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Paperfolding and modular functions

Dedicated to Professor Takashi Aoki on the occasion of his 60th birthday

By

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Abstract

The two lacunary series $\chi_+(z) = \sum_{n\geq 0} z^{2^n}$, $\chi_-(z) = \sum_{n\geq 0}(-1)^n z^{2^n}$ and $\xi(z) = \sum_{n\geq 1} s_n z^n$, $s_{2n} = s_n$,$s_{2n+1} = (-1)^n$ are all connected to the paperfolding. We study their behaviours at $z = 1$ and some of their relations to modular functions. We also investigate the zeros of their partial sums.

§ 1. Introduction

This work concerns some analytic questions behind the paperfolding. One of its main objectives is the study of the generating series

$$\xi(z) = \sum_{n\geq 1} s_n z^n, \quad |z| < 1 \tag{1.1}$$

where (s_n) is the paperfolding sequence defined by

$$s_{2n} = s_n, \quad s_{2n+1} = (-1)^n, \quad n \in \mathbb{Z}_+.$$

The origin of the function $\xi(z)$ has to be found in the two functions

$$\chi_+(z) = \sum_{n\geq 0} z^{2^n}, \quad \chi_-(z) = \sum_{n\geq 0} (-1)^n z^{2^n}, \quad |z| < 1, \tag{1.2}$$

which in turn are related to the paperfolding through their expansions in continued fractions [4], [1], [15]. The function $\chi_+(z)$ is sometimes called Fredholm series though its

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use is probably much more older. To study the values of the Fredholm series at algebraic points, the classical Liouville's theorem is not powerful enough. Mahler introduced a new approach for studying arithmetic properties of certain functions satisfying a functional equation $f(z^2) = R(z, f(z))$. With his method he was able to show that for an algebraic number α with $0 < \alpha < 1$, $\sum_{n=0}^{\infty} \alpha^{2^n}$, is transcendental. For example $\kappa = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2^n}}$ is transcendental. This result was first proved by Kempner in 1913. The series (1.2) verify the functional equations

$$\chi_+(z) = z + \chi_+(z^2), \quad \chi_-(z) = z - \chi_-(z^2). \tag{1.3}$$

Our interest in χ_+, χ_-, ξ will be from the angle of continued fractions, already considered in [4], [1], [15] and also by means of modular forms and the main objective is to study their singularities on the unit circle as well as their oscillations when $x \to 1^-$. The relationship between $\chi_\pm$ and automorphic functions is as follows: the first power series χ_+ is related to the fourth powers of the Jacobi theta functions and the second χ_- is related to an eta product and to the modular curve $X_0(14)$. This is the content of theorems (6.4), (6.5) of this work. Consequently χ_+, χ_- which seem very different, are, in some sense, connected between them: the η-function is a θ-function. We will present at the end of section (6) an argument due to P. Bundschuh [3] showing that the presumed relation between χ_+ and χ_- cannot be in any way of algebraic nature. The theorem (4.1) gives a new presentation of a celebrated identity of Hardy. We give it here for two reasons: to insist on the fact that the oscillations are essentially due to the functional equations (1.3), the associated homogeneous equations of which is just $f(x^2) = f(x)$ whose solutions are the periodic functions of the variable $\dfrac{1}{\log 2} \log\log \dfrac{1}{x}$ of period $\log 2$, hence the evoked oscillations. The second reason is that the shape of Hardy's relation is not due to the lacunarity in an essential way. The series (1.1) is not lacunar but presents similar aspect, maybe more elaborate.

Our main references for modular forms and theta functions are Zagier [24] and Ono [17]. In what follows, we set $z = e^{2i\pi\tau}$, $\Im\tau > 0$.

Definition 1.1. The theta function, associated to the Dirichlet character ψ is the series given by

$$\theta_\psi(\tau) = \theta_\psi(z) = \sum_n \psi(n) e^{2i\pi n^2 \tau} = \sum_n \psi(n) z^{n^2}$$

if ψ is even, and if ψ is odd, by

$$\theta_\psi(\tau) = \theta_\psi(z) = \sum_n \psi(n) n e^{2i\pi n^2 \tau} = \sum_n \psi(n) n z^{n^2}.$$

The summations are over positive integers, unless ψ is the trivial character, in which case the summation is over all integers.

