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On the WKB theoretic structure of a Schrödinger operator with a Stokes curve of loop type

Dedicated to Professor T. Aoki on his 60th birthday

By

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Abstract

We construct a formal transformation from a Schrödinger type equation that has a Stokes curve of loop type to a modified Bessel type equation near the loop. We also prove the transformation series are formal series of Gevrey 1.

§ 1. Introduction

The purpose of this article is to construct a WKB theoretic transformation from a Schrödinger equation with a Stokes curve of loop type to a modified Bessel-type equation near the Stokes curve. In the exact WKB analysis, several transformation theories to canonical forms had been established in these two decades. In [AKT1], Aoki, Kawai and Takei gave a transformation from a Schrödinger equation to the Airy equation near a simple turning point. Using this transformation, they analyzed the discontinuity of the Borel transform of a WKB solution of the Schrödinger equation at its movable singularities and gave another proof of the Voros connection formula ([V], §6). They also considered the case where two simple turning points are connected by a Stokes curve and constructed a transformation from the Schrödinger equation to the Weber equation near the Stokes curve. The singularity structure of the Borel transformed WKB solutions of those equations was analyzed in the second part [AKT2] of their papers under the condition that the two turning points are sufficiently close together. On the other hand, Koike established a transformation theory near a simple pole of

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the potential of the Schrödinger equation [K]. Recently, Kamimoto, Kawai and Takei have investigated the so-called M2P1T operator by a careful and heavy computation [KKT]. We note that all those theories have not treated the case where a Stokes curve forms a loop. This type of degeneration of Stokes curves is observed as frequent as the degeneration of the Weber type treated in [AKT1] (cf. [T]) and plays a role in the theory of the cluster algebra [IN]. A modified Bessel-type equation is the simplest differential equation whose Stokes curve forms a loop around one of the poles. In this article, we construct a WKB theoretic transformation from a Schrödinger equation that has a loop-type Stokes curve to the modified Bessel equation on an open set containing the Stokes curve.

If we employ the modified Bessel equation as a canonical form, we expect that we can construct a transformation in the simply connected open set containing the loop when the pre-transformed equation has one and only one double pole inside the loop. It will be discussed in the forthcoming paper as well as an investigation of connection automorphisms concerning the loop-type degeneration of Stokes curves.

The plan of this paper is as follows. In section 2, we state our main results. We construct the transformation series in Section 3. In Section 4, we give estimation of the transformation series constructed in Section 2. This estimation ensures Borel transformability of the series.

§ 2. Statements of the main results

We consider the following one-dimensional Schrödinger equation with a large parameter η:

$$\Big(-\frac{d^2}{d\tilde{x}^2}+\eta^2 Q(\tilde{x})\Big)\tilde{\psi}(\tilde{x},\eta)=0. \tag{2.1}$$

Here $Q(\tilde{x})$ is a meromorphic function defined on an open set $U\subset\mathbb{C}$. We assume that (2.1) has a simple turning point p_0 (i.e., a zero of $Q(\tilde{x})$ of order 1). A Stokes curve emanating from the turning point p_0 is a curve defined by

$$\operatorname{Im}\int_{p_0}^{\tilde{x}}\sqrt{Q(\tilde{x})}d\tilde{x}=0.$$

It is well known that there are three Stokes curves emanating from p_0 (cf.[KT]). We define the notion of the loop formed by the Stokes curves.

Definition 2.1. Let s_0, s_1 and s_2 be germs at p_0 of the Stokes curves emanating from p_0. Let S_0 and S_1 be the Stokes curves which are extensions of the germs s_0 and s_1, respectively. We may assume that S_0 and S_1 are contained in U. We say that s_0 and

s_1 form a loop if $S_0 = S_1$ as a set. Further, we say that the Stokes curves emanating from the turning point p_0 form a loop if two of the germs emanating from p_0 form a loop.

Under the above notation, we have the following proposition.

Proposition 2.2. *Suppose that the Stokes curves of the equation* (2.1) *emanating from* p_0 *form a loop* γ. *Then there exist a small open neighborhood* V *of the loop and two Gevrey 1 series*

$$x = \sum_{j\geq 0} x_j(\tilde{x})\eta^{-j}, \qquad c = \sum_{j\geq 0} c_j\eta^{-j}$$

which enjoy the following properties:

(1) *Each* $x_j(\tilde{x})(j \geq 1)$ *is holomorphic on* V.
(2) *The function* $x_0(\tilde{x})$ *is biholomorphic on* V *and satisfies*

$$x_0(p_0) = c_0,$$

where c_0 *is a real positive number defined by*

$$c_0 = \frac{1}{4\pi^2}\Big(\int_\gamma \sqrt{Q(\tilde{x})}d\tilde{x}\Big)^2. \tag{2.2}$$

