(1+\omega'(1)\varepsilon)^{\mathrm{ord}_\mathfrak{p}\mathfrak{m}}, \tag{3.13}$$

where $\psi_1 = 1$ in this case. Therefore the fact that $\mathcal{H}_{1+\varepsilon}$ is a simultaneous eigenform can be shown similarly to the case of the usual Eisenstein series $E_k(\eta,\psi)$ whose $\mathfrak{m}$-th Fourier coefficient is $c(\mathfrak{m},E_k(\eta,\psi)) = \sum_{\mathfrak{r}|\mathfrak{m}}\eta(\mathfrak{m}/\mathfrak{r})\psi(\mathfrak{r})N\mathfrak{r}^{k-1}$. The remaining assertions also follow from (3.13). □

We note that we have $\omega'(1) = u(1)\mathcal{L}_{\mathrm{an}}(\chi)$ since $\omega(k) = u(k)\mathcal{L}_{\mathrm{an}}(k,\chi)$. Then we get the following main result in this subsection.

**Theorem 3.18** ([DDP, Theorem 3.7])**.** *Under the assumption (3.1):*

*If* $|S_p| > 1$*, assume that Leopoldt's conjecture is true for* $F$*.*

*If* $|S_p| = 1$*, assume that Leopoldt's conjecture is true for* $F$*,*

*and that* $\mathrm{ord}_{s=1}(\mathcal{L}_{\mathrm{an}}(s,\chi) + \mathcal{L}_{\mathrm{an}}(s,\chi^{-1})) = \mathrm{ord}_{s=1}\mathcal{L}_{\mathrm{an}}(s,\chi^{-1})$, *there exists a homomorphism*

$$\phi_{1+\varepsilon}\colon \mathbf{T}_{(1)} \to E[\varepsilon]/(\varepsilon^2)$$

*satisfying*

$$\begin{aligned}\phi_{1+\varepsilon}(T_{\mathfrak{l}}) &= \psi_1(\mathfrak{l}) + \psi_2(\mathfrak{l}) \quad (\forall \mathfrak{l} \notin S),\\ \phi_{1+\varepsilon}(U_{\mathfrak{l}}) &= \psi_1(\mathfrak{l}) + \psi_2(\mathfrak{l}) \quad (\forall \mathfrak{l} \in S),\\ &= \begin{cases} \psi_1(\mathfrak{l}) & (\forall \mathfrak{l} \in R),\\ 1 + u(1)\mathcal{L}_{\mathrm{an}}(\chi)\varepsilon & (\mathfrak{l} = \mathfrak{p}).\end{cases}\end{aligned}$$

## § 3.6. Construction of a cocycle.

Consider the product of Galois representations $\rho_i$ in Theorem 3.14

$$\rho_{(1)} := \prod_i \rho_i\colon G_F \to \mathrm{GL}_2(\mathcal{F}_{\mathbf{T}_{(1)}}).$$

Taking the basis as in Theorem 3.15, we write

$$\begin{pmatrix} a(\sigma) & b(\sigma)\\ c(\sigma) & d(\sigma)\end{pmatrix} := \rho_{(1)}(\sigma).$$

We summarize the properties of continuous maps $a, b, c, d\colon G_F \to \mathcal{F}_{\mathbf{T}_{(1)}}$ which we have seen:

- $a(\sigma), d(\sigma) \in \mathbf{T}_{(1)}^\times$ $(\forall \sigma \in G_F)$ (by Theorem 3.15-1).
- By Theorem 3.14-3, for each $\mathfrak{q} \in S_p$, there exists $\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}}\\ C_{\mathfrak{q}} & D_{\mathfrak{q}}\end{pmatrix} \in \mathrm{GL}_2(\mathcal{F}_{\mathbf{T}_{(1)}})$ satisfying

$$\begin{pmatrix} a(\sigma) & b(\sigma)\\ c(\sigma) & d(\sigma)\end{pmatrix}\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}}\\ C_{\mathfrak{q}} & D_{\mathfrak{q}}\end{pmatrix} = \begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}}\\ C_{\mathfrak{q}} & D_{\mathfrak{q}}\end{pmatrix}\begin{pmatrix} \chi\epsilon_{\mathrm{cyc}}\eta_{\mathfrak{q}}^{-1}(\sigma) & *\\ 0 & \eta_{\mathfrak{q}}(\sigma)\end{pmatrix} \quad (\forall \sigma \in G_{F_{\mathfrak{q}}}).$$

  Moreover we see that $A_{\mathfrak{p}} \in \mathcal{F}_{\mathbf{T}_{(1)}}^\times$ (by Theorem 3.15-3), $C_{\mathfrak{q}} \in \mathcal{F}_{\mathbf{T}_{(1)}}^\times$ $(\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\})$ (by Theorem 3.15-2).