In the case of a trivial character, we write $\theta_\psi = \theta$ and we know that

$$\theta(\tau) = \prod_{n=1}^{\infty}(1-z^{2n})(1+z^{2n-1})^2 = \sum_{-\infty}^{\infty} z^{n^2}.$$

We recall first the definition of the Jacobi-Legendre symbol. If p is an odd prime and $a \in \mathbb{Z}$, then

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if} \quad p \nmid a \quad \text{and } x^2 \equiv a (\operatorname{mod} \mathrm{p}) \quad \text{has a solution,} \\ 0 & \text{if} \quad p \mid a, \\ -1 & \text{if} \quad p \nmid a \quad \text{and } x^2 \equiv a (\operatorname{mod} \mathrm{p}) \quad \text{does not have a solution.} \end{cases}$$

If b is a positive odd integer, we extend the definition of the Legendre symbol by

$$\left(\frac{a}{b}\right) = \prod_{j=1}^{r} \left(\frac{a}{p_j}\right)$$

where $b = \prod_{j=1}^{r} p_j$ is the prime decomposition. We retain that the sequence $\left(\chi_{-4}(n) = \left(\frac{-4}{n}\right)\right)_{n\geq 1}$ is 4-periodic

$$\left(\left(\frac{-4}{n}\right)\right)_{n\geq 1} = (1, 0, -1, 0, 1, 0, -1, 0, \cdots).$$

The connection of the character χ_{-4} with the Gaussian integers $\mathbb{Z}[i] = \{a+ib : a, b \in \mathbb{Z}\}$ will also be important in this work.

For later use we remind that the Dedekind eta function is

$$\eta(\tau) = z^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-z^n) = z^{\frac{1}{24}} \sum_{-\infty}^{\infty}(-1)^n z^{\frac{3n^2+n}{2}} = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) z^{\frac{n^2}{24}}. \tag{1.4}$$

Hence the eta function is a theta function and in fact if $\chi_{12} = \left(\frac{12}{\bullet}\right)$, then

$$\eta = \theta_{\chi_{12}}, \tag{1.5}$$

$$\chi_{12}(m) = \left(\frac{12}{m}\right) = \begin{cases} 1 & \text{for } m \equiv \pm 1 \ (\operatorname{mod}\ 12\ ) \\ \text{-1} & \text{for } m \equiv \pm 5 \ (\operatorname{mod}\ 12)\ . \end{cases}$$

On the other hand, we have the identity [17] (p.17):

$$\theta(z) = \theta(\tau) = \eta^5(2\tau)\eta^{-2}(\tau)\eta^{-2}(4\tau),$$

which can be considered as an inversion of (1.5).

§ 2. Various linked series and the High indices theorem

In general the singularities or the behaviors of a Dirichlet series

$$f(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n s}, \quad \lambda_0 < \lambda_1 < \dots$$

depend on the coefficients (a_n) and the exponents (λ_n). However there are circumstances where only the exponents determine the behavior of the sum or the singularities of the series on the boundary of the domain of convergence. An example of a such situation is given by the outstanding Tauberian theorem of Hardy and Littelwood [9] (Theorem 114, p.173):

Theorem 2.1 (High-Indices Theorem). *If $f(x) = \sum_{k=0}^{\infty} a_k x^{n_k}$ converges for $0 < x < 1$ where the exponents n_k are positive integers satisfying the lacunarity condition*

$$\frac{n_{k+1}}{n_k} \geq q > 1$$

and if $f(x) \to A$ as $x \to 1^-$, then the infinite series $\sum_{k=0}^{\infty} a_k$ converges and its sum is A.