(3) *The following relation is formally satisfied*:

$$\eta^2 Q(\tilde{x}) = \eta^2\Big(\frac{\partial x}{\partial \tilde{x}}\Big)^2\frac{x-c}{x^2} - \frac{1}{2}\{x;\tilde{x}\}. \tag{2.3}$$

Here $\{x;\tilde{x}\}$ *designates the Schwarzian derivative defined as follows:*

$$\{x;\tilde{x}\} = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2.$$

Here and hereafter we denote $d^\alpha x_j/d\tilde{x}^\alpha$ by $x_j^{(\alpha)}(\alpha = 0,1,2,\dots)$. We also denote, as usual, $x_j^{(1)}, x_j^{(2)}, \dots$ by $x_j', x_j'', \dots$, respectively. Throughout this paper, we consider a relation between WKB solutions (see [KT]).

Theorem 2.3. *Suppose that the Stokes curves of the equation* (2.1) *emanating from* p_0 *form a loop* γ. *Let* $\tilde{\psi}(\tilde{x},\eta)$ *be a WKB solution of the equation*

$$\Big(-\frac{d^2}{d\tilde{x}^2} + \eta^2 Q(\tilde{x})\Big)\tilde{\psi}(\tilde{x},\eta) = 0.$$

Let $x(\tilde{x},\eta)$ *and* c *be the series given in Proposition* 2.2. *Then there exists a WKB solution* $\psi(x,\eta)$ *which satisfies the following two conditions:*

$$\tilde{\psi}(\tilde{x},\eta) = \Big(\frac{\partial x(\tilde{x},\eta)}{\partial \tilde{x}}\Big)^{-\frac{1}{2}}\psi(x(\tilde{x},\eta),\eta),$$

$$\left(-\frac{d^2}{dx^2}+\eta^2\frac{x-c}{x^2}\right)\psi(x,\eta)=0. \tag{2.4}$$

The theorem can be proved by using Proposition 2.2. We note that Stokes curves of (2.4) form a loop (Fig. 1).

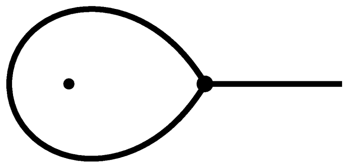

Fig. 1

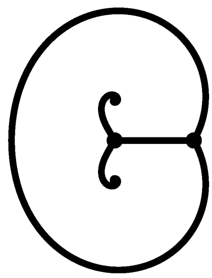

Fig. 2

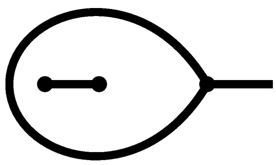

Fig. 3

The Fig. 2 and Fig. 3 are the examples that the Stokes curves which enjoy the hypothesis in Theorem 2.3. When a potential has the form $-x(x-2)/((x+i)(x-i)(x+10))^2$ (resp., $(x-1/2)/((x-1/4)(x-1/3))$), Stokes curves form the loop as in Fig. 2 (resp., Fig. 3). It is noteworthy that there are two poles counting multiplicity in the domain surrounded by the Stokes curve which forms a loop (see Lemma 61.1 in [S]).

§ 3. Construction of the transformation series

Let us prove Proposition 2.2. Comparing the coefficients of $\eta^{-n+2}(n\in\mathbb{Z}_{\geq 0})$ in (2.3), we have

$$Q(\tilde{x})=\frac{x_0(\tilde{x})-c_0}{x_0^2(\tilde{x})}(x_0'(\tilde{x}))^2, \tag{3.1}$$

$$(x_0-c_0)x_1'+\frac{2c_0-x_0}{2x_0}x_0'x_1=\frac{x_0'}{2}c_1, \tag{3.2}$$

$$(x_0-c_0)x_n'+\frac{2c_0-x_0}{2x_0}x_0'x_n=\frac{x_0'}{2}c_n+R_n(x_{k_1}^{(\alpha)},c_{k_2}:0\leq\alpha\leq 3,0\leq k_1,k_2\leq n-1), \tag{3.3}$$

where $n \geq 2$ and R_n is given by the following formula:

$$\begin{aligned}
R_n = & \frac{x_0'}{2}(x_0 - c_0) \sum_{\substack{j+k=n \\ 1 \leq j,k \leq n-1}} \frac{x_j x_k}{x_0^2} \\
& + \frac{x_0^2}{4(x_0')^2} \sum_{\substack{k+l+\mu=n-2 \\ 0 \leq k,l,\mu \leq n-2}} (-1)^l \sum_{\mu_1+\cdots+\mu_l=\mu} \frac{x_k'''}{(x_0')^l} x'_{\mu_1+1} \cdots x'_{\mu_l+1} \\
& - \frac{1}{2x_0'} \sum_{\substack{n_1+m_1+j+t+l_1=n \\ 0 \leq n_1,m_1,j,t \leq n-1}} (-1)^{l_1} x'_{m_1} x'_{n_1} (x_j - c_j) \sum_{s_1+s_2+\cdots+s_{l_1}=t} \prod_{1 \leq h \leq l_1} \sum_{\substack{j+k=s_h+1 \\ 0 \leq j,k \leq n-1}} \frac{x_j x_k}{x_0^2} \\
& - \frac{3x_0^2}{8(x_0')^3} \sum_{k_1+k_2+l+\mu=n-2} \sum_{\mu_1+\cdots\mu_l=\mu} \frac{(-1)^l(l+1)}{(x_0')^l} x''_{k_1} x''_{k_2} x'_{\mu_1+1} \cdots x'_{\mu_l+1}.
\end{aligned}$$

We first show that if the equation (3.1) has a holomorphic solution x_0, then c_0 is given by (2.2). Taking the square root of (3.1), we obtain

$$\sqrt{Q(\tilde{x})} = \frac{\sqrt{x_0 - c_0}}{x_0} x_0'. \tag{3.4}$$

We choose the branch of $\sqrt{x - c_0}/x$ as $\sqrt{x - c_0} > 0$ holds on the half line

$$\{x \in \mathbb{C}; \mathrm{Re}(x - c_0) > 0, \mathrm{Im}(x - c_0) = 0\}$$

and take the half line

$$\{c_0 + re^{i\vartheta}; r > 0\} \qquad (-2\pi/3 < \vartheta < 0)$$

as the branch cut. On the other hand, the branch of $\sqrt{Q(\tilde{x})}$ is chosen as follows: Let s_0, s_1 and s_2 be the germs of Stokes curves of (2.1) emanating from p_0 for which the initial directions are $\theta_0, \theta_0 + 2\pi/3$ and $\theta_0 - 2\pi/3$, respectively. This means, for example, s_0 is tangent to the half line $\{p_0 + re^{i\theta_0}; r > 0\}$ at p_0. We take the branch cut for $\sqrt{Q(\tilde{x})}$ so that it coincides with $\{p_0 + re^{i(\vartheta+\theta_0)}; r > 0\}$ near p_0 and that it does not intersects the loop. We choose the branch of $\sqrt{Q(\tilde{x})}$ so that $\mathrm{Re}\sqrt{Q(\tilde{x})}dx > 0$ on s_0. Then we have $\mathrm{Re}\sqrt{Q(\tilde{x})}dx < 0$ on s_1. We may assume that s_1 and s_2 form the loop γ. The orientation of γ is chosen to be consistent with s_1. Integrating (3.4) on γ, we find c_0, namely (2.2). Conversely, if c_0 is given by (2.2), then we can find the holomorphic solution x_0 of (3.4) as follows. To discuss the existence of $x_0(\tilde{x})$, we make use of the following two functions:

$$w_1(z) = z - \tan^{-1} z, w_2(\tilde{x}) = \frac{1}{2\sqrt{c_0}} \int_{p_0}^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x}, \tag{3.5}$$

where we set

$$z = \sqrt{\frac{x}{c_0} - 1}.$$

Then, by integrating the both sides of the equation (3.4), we obtain

$$w_1(z) = w_2(\tilde{x}). \tag{3.6}$$

We can easily verify that $w_1(z)$ maps the loop of the Stokes curve of the equation (2.4) onto the closed interval $[-2\pi\sqrt{c_0}, 0]$ bijectively and that its derivative never vanishes on the loop except for c_0. Similarly $w_2(\tilde{x})$ maps γ onto the same interval bijectively. Further, their derivatives never vanish on the loops except for p_0 and c_0. Let V' be an open set which contains $\gamma - \{p_0\}$ as a relatively closed subset and satisfy $w_2{}' \neq 0$ on V'. The above arguments imply that if we consider the following function

$$z(\tilde{x}) = w_1^{-1}(w_2(\tilde{x})),$$

then the function

$$x_0(\tilde{x}) = c_0(z(\tilde{x})^2 + 1) \tag{3.7}$$

is the holomorphic solution which satisfies $x_0(p_0) = c_0$ on V'. Furthermore the function $z(\tilde{x})$ has the following expansion at $\tilde{x} = p_0$:

$$z(\tilde{x}) = A_0(\tilde{x} - p_0)^{\frac{1}{2}}(1 + O(\tilde{x} - p_0)),$$

where A_0 is a non-zero constant. Thus we have found the function (3.7) is single-valued, holomorphic even at the starting point $\tilde{x} = p_0$ and $(dx_0/d\tilde{x})(p_0) \neq 0$. This means that we may define V as the union of V' and an open small disk with the center p_0. Next we consider the analytic continuation of $x_0(\tilde{x})$ along γ. By the choice of c_0, we have the same equation after the continuation as (3.6) because, after the continuation, the constants which are added to the both sides of (3.6) cancel out. This means that the germ of $x_0(\tilde{x})$ at the end point p_0 of the analytic continuation coincides with the germ of $x_0(\tilde{x})$ at the starting point p_0. This completes the proof of the existence of (x_0, c_0).

Next we try to find $(x_n(\tilde{x}), c_n)(n \geq 1)$. The task is to show the differential equation

$$(x_0 - c_0)x_n' + \frac{2c_0 - x_0}{2x_0}x_0' x_n = \frac{x_0'}{2}c_n + R(\tilde{x}) \tag{3.8}$$

has a holomorphic solution on V. Here $R(\tilde{x})$ denotes a given holomorphic function on V. We decompose (3.8) into the following two equations:

$$(x_0 - c_0)\frac{du}{d\tilde{x}} + \frac{2c_0 - x_0}{2x_0}x_0' u = \frac{x_0'}{2}, \tag{3.9}$$

$$(x_0 - c_0)\frac{dv}{d\tilde{x}} + \frac{2c_0 - x_0}{2x_0}x_0' v = R(\tilde{x}). \tag{3.10}$$

Both equations have singularity only at p_0 in V. Furthermore, they have the unique holomorphic solution at p_0 as the characteristic exponent is equal to $-\frac{1}{2}$. We denote those holomorphic solutions by $u^0(\tilde{x})$ and $v^0(\tilde{x})$. Note that $x_n(\tilde{x}) = c_n u^0(\tilde{x}) + v^0(\tilde{x})$ gives a holomorphic solution of (3.8) at p_0. Now we consider the analytic continuation of $x_n(\tilde{x})$ along γ. Using $t = x_0(\tilde{x})$ as a new coordinate in V, the equation (3.9) can be expressed as

$$(c_0 - t)\frac{du}{dt} + \frac{t - 2c_0}{2t}u = -\frac{1}{2}. \tag{3.11}$$

The holomorphic solution u^0 of (3.11) at c_0 is written as

$$u^0(t) = \frac{t}{2(t - c_0)^{\frac{1}{2}}} \int_{c_0}^{t} \frac{1}{s(s - c_0)^{\frac{1}{2}}} ds. \tag{3.12}$$

We denote the loop formed by the Stokes curves of (2.4) by γ_1. By the construction of x_0, we have $x_0(\gamma) = \gamma_1$. The orientation of γ_1 is chosen to be consistent with γ. Then the analytic continuation u^1 of u^0 along γ_1 is given by

$$u^1(t) = \frac{t}{2(t - c_0)^{\frac{1}{2}}} \left(\int_{\gamma_1} \frac{1}{s(s - c_0)^{\frac{1}{2}}} ds + \int_{c_0}^{t} \frac{1}{s(s - c_0)^{\frac{1}{2}}} ds \right) \tag{3.13}$$

$$= u^0(t) + \lambda H(t), \tag{3.14}$$

where

$$\lambda = \int_{\gamma_1} \frac{1}{2s(s - c_0)^{\frac{1}{2}}} ds,$$

and $H(t) = t/(t - c_0)^{\frac{1}{2}}$ is a solution of the homogeneous equation

$$(c_0 - t)\frac{du}{dt} + \frac{t - 2c_0}{2t}u = 0.$$

Similarly we can get the analytic continuation v^1 of v^0 as follows:

$$v^1(t) = v^0(t) + \tilde{\lambda} H,$$

where we set $\tilde{\lambda}$ as

$$\tilde{\lambda} = \int_{\gamma_1} \frac{R_n(x_0^{-1}(s))}{s(s - c_0)^{\frac{1}{2}}} ds.$$

Note that λ is different from zero. Thus we conclude that the analytic continuation of $x_n(\tilde{x})$ is given by

$$\begin{aligned} x_n(\tilde{x}) &= c_n u^1 + v^1 \\ &= (c_n u^0 + v^0) + (c_n \lambda + \tilde{\lambda})H. \end{aligned} \tag{3.15}$$

To construct a holomorphic solution x_n on V, $c_n\lambda + \tilde{\lambda}$ must be equal to zero since $H(t)$ is not holomorphic at p_0. Therefore we have got the constant $c_n = -\lambda^{-1}\tilde{\lambda}$ and the holomorphic solution x_n.