Therefore by putting

$$\begin{aligned} K(\sigma) &:= \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}}\frac{b(\sigma)}{d(\sigma)} \quad (\sigma \in G_F),\\ x_{\mathfrak{q}} &:= \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}}\frac{A_{\mathfrak{q}}}{C_{\mathfrak{q}}} \quad (\mathfrak{q} \in S_p - \{\mathfrak{p}\}),\\ \mathbf{B} &:= \sum_{\sigma \in G_F} \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}} b(\sigma) \cdot \mathbf{T}_{(1)} \subset \mathcal{F}_{\mathbf{T}_{(1)}},\end{aligned}$$

we can write

$$K|_{G_{F_\mathfrak{q}}} = \begin{cases} x_\mathfrak{q} \cdot \dfrac{\chi \underline{\epsilon}_{\mathrm{cyc}} \eta_\mathfrak{q}^{-1} - a}{d} & (\mathfrak{q} \in S_p - \{\mathfrak{p}\}) \\ \dfrac{\underline{\epsilon}_{\mathrm{cyc}} \eta_\mathfrak{q}^{-1} - a}{d} & (\mathfrak{q} = \mathfrak{p}). \end{cases} \tag{3.14}$$

As preparation, we prove a claim in the proof of [DDP, Theorem 4.4]. Put $\mathbf{m} = \operatorname{Ker} \phi_1 \colon \mathbf{T}_{(1)} \to E$.

**Lemma 3.19.** *We have* $\mathbf{B} \subset \mathbf{m}$.

*Proof.* Put $\mathbf{B}^\sharp := (\mathbf{B} + \mathbf{m})/\mathbf{m}$, $\overline{\mathbf{B}}^\sharp := \mathbf{B}^\sharp/\mathbf{m}\mathbf{B}^\sharp$, and $\overline{K}^\sharp \colon G_F \to \overline{\mathbf{B}}^\sharp$ to be the associated map to $K \colon G_F \to \mathbf{B}$. Then the local triviality of $\overline{K}^\sharp$ implies the global triviality of $\overline{K}^\sharp$. This can be seen similarly as in the proof of Theorem 3.15-3. In fact, we see the following.

1. $[\overline{K}^\sharp|_{G_{F_\mathfrak{q}}}] = 0$ $(\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\})$, $\overline{K}^\sharp|_{G_{F_\mathfrak{p}}} = 0$. In particular, by (3.5), we see that $[\overline{K}^\sharp] \in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, \overline{\mathbf{B}}^\sharp(\chi^{-1})) = H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1})) \otimes_E \overline{\mathbf{B}}^\sharp$.

   *Proof.* It follows from (3.12),(3.14). □

2. $[\overline{K}^\sharp|_{G_{F_\mathfrak{p}}}] = 0$ implies $[\overline{K}^\sharp] = 0$.

   *Proof.* It is clear since $\mathrm{res}_\mathfrak{p}$ is injective on $H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, \overline{\mathbf{B}}^\sharp(\chi^{-1}))$. □

3. $[\overline{K}^\sharp] = 0$ implies $\overline{K}^\sharp = 0$.

   *Proof.* Similarly to the proof of step 3 in the proof of Theorem 3.15-3. □

4. $\overline{K}^\sharp = 0$ implies $\overline{\mathbf{B}}^\sharp = \mathbf{B}^\sharp/\mathbf{m}\mathbf{B}^\sharp = 0$. That is, $\mathbf{B}^\sharp = \mathbf{m}\mathbf{B}^\sharp$.

Now Nakayama's Lemma states $\mathbf{B}^\sharp = 0$, so we get $\mathbf{B} + \mathbf{m} = \mathbf{m}$. Hence the assertion is clear. □