An immediate consequence of this result is that, due to the divergence of the series $\sum a_n$, $a_n = \pm 1$ if n is a power of 2 and $a_n = 0$ otherwise, the radial limits

$$\lim_{r \to 1^-} \chi_{\pm}(re^{i\alpha}), \ 0 \leq \alpha \leq 2\pi$$

do not exist. Moreover, while the series $\chi_+(x)$ is unbounded in $[0, 1)$, the alternating series $\chi_-(x)$ verifies $0 \leq x - x^2 \leq \chi_-(x) \leq x < 1$ in that interval. Actually [8, 10] as $x \to 1^-$ the series $\chi_-(x)$ oscillates round $x = \frac{1}{2}$ with amplitude 2.75×10^{-3}, and is asymptotically periodic in $\dfrac{1}{\log 2} \log\log \dfrac{1}{x}$ of period $\log 2$. The series $\chi_-(z)$ has another attractive peculiarity. By means of formal manipulations we can find, starting from $\chi_-(z)$, another solution of the functional equation $f(z) = z - f(z^2)$. In fact

$$\chi_-(z) = \sum_{n \geq 0} (-1)^n z^{2^n}$$

$$= \sum_{n\ge 0}(-1)^n \sum_{m\ge 0} \frac{2^{nm}(\log z)^m}{m!}.$$

But very formally

$$\begin{aligned}\sum_{n\ge 0}(-1)^n \sum_{m\ge 0} \frac{2^{nm}(\log z)^m}{m!} &= \sum_{m\ge 0}\frac{(\log z)^m}{m!}\sum_{n\ge 0}(-2^m)^n \\ &= \sum_{m\ge 0}\frac{(\log z)^m}{m!(1+2^m)}.\end{aligned}$$

We obtain in this way another solution $\chi_{--}(z)$ of $f(z) = z - f(z^2)$ which is, furthermore, an entire function of $\log z$ and a holomorphic function in the disc $\{|z-1| < 1\}$ if, for example, the principal branch of the logarithm is chosen.

Proposition 2.2. *The infinite product*

$$h(\tau) = \prod_{n=0}^{\infty}\left(1 - \frac{\tau}{2^n}\right)$$

is the unique solution of the functional equation

$$h(\tau) = (1-\tau)h(\frac{\tau}{2}), \quad h(0) = 1.$$

It represents an entire function of zero exponential type with the following expansion

$$h(\tau) = 1 + \sum_{n=1}^{\infty}\frac{2^n}{(1-2^n)(1-2^{n-1})\cdots(1-2)}\tau^n$$

and hence

$$h\left(\frac{d}{d\tau}\right) = 1 + \sum_{n=1}^{\infty}\frac{2^n}{(1-2^n)(1-2^{n-1})\cdots(1-2)}\frac{d^n}{d\tau^n}$$

is a well defined infinite order differential operator with

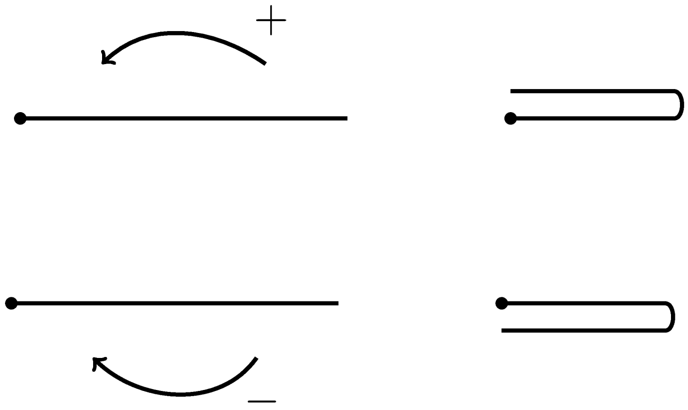
$$h(\frac{d}{d\tau})\chi_+(z) = 0, \quad h(\frac{d}{d\tau})\chi_-(z) = 0.$$

The functions $\chi_\pm$ share the following analytic property: the associated Dirichlet series are solutions of infinite order differential operators, hence the line of convergence are natural boundaries. It is possible to show that actually, the Dirichlet series associated to (1.2) cannot be continued analytically across the line of convergence $\Im\tau = 0,\ z = e^{2i\pi\tau}$. This means that the power series have the unit circle as a natural boundary [15], [20]. This is another reading of the High Indices theorem.

The functions $\chi_+(z)$ have also some deep arithmetical properties. K.Mahler showed in [12] that for every $d \geq 2$, the value of $\sum_{n\geq 0} z^{-d^n}$ at any no zero algebraic number is transcendental.