§ 4. Estimation of $x_n(\tilde{x})$ and c_n

In this section, we investigate Borel transformability of the function $x(\tilde{x}, \eta)$ and the series c constructed in the previous section. It can be proved in a similar way as in the case of the discussion about a canonical form near one simple turning point (see [AKT1]). That is, in order to estimate $x_n(\tilde{x})$, we estimate the right hand side of (3.3).

First we define an open set V as

$$V = \bigcup_{\tilde{x} \in \gamma} B_{r_0}(\tilde{x})$$

for a sufficiently small number $r_0 > 0$, where B_{r_0} denotes the open disk of radius r_0 with the center $\tilde{x}$. We can find positive constants C_1 and $r < r_0$ so that

$$|x_0(\tilde{x})|, \quad \frac{1}{|x_0(\tilde{x})|}, \quad |x_0'(\tilde{x})|, \quad \frac{1}{|x_0'(\tilde{x})|} \leq C_1$$

hold on $V(r) = \cup_{\tilde{x}\in\gamma}\overline{B_r(\tilde{x})}$ as $x_0'(p_0) \neq 0$ and the image of γ by x_0 coincides with γ_1, where $\overline{B_r(\tilde{x})}$ denotes the closure of the open ball $B_r(\tilde{x})$. Then, for each $\epsilon > 0$, we have

$$|x_0''(\tilde{x})| \leq C_1\epsilon^{-1} \tag{4.1}$$

$$|x_0'''(\tilde{x})| \leq 2C_1\epsilon^{-2} \tag{4.2}$$

on $V(r-\epsilon)$. We want to prove the following proposition by the induction on $n \in \mathbb{N}$.

Proposition 4.1. *There exists a positive constant C for which the following inequalities hold for each $\epsilon > 0$ and $n \in \mathbb{N}$:*

$$|c_n| \leq n!C^{n-1}\epsilon^{-n}, \tag{4.3}$$

$$\sup_{\tilde{x}\in V(r-\epsilon)} |x_n(\tilde{x})| \leq n!C^{n-1}\epsilon^{-n}, \tag{4.4}$$

$$\sup_{\tilde{x}\in V(r-\epsilon)} |x_n'(\tilde{x})| \leq n!C^{n-1}\epsilon^{-n}. \tag{4.5}$$

Note that above inequalities hold for $n = 1$ as x_1 and c_1 are equal to zero. To prove the proposition we prepare the following two lemmas:

Lemma 4.2. *Let U be an open small neighborhood of the loop γ_1, $v(t)$ a given holomorphic function on U. Let $v(t)$ be a bounded holomorphic function on U, r_1 the* *maximum radius so that* $B_{r_1}(c_0)$ *is contained in* U. *We assume that the following differential equation has a holomorphic solution on* U.

$$(t-c_0)\frac{du}{dt}+\frac{2c_0-t}{2t}u=\frac{1}{2}c+v(t), \tag{4.6}$$

where c *is a constant. Then there exist positive constants* N_1, N_2 *and* N_3 *so that the following inequalities hold :*

$$|c| \leq N_1 \sup_{t\in U}|v(t)|, \tag{4.7}$$

$$\sup_{t\in B_{r_1}(c_0)}|u(t)| \leq N_2 \sup_{t\in U}|v(t)|, \tag{4.8}$$

$$\sup_{t\in B_{r_1}(c_0)}\left|\frac{du(t)}{dt}\right| \leq \frac{N_3}{r_1}\sup_{t\in U}|v(t)|. \tag{4.9}$$

Proof. By the argument in the previous section, to have the holomorphic solution, c must be determined as follows:

$$c=-\lambda^{-1}\tilde{\lambda}, \tag{4.10}$$

$$\lambda=\int_{\gamma_1}\frac{1}{2s(s-c_0)^{\frac{1}{2}}}ds, \tag{4.11}$$

$$\tilde{\lambda}=\int_{\gamma_1}\frac{v(s)}{s(s-c_0)^{\frac{1}{2}}}ds. \tag{4.12}$$