The fact that $\mathbf{B} \subset \mathbf{m}$ implies $\phi_1 \circ K = 0$, so $(\phi_{1+\varepsilon} \circ K)(G_F) \subset E \cdot \varepsilon$. That is, there exists a continuous map $\kappa \colon G_F \to E$ satisfying

$$\kappa \cdot \varepsilon = \phi_{1+\varepsilon} \circ K. \tag{3.15}$$

We have

$$x_\mathfrak{q} \in \mathbf{B} \subset \mathbf{m},$$

$$\phi_{1+\varepsilon} \circ \underline{\epsilon}_{\mathrm{cyc}} = 1 + \kappa_{\mathrm{cyc}} \varepsilon$$

by definition and the above Lemma. Moreover we see that

$$\phi_{1+\varepsilon} \circ \eta_{\mathfrak{p}} = 1 + u(1)\mathcal{L}_{\mathrm{an}}(\chi)\kappa_{\mathrm{nr}}\varepsilon$$

by (3.8), Theorem 3.18, and that

$$\begin{aligned} \phi_{1+\varepsilon} \circ a &= \psi_1 = 1 + v(1)\kappa_{\mathrm{cyc}}\varepsilon, \\ \phi_{1+\varepsilon} \circ d &= \psi_2 = \chi(1 + u(1)\kappa_{\mathrm{cyc}}\varepsilon) \end{aligned}$$

by (3.10), Theorem 3.14-1, Theorem 3.18, Definition 3.16. Summarizing the above, we have the following Theorem.

**Theorem 3.20** ([DDP, Theorem 4.4])**.** *We have*

$$\begin{aligned} [\kappa|_{G_{F_{\mathfrak{q}}}}] &= 0 \in H^1(G_{F_{\mathfrak{q}}}, E(\chi^{-1})) \quad (\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\}), \\ \kappa|_{G_{F_{\mathfrak{p}}}} &= u(1)(-\mathcal{L}_{\mathrm{an}}(\chi) \cdot \kappa_{\mathrm{nr}} + \kappa_{\mathrm{cyc}}). \end{aligned}$$

*Proof.* By (3.14) and $x_{\mathfrak{q}} \in \mathbf{m}$, we can write

$$\kappa|_{G_{F_{\mathfrak{q}}}} = x'_{\mathfrak{q}} \cdot \phi_1 \circ \frac{\chi\epsilon_{\mathrm{cyc}}\eta_{\mathfrak{q}}^{-1} - a}{d} = x'_{\mathfrak{q}} \cdot (1 - \chi^{-1})$$

with an element $x'_{\mathfrak{q}} \in E$ for $\mathfrak{q} \in S_p - \{\mathfrak{p}\}$. Then the first assertion is clear. Similarly we can write

$$\begin{aligned} \kappa|_{G_{F_{\mathfrak{p}}}} \cdot \varepsilon &= \phi_{1+\varepsilon} \circ \frac{\epsilon_{\mathrm{cyc}}\eta_{\mathfrak{q}}^{-1} - a}{d} \\ &= \frac{(1 + \kappa_{\mathrm{cyc}} \cdot \varepsilon)(1 - u(1)\mathcal{L}_{\mathrm{an}}(\chi)\kappa_{\mathrm{nr}} \cdot \varepsilon) - (1 + v(1)\kappa_{\mathrm{cyc}} \cdot \varepsilon)}{1 + u(1)\kappa_{\mathrm{cyc}} \cdot \varepsilon} \\ &= (-u(1)\mathcal{L}_{\mathrm{an}}(\chi)\kappa_{\mathrm{nr}} + \kappa_{\mathrm{cyc}} - v(1)\kappa_{\mathrm{cyc}}) \cdot \varepsilon. \end{aligned}$$

By definition of $u, v$ in Corollary 3.10 and the assumption (3.1), we have

$$u(1) = \begin{cases} 1 & \text{if } |S_p| > 1, \\ \dfrac{\mathcal{L}_{\mathrm{an}}^{(t)}(1, \chi^{-1})}{\mathcal{L}_{\mathrm{an}}^{(t)}(1, \chi) + \mathcal{L}_{\mathrm{an}}^{(t)}(1, \chi^{-1})} & \text{if } |S_p| = 1, \end{cases}$$

$$v(1) = \begin{cases} 0 & \text{if } |S_p| > 1, \\ \dfrac{\mathcal{L}_{\mathrm{an}}^{(t)}(1, \chi)}{\mathcal{L}_{\mathrm{an}}^{(t)}(1, \chi) + \mathcal{L}_{\mathrm{an}}^{(t)}(1, \chi^{-1})} & \text{if } |S_p| = 1 \end{cases}$$

with $t := \mathrm{ord}_{s=1}\mathcal{L}_{\mathrm{an}}(s, \chi^{-1})$. Then the second assertion is clear. □

By (3.5) and the above Theorem, we see that $\kappa \in H^1_{\mathfrak{p},\mathrm{cyc}}(G_F, E(\chi^{-1}))$ and that Conjecture 3.3 holds true under the assumption (3.1).

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