§ 3. Continued fractions

We will refer to [4], [1], [15] for a large presentation of paperfolding. A sheet of paper can be folded once right half over left or left half over right



$(\varepsilon_1 = 1)$

$(\varepsilon_1 = -1)$

We start with a sheet of paper and fold it in half in the positive sense. The resulting piece is now folded in a half, always in the positive sense and so forth. After its unfolding the sheet exhibits the pattern

$$\vee \vee \wedge \vee \vee \wedge \wedge \vee \vee \vee \wedge \wedge \vee \wedge \wedge \cdots \tag{3.1}$$

If we encode the binary sequence by $\vee = +1$, $\wedge = -1$, we obtain the paperfolding sequence $(s_n)_{n\geq 1}$ satisfying

$$s_{2n} = s_n, n \geq 1; \quad s_{2n+1} = (-1)^n, n \geq 0$$

or, equivalently

$$s_n = (-1)^b \ \text{ if } \ n = 2^a(1+2b). \tag{3.2}$$

We have a general result concerning the expansion of formal power series in continued fractions:

Theorem 3.1 (Artin). *Let n be an integer. Any formal power series*

$$f(X) = \sum_{-\infty < i \leq n} b_i X^i \in \mathbb{Q}((X^{-1}))$$

can be expanded uniquely into continued fractions

$$f(X) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

$$= [a_0, a_1, a_2, \cdots]$$

where $a_i \in \mathbb{Q}[X]$ *and* $\deg a_i > 0$ *for* $i > 0$.

The expansion in continued fractions of $\tilde{\chi}_+(z) = \sum_{n\geq 0} z^{-2^n}$ is of particular interest. It begins as follows [11], [21] :

$$\frac{1}{z} + \frac{1}{z^2} = \cfrac{1}{z - 1 + \cfrac{1}{z+1}} = [0, z-1, z+1]$$

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^4} = [0, z-1, z+2, z, z-1].$$

More generally if $n = 2^m$ and

$$\frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^{2^m}} = \frac{\tilde{p}_n}{\tilde{q}_n} = [0, a_1, a_2, \cdots, a_n].$$

with $\tilde{q}_n(z) = z^{2^n}$, then

$$\frac{1}{z} + \frac{1}{z^2} + \cdots + \frac{1}{z^{2^m}} + \frac{1}{z^{2^{m+1}}} = \frac{\tilde{p}_n}{\tilde{q}_n} + \frac{1}{\tilde{q}_n^2} = [0, a_1, a_2, \cdots, a_{n-1}, a_n + 1, a_n - 1, a_{n-1} \cdots, a_1].$$

In other words if $\overrightarrow{w}$ is the word $a_1, a_2, \cdots, a_n$, that is $\frac{\tilde{p}_n}{\tilde{q}_n} = [0, \overrightarrow{w}]$, then

$$\frac{\tilde{p}_n}{\tilde{q}_n} + \frac{1}{\tilde{q}_n^2} = [0, \overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w}]$$

with the notation

$$\overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w} = a_1, a_2, \cdots, a_{n-1}, a_n + 1, a_n - 1, \cdots, a_1.$$

This defines an operator $S : \overrightarrow{w} \to \overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w}$, so that

(3.3) $$\overrightarrow{w} \xrightarrow{S} \overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w} \xrightarrow{S} \overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w}\,\overrightarrow{p}\,\overrightarrow{w}\,\overleftarrow{p}\,\overleftarrow{w} \xrightarrow{S} \overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w}\,\overrightarrow{p}\,\overrightarrow{w}\,\overleftarrow{p}\,\overleftarrow{w}\,\overrightarrow{p}\,\overrightarrow{w}\,\overrightarrow{p}\,\overleftarrow{w}\,\overleftarrow{p}\,\overrightarrow{w}\,\overleftarrow{p}\,\overleftarrow{w} \xrightarrow{S} \cdots$$

and the expansion in continued fractions of $\tilde{\chi}_+(z) = \sum_{n\geq 0} z^{-2^n}$ is formally given by

$$\sum_{n\geq 0} z^{-2^n} = S^{\infty}[z-1, z+1] = \lim_{n\to\infty} S^n[z-1, z+1], \quad S^n = S \circ S \circ \cdots \circ S \ (n\text{–times}).$$