The first inequality (4.7) holds if we set the positive constant N_1 as

$$N_1=|\lambda|^{-1}\int_{\gamma_1}|ds|\sup_{t\in\gamma_1}\frac{1}{|s||s-c_0|^{\frac{1}{2}}},$$

for we have

$$\begin{aligned}|c| &= |\lambda|^{-1}|\tilde{\lambda}| \\ &\leq |\lambda|^{-1}\sup_{t\in U}|v(t)|\int_{\gamma_1}\frac{|ds|}{|s||s-c_0|^{\frac{1}{2}}} \\ &\leq N_1\sup_{t\in U}|v(t)|.\end{aligned}$$

It is easy to verify that, by the definition of c,

$$u(t)=\frac{t}{2}\int_0^1\frac{c+2v(l(t-c_0)+c_0)}{l^{\frac{1}{2}}(l(t-c_0)+c_0)}dl$$

is the unique holomorphic solution of (4.6). Then we have

$$|u(t)| \leq \frac{|t|}{2}(|c|+2\sup_{s\in B_{r_1}(c_0)}|v(s)|)\int_0^1\frac{dl}{l^{\frac{1}{2}}|l(t-c_0)+c_0|} \qquad (t\in B_{r_1}(c_0)). \tag{4.13}$$

Since $|l(t-c_0)+c_0|^{-1}$ is bounded in $B_{r_1}(c_0)$, we can conclude that there exists a positive constant N_2 for which the following inequality holds:

$$|u(t)| \leq N_2 \sup_{t\in U} |v(t)|.$$

Rewriting the equation (4.6) as

$$(t-c_0)\frac{du}{dt} = \frac{1}{2}c + v(t) - \frac{2c_0-t}{2t}u(t),$$

we have

(4.14)
$$\begin{aligned}\sup_{t\in B_{r_1}(c_0)} \left|(t-c_0)\frac{du(t)}{dt}\right| &\leq \frac{1}{2}|c| + \sup_{t\in B_{r_1}(c_0)} |v(t)| + \sup_{t\in B_{r_1}(c_0)} \left|\frac{2c_0-t}{2t}\right| \sup_{t\in B_{r_1}(c_0)} |u(t)| \\ &\leq N_3 \sup_{t\in U} |v(t)|,\end{aligned}$$

where we set

$$N_3 = \frac{N_1+2}{2} + N_2 \sup_{t\in U}\left|\frac{2c_0-t}{2t}\right|.$$

Since

$$\frac{1}{2}c + v(c_0) - \frac{1}{2}u(c_0) = 0$$

holds, (4.14) combined with Schwarz lemma entails (4.9). This completes the proof of the lemma. □

Lemma 4.3. *Let r, U, $u(t)$, $v(t)$ and c be as in Lemma 4.2. Then there exists a positive constant N' that satisfies the following inequalities:*

(4.15)
$$\sup_{t\in U} |u(t)| \leq N' \sup_{t\in U} |v(t)|,$$

(4.16)
$$\sup_{t\in U} \left|\frac{du(t)}{dt}\right| \leq N' \sup_{t\in U} |v(t)|.$$

Proof. Let $t_1 \in B_{r_1}(c_0) \setminus B_{\frac{r_1}{2}}(c_0)$ and $t \in U \setminus B_{r_1}(c_0)$. The holomorphic solution of (4.6) can be expressed as

$$\begin{aligned}u(t) &= \frac{t}{(t-c_0)^{\frac{1}{2}}} \int_{c_0}^{t} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}}ds \\ &= \frac{t}{(t-c_0)^{\frac{1}{2}}} \left\{\int_{c_0}^{t_1} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}}ds + \int_{t_1}^{t} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}}ds\right\},\end{aligned}$$

and we get the following inequality :

$$|u(t)| \leq \left|\frac{t}{(t-c_0)^{\frac{1}{2}}}\right| \left\{\left|\int_{c_0}^{t_1} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}}ds\right| + \left|\int_{t_1}^{t} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}}ds\right|\right\}.$$

Here we have taken suitably the path of integration in U. Obviously, $|s - c_0|^{-\frac{1}{2}}$ is bounded on the path of integration. Hence we can get positive constants L_1 and L_2 which satisfy the following inequalities.

$$\sup_{t\in U\setminus B_{r_1}(c_0)} \left| \frac{t}{(t-c_0)^{\frac{1}{2}}} \int_{t_1}^{t} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}} ds \right| \leq L_1 \left(|c| + 2\sup_{t\in U} |v(t)| \right) \tag{4.17}$$
$$\leq L_2 \sup_{t\in U} |v(t)|.$$

Since $t_1 \in B_{r_1}(c_0) \setminus B_{\frac{r_1}{2}}(c_0)$, we may use Lemma 4.2. Thus we find that there exists a positive constant L_3 and the first term

$$\left| \frac{t}{(t-c_0)^{\frac{1}{2}}} \int_{c_0}^{t_1} \frac{c+2v(s)}{2s(s-c_0)^{\frac{1}{2}}} ds \right|$$

is dominated by

$$L_3 \sup_{t\in U} |v(t)|.$$

Hence we obtain the required inequality. Next we prove that the inequality (4.16) holds. Since $|t-c_0|^{-\frac{1}{2}}$ is bounded in $t \in U\setminus B_{r_1}(c_0)$, we can prove it without using the Schwarz lemma. That is,

$$\left| \frac{du}{dt} \right| \leq \frac{1}{|t-c_0|} \left(\frac{|c|}{2} + \frac{|2c_0 - t|}{2} |u(t)| + |v(t)| \right) \tag{4.18}$$
$$\leq N' \sup_{t\in U\setminus B_{r_1}(c_0)} |v(t)|$$

holds over $U \setminus B_{r_1}(c_0)$. This fact and Lemma 4.2 entail the second inequality. This completes the proof of the lemma. $\square$

Proof of Proposition 4.1. The assumption of induction and the above lemmas imply that there exists a positive constant N such that

$$|c_n| \leq N \sup_{\tilde{x}\in V(r-\epsilon)} |R_n(\tilde{x})|, \tag{4.19}$$

$$\sup_{\tilde{x}\in V(r-\epsilon)} |x_n(\tilde{x})| \leq N \sup_{\tilde{x}\in V(r-\epsilon)} |R_n(\tilde{x})|, \tag{4.20}$$

$$\sup_{\tilde{x}\in V(r-\epsilon)} \left| \frac{dx_n(\tilde{x})}{d\tilde{x}} \right| \leq N \sup_{\tilde{x}\in V(r-\epsilon)} |R_n(\tilde{x})| \tag{4.21}$$

hold for any sufficiently small $\epsilon > 0$, where we apply the lemmas regarding U as $V(r-\epsilon)$. Using this fact, we estimate $x_n(\tilde{x})$ under the induction hypothesis that (4.19), (4.20) and (4.21) hold for $k \leq n$. To simplify the notations, we rewrite R_n as $R_n = R_{n,1} -$

$R_{n,2}+R_{n,3}-R_{n,4}$, where $R_{n,j}(j\in\{1,2,3,4\})$ are given by

$$R_{n,1}=\frac{x_0'}{2}(x_0-c_0)\sum_{j+k=n}\frac{x_jx_k}{x_0^2},$$

$$R_{n,2}=\frac{1}{2x_0'}\sum_{n_1+m_1+j+t+l_1=n}(-1)^{l_1}x'_{m_1}x'_{n_1}(x_j-c_j)\sum_{s_1+s_2+\cdots+s_{l_1}=t}\prod_{1\le h\le l_1}\sum_{\substack{j+k=s_h+1\\0\le j,k\le n-1}}\frac{x_jx_k}{x_0^2},$$

$$R_{n,3}=\frac{x_0^2}{4(x_0')^2}\sum_{k+l+\mu=n-2}(-1)^l\sum_{\mu_1+\cdots+\mu_l=\mu}\frac{x_k'''}{(x_0')^l}x'_{\mu_1+1}\cdots x'_{\mu_l+1},$$

$$R_{n,4}=\frac{3x_0^2}{8(x_0')^3}\sum_{k_1+k_2+l+\mu=n-2}\sum_{\mu_1+\cdots\mu_l=\mu}\frac{(-1)^l(l+1)}{(x_0')^l}x''_{k_1}x''_{k_2}x'_{\mu_1+1}\cdots x'_{\mu_l+1}.$$

Since $(2x_0'/x_0^2)R_{n,3}$ and $(2x_0'/x_0^2)R_{n,4}$ respectively have the same form as I and II in (A.2.18) of [AKT1], we can verify that the following inequalities hold if we assume that $C>C_1$:

$$\sup_{\tilde{x}\in V(r-\epsilon)}|R_{n,3}|\le n!C^{n-1}\epsilon^{-n}\frac{C_1^3}{2}(1-C_1C^{-1})^{-1}(e^2C^{-1}+C_1)C^{-1}, \tag{4.22}$$

$$\sup_{\tilde{x}\in V(r-\epsilon)}|R_{n,4}|\le n!C^{n-1}\epsilon^{-n}\frac{3}{8}C_1^5C^{-1}(1-C_1C^{-1})^{-2}(eC^{-1}+C_1)^2. \tag{4.23}$$

And it is also easy to get the estimation of $R_{n,1}$:

$$\sup_{\tilde{x}\in V(r-\epsilon)}|R_{n,1}|\le n!C^{n-1}\epsilon^{-n}C_1^3(C_1+|c_0|)C^{-1}.$$

Now let us estimate $R_{n,2}$. To get it, we use the following lemma([AKT1], Sublemma 2.2). Since its proof is not given in [AKT1], we prove the lemma here.