Thus we have the following table

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
folds	$\vee$	$\vee$	$\wedge$	$\vee$	$\vee$	$\wedge$	$\wedge$	$\vee$	$\vee$	$\vee$	$\wedge$	$\wedge$	$\vee$	$\wedge$	$\wedge$	$\cdots$
arrows	$\rightarrow$	$\rightarrow$	$\leftarrow$	$\rightarrow$	$\rightarrow$	$\leftarrow$	$\leftarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\leftarrow$	$\leftarrow$	$\rightarrow$	$\leftarrow$	$\leftarrow$	$\cdots$
$(-1)^{n_0}$	1	1	-1	1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	$\cdots$

The three lines are identical if we attribute to $\vee$ and $\rightarrow$ the value 1 and to $\wedge$ and $\leftarrow$ the value -1.

§ 4. Hardy's identity through differences equations

We give an interpretation of an identity of Hardy by using some differences equations [8]. This identity reads, for $0 < x < 1$:

$$\sum_{n\geq 0} x^{2^n} = \frac{1}{\log 2}\ \log(\log 1/x) + \frac{1}{2} - \frac{\gamma}{\log 2}\sum_{n\geq 1}\frac{(-1)^{n-1}(\log 1/x)^n}{n!(2^n-1)} - \frac{1}{\log 2}\ \sum_{k\neq 0}\Gamma\left(\frac{-2ki\pi}{\log 2}\right)(\log 1/x)^{2ki\pi/\log 2}. \tag{4.1}$$

We consider the power series

$$G(z) = G(z,a) = \sum_{n\geq 1}\frac{1}{a^n-1}\frac{z^n}{n!} \tag{4.2}$$

where $a \neq 1$ is a given positive real number. The function G is an entire function solution of the functional equation

$$G(z,a) = -G(\frac{z}{a},\frac{1}{a}). \tag{4.3}$$

We introduce the new variable u, $z = -a^u$ and the function

$$\mathcal{H}(u) = -zG'(z). \tag{4.4}$$

Then

$$\mathcal{H}(u+1) - \mathcal{H}(u) = j_a(u) = a^u e^{-a^u}. \tag{4.5}$$

The function j_a belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ of infinitely differentiable functions f such that $u^n f^{(m)}$ is bounded on $\mathbb{R}$. In particular each of the two series $\sum_{m\geq 0} j_a(u+m)$, $\sum_{m\geq 1} j_a(u-m)$ converges uniformly on compact sets of $\mathbb{R}$.

Theorem 4.1. *The equation* (4.5) *has three entire solutions* $\mathcal{K}(u) = \mathcal{K}(u,a)$, $\mathcal{K}_1(u) = \mathcal{K}_1(u,a)$ *and* $\mathcal{K}_2(u) = \mathcal{K}_2(u,a)$, *with*

(1) $\mathcal{K}(u) = -\sum_{n\geq 0} \frac{(-1)^{n+1}}{a^{n+1}-1} \frac{a^{(n+1)u}}{n!}$.

(2) $\mathcal{K}_1(u) = \sum_{m\geq 1} j_a(u-m)$.

(3) $\mathcal{K}_2(u) = -\sum_{m\geq 0} j_a(u+m)$.

(4) $\mathcal{K}(u) = \mathcal{K}_1(u)$ *if* $a > 1$; $\mathcal{K}(u) = \mathcal{K}_2(u)$ *if* $a < 1$.

(5) $\mathcal{K}_1(u) - \mathcal{K}_2(u) = P(u)$ *is a periodic function of period 1 given by*

$$P(u) = \sum_{-\infty}^{+\infty} \Gamma\left(1 - \frac{2in\pi}{\log a}\right) e^{2n\pi iu}.$$

(6) *The map which carries* $\mathcal{K}_1(u)$ *on* $\mathcal{K}_2(u)$ *is represented by the functional relation* (4.3)

$$G(z,a) = -G(\frac{z}{a}, \frac{1}{a}).$$

(7) *The following identity holds for all* $a > 1$

$$\sum_{n\geq 1} \frac{(-1)^n}{a^n - 1} \frac{a^{nu}}{n!} = \sum_{m\geq 1} \left(e^{-a^{u-m}} - 1\right).$$