Lemma 4.4. *The following inequality holds for all positive integers n and l satisfying $l\le n$*

$$\sum_{\substack{n_1+\cdots+n_l=n\\n_1,\cdots,n_l\ge1}}n_1!\cdots n_l!\le n!$$

Proof. We prove the inequality by induction on l. Since

$$\sum_{n_1=n}n_1!=n!$$

hold, the inequality holds when $l=1$. We assume that the inequality holds for $l\le k$. Substituting n_{k+1} with s and using induction hypothesis, we have the following inequality:

$$\begin{aligned}\sum_{\substack{n_1+n_2+\cdots+n_k+n_{k+1}=n\\ n_1,\cdots,n_{k+1}\geq 1}} n_1!\cdots n_k! n_{k+1}! &= \sum_{s=1}^{n-k} s! \sum_{\substack{n_1+n_2+\cdots+n_k=n-s\\ n_1,\cdots,n_k\geq 1}} n_1!\cdots n_k!\\ &\leq \sum_{s=1}^{n-k} s!(n-s)!\\ &= n!\sum_{s=1}^{n-k}\frac{s!(n-s)!}{n!}.\end{aligned}$$

Since

$$\frac{s!(n-s)!}{n!}\leq\frac{1}{n}\qquad(1\leq s\leq n),$$

we have

$$\sum_{s=1}^{n-k}\frac{s!(n-s)!}{n!}\leq 1-\frac{k}{n}.$$

Therefore, we have the required inequality:

$$\sum_{n_1+\cdots+n_l=n} n_1!\cdots n_l!\leq n!.$$

Thus, induction proceeds. We have finished the proof of the lemma. □

By (4.4) and (4.5), we have

$$\sup_{\tilde{x}\in V(r-\epsilon)}|x_k''(\tilde{x})|\leq(k+1)!C^{k-1}e\epsilon^{-k-1},\tag{4.24}$$

$$\sup_{\tilde{x}\in V(r-\epsilon)}|x_k'''(\tilde{x})|\leq(k+2)!C^{k-1}e^2\epsilon^{-k-2}.\tag{4.25}$$

The hypothesis also allows us to get the following inequality for each j:

$$\begin{aligned}|a_{s_j}| &\leq C_1^2\sum_{j+k=s_j+1} j!k!C^{j+k-2}\epsilon^{-(j+k)}\\ &\leq C_1^2C^{s_j-1}\epsilon^{-(s_j+1)}(s_j+1)!.\end{aligned}$$

Then we obtain

$$\begin{aligned}\sum_{s_1+\cdots+s_{l_1}=t}|a_{s_1}|\cdots|a_{s_{l_1}}| &\leq C_1^{-2l_1}\sum_{s_1+\cdots+s_{l_1}=t}\Pi_{h=1}^{h=l_1}C^{s_h-1}\epsilon^{-(s_h+1)}(s_h+1)!\\ &\leq C_1^{-2l_1}C^{t-l_1}\epsilon^{-t-l_1}(t+l_1)!.\end{aligned}$$

We also obtain the following inequality:

$$\sup_{\tilde{x}\in V(r-\epsilon)}|R_{n,2}|\leq n!C^{n-1}\epsilon^{-n}4C_1C^{-2}(1-C_1^2C^{-2})^{-1}.$$

The above inequalities about $R_{n,j}$ show that R_n is dominated by

$$\sup_{\tilde{x}\in V(r-\epsilon)} |R_n| \leq n! C^{n-1}\epsilon^{-n} A,$$

where

$$\begin{aligned} A =& C_1^3(C_1+|c_0|)C^{-1} + 4C_1C^{-2}(1-C_1^2C^{-2})^{-1} \\ &+ \frac{1}{2}C^{-1}C_1^3(1-C_1C^{-1})^{-1}(eC^{-1}+C_1)\left(\frac{3}{4}C_1^2(1-C_1C^{-1})^{-1}(eC^{-1}+C_1)+1\right). \end{aligned}$$

Therefore if we choose C so that $NA \leq 1$, the induction proceeds. This completes the estimation over V.

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