The proof of the theorem is based on the difference equation (4.5). Since the second member j_a is in the Schwartz space $\mathcal{S}(\mathbb{R})$, we can solve this equation by right iteration, left iteration or localization. More explicitly, let D be the differentiation operator $\dfrac{d}{dx}$. By Taylor's formula the equation (4.5) takes the form

$$\left(e^D - 1\right)\mathcal{H}(u) = j_a(u)$$

or

$$\left(1 - e^{-D}\right)\mathcal{H}(u) = j_a(u-1).$$

The formal inverse of the operator $(1 - e^{-D})$ is $\displaystyle\sum_{n\geq 0} e^{-nD}$ so that the equation (4.5) has the solution $\mathcal{K}_1(u) = \displaystyle\sum_{m\geq 1} j_a(u-m)$ which is in fact an entire function, the series being a series of holomorphic functions on $\mathbb{C}$ uniformly convergent on compact sets.
The equation (4.5) written in the form $\left(1 - e^D\right)\mathcal{H}(u) = -j_a(u)$ can be solved formally by right iteration $\mathcal{K}_2(u) = -\sum_{m\geq 0} j_a(u+m)$ and we obtain an another entire solution.
The third method to solve (4.5) is by the localization method. We define the operator $D^{-1} = \dfrac{1}{D}$ by

$$D^{-1}g(u) = \int_0^u g(x)dx,$$

hence the equation (4.5) gives

$$\left(\frac{e^D - 1}{D}\right)\mathcal{H}(u) = \int_0^u j_a(x)\,dx. \tag{4.6}$$

The function $\dfrac{z}{e^z - 1}$ is holomorphic in the disc centered at the origin and of radius 2π and has the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n, \quad |z| < 2\pi.$$

The coefficients b_n are the Bernoulli numbers

$$b_0 = 1,\ b_1 = -\frac{1}{2},\ b_2 = \frac{1}{6},\ b_3 = 0,\ b_4 = -\frac{1}{30}, \cdots$$

A formal solution of the equation (4.6) is

$$\mathcal{K}_3(u) = D^{-1} j_a(u) + \sum_{n \leq 1} \frac{b_n}{n!} D^n D^{-1} j_a(u)$$

and an easy calculation shows that

$$\mathcal{K}_3(u) = a^u \sum_{m \geq 0} \frac{(-1)^m a^{um}}{m!(a^{m+1} - 1)}.$$

We observe that we have recovered the function $\mathcal{K}_3(u) = \mathcal{K}(u)$ from which we derived the difference equation (4.5).

The difference $\mathcal{K}_1(u) - \mathcal{K}_2(u)$ is a solution of the homogenous equation

$$\mathcal{H}(u+1) - \mathcal{H}(u) = 0$$

so it is a periodic function of period 1. The Fourier coefficients can be computed as in the preceding section. The map $a \mapsto \dfrac{1}{a}$ is a homeomorphism from (0, 1) onto $(1, +\infty)$, so we can suppose in the sequel $a > 1$, $z = a^u$. The relation (4.3) gives $-zG'(z,a) = \dfrac{z}{a}G'(\dfrac{z}{a}, \dfrac{1}{a})$ which is equivalent to $\mathcal{K}_1(u) = \mathcal{K}_2(1-u, \dfrac{1}{a})$. Finally if we integrate from $-\infty$ to u the identities

$$\sum_{m \geq 1} j_a(x - m) = \mathcal{K}_1(x) = \mathcal{K}(x)$$

and use

$$\int_{-\infty}^{u} j_a(x-m)\,dx = \frac{-1}{\log a}\left(e^{-a^{u-m}} - 1\right)$$

we obtain the last statement of the proposition.

Remark 4.1. We have seen that Hardy's identity can be considered as a connexion formula between two solutions to the same difference equation. In principle other connexion formulas could be obtained by considering other solutions. However, according to the asymptotic formula

$$|\Gamma(x+iy)| = \sqrt{2\pi} e^{-\frac{1}{2}\pi|y|} |y|^{x-\frac{1}{2}} \left(1 + r(x,y)\right),$$

where as $|x+iy|$ tends to ∞, $r(x,y)$ tends to 0, uniformly in the strip $|x| \leq \alpha$ where α is a constant, the connexion

$$\mathcal{K}_1(u) - \mathcal{K}_2(u) = P(u) = \sum_{-\infty}^{+\infty} \Gamma(1 - 2n\pi i A) e^{2n\pi i u},$$

with $A = \dfrac{1}{\log a}$, is very small on the real axis; two different solutions of the same difference equation can be very close. This has been pointed out, in another context, by Ramanujan and Hardy (see the discussion in [2]; Entry 17) and also, more recently, by Tricomi [23] who found that for $a = 2$ or $A = \dfrac{1}{\log 2} = 1.442695...$ one has for every real value x

$$|P(x) - 1| < 0.0000099.$$

On the other hand Berndt quotes in [2] the approximation

$$\log 2 \left(\sum_{n=0}^{\infty} 2^n e^{-2^n x} + \sum_{n=0}^{\infty} \frac{(-x)^n}{(2^{n+1}-1)n!} \right) \approx$$
$$\frac{1}{x}\left(1 + 0.0000098844 \cos\left(\frac{2\pi \log x}{\log 2} + 0.872711\right)\right).$$

We would like to investigate how the functional relations (4.3) and (4.4) can be linked to modular properties of some Eisenstein series. The function (4.2)

$$G(z) = \sum_{n \geq 1} \frac{1}{a^n - 1} \frac{z^n}{n!}$$

is an entire function of order 1 and of exponential type $\dfrac{1}{a}$. Its Borel transform is

$$B(z,\, a) = \sum_{n \geq 1} \frac{1}{a^n - 1} z^{-(n+1)}, \quad |z| > \frac{1}{a}. \tag{4.7}$$

Now we consider

$$\xi(z,\, a) = \xi(z) = \sum_{n \geq 1} \frac{1}{a^n - 1} z^n, \; |a| > 1. \tag{4.8}$$

This function has some interesting properties. We first prove

Proposition 4.2. *The function $\xi(z,\,a)$ has a meromorphic continuation to the whole plane with simple poles at a^n, $n \in \mathbb{Z}_+$ and with corresponding residues a^n.*

In fact, from the definition, we see that for $|z| < 1$

$$\xi(z,\,a) - \xi(\frac{z}{a},\,a) = \frac{z}{a-z}$$

so that

$$\xi(z,\,a) = z \sum_{n\geq 1} \frac{1}{a^n - z}.$$

This last series converges uniformly on compact sets of $\mathbb{C} \setminus \{a^n, n \in \mathbb{Z}_+\}$; the result follows.

Let $B(t)$ be the function defined for $0 < t < \infty$ by

$$B(t) := \frac{1}{4} - \frac{\pi}{12}t + 2\pi t \frac{\partial \xi}{\partial z}(1,\, e^{2\pi t}).$$

Proposition 4.3. *The function B verifies the functional relation*

$$B(t) + B(\frac{1}{t}) = 0.$$

In particular $B(1) = 0$ or

$$\sum_{n\geq 1} \frac{n}{e^{2\pi n} - 1} = \frac{1}{24} - \frac{1}{8\pi} = 0.0018779...$$

The proof of the proposition is an immediate consequence of the modular transformation law of the Eisenstein series

$$G_2(\tau) = -\frac{1}{24} + \sum_{n\geq 1} \frac{n}{e^{-2i\pi n\tau} - 1} = -\frac{1}{24} + \sum_{n\geq 1} \sigma_1(n) e^{2i\pi n\tau},\ \sigma_1(n) = \sum_{d|n} d \tag{4.9}$$

which satisfies, for all $\begin{pmatrix} \alpha\,\beta \\ \gamma\,\delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$G_2\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = (\gamma\tau + \delta)^2 G_2(\tau) - \frac{\gamma(\gamma\tau + \delta)}{4i\pi}.$$

We obtain the desired formula by choosing $\alpha = \delta = 0,\ \beta = -1, \gamma = 1$ and $\tau = it,\ t > 0$.

Now we show that the basic function (4.8) is related to the Eisenstein series G_2. We shall suppose $q \in \mathbb{C}$, $|q| < 1$ and introduce

$$\chi(z,\,q) = \sum_{n\geq 1} \frac{z^n}{1 - q^n} \tag{4.10}$$

and

$$f(z,\, q) = \sum_{n\geq 1} \frac{z^n}{n(1-q^n)}. \tag{4.11}$$

We clearly have $\chi(z,\, q) = z\dfrac{\partial}{\partial z} f(z,\, q)$. For $|z| < 1$

$$\chi(z,\, q) = \xi(\frac{z}{q},\, \frac{1}{q})$$

and for $|z| > q$ and $a = \dfrac{1}{q}$, and from (4.7)

$$B(z,\, a) = \frac{1}{z}\xi(\frac{1}{z},\, a) = \frac{1}{z}\chi(\frac{q}{z},\, q)$$

Theorem 4.4. *Let $F(z,q)$ be defined for $|z| < 1$ by*

$$F(z,q) = \frac{1}{(1-q)(1-qz)(1-q^2z)...} = 1 + \sum_{n\geq 1} \frac{1}{(1-q)(1-q^2)...(1-q^n)} z^n$$

Then

$$\chi(z,\, q) = z\frac{F'(z,q)}{F(z,q)}$$

The argument is classical and we repeat it here for the sake of completeness, we refer to ([18], Chapter 12) for more developments on $F(z,q)$. First we have

$$e^{f(z,q)} = F(z,q)$$

for

$$\begin{aligned} f(z,q) &= \sum_{n\geq 1} \frac{z^n}{n(1-q^n)} = \sum_{n\geq 1} \frac{z^n}{n} \sum_{m\geq 0} q^{mn} \\ &= \sum_{m\geq 0}\sum_{n\geq 1} \frac{(zq^m)^n}{n} = \sum_{m\geq 0} \log \frac{1}{1-zq^m} = \log F(z,q). \end{aligned}$$

On the other hand if $F(z,q) = \sum_{n\geq 0} c_n z^n$, then the relation $F(z,qz) = (1-z)F(z,q)$ gives

$$c_0 = 1, \quad c_n(1-q^n) = c_{n-1}, \quad n \geq 1,$$

so that

$$F(z,q) = 1 + \sum_{k\geq 1} B_k(q) z^k, \quad B_k(q) = \frac{1}{(1-q)(1-q)(1-q^2)...(1-q^n)}.$$

Remark 4.2. The power series (4.10) and (4.11) defined initially for $a \in \mathbb{C}$, $|a| \neq 1$ can also be considered for $a = e^{i\pi\alpha}$ for irrational α. Although denominators $1 - e^{i\pi n\alpha}$ never vanish, their modulus $|1 - e^{i\pi n\alpha}|$ can become very small (phenomenon of small divisors).

§ 5. Two families of polynomials, The Erdös-Turan theorem

Jentzsch showed that if a power series has a finite radius of convergence then every point of its circle of convergence is a limit point of zeros of its partial sums. Szegö sharpened the theorem of Jentzch by proving that for a power series with finite radius of convergence there is an infinite sequence of partial sums, the zeros of which are equi-distributed in Hermann Weyl sense. The meaning of this term is as follows: let $P_n(z)$ be a sequence of polynomials, let the degree of $P_n(z)$ be m_n and suppose that $\lim_{n\to\infty} m_n = \infty$. The number of the zeros of the polynomial $P_n(z)$ in the sector

$$\{z \in \mathbb{C},\ \alpha < \arg z < \beta\}$$

is called $N_n(\alpha, \beta)$. We say that the zeros of the sequence $P_n(z)$ are equi-distributed if for all $\alpha, \beta \in \mathbb{R}$, $\beta - \alpha < 2\pi$:

$$N_n(\alpha, \beta) = \frac{\beta - \alpha}{2\pi} m_n + \mathrm{o}(m_n).$$

For a polynomial $P(z) = a_0 + a_1 z + \cdots a_n z^n \in \mathbb{C}[z]$, $a_0 a_n \neq 0$, we introduce

$$\|P\| = \frac{|a_0| + |a_1| + \cdots |a_n|}{\sqrt{|a_0 a_n|}}$$

and $N(\alpha, \beta)$ as above.

Theorem 5.1 (Erdös-Turan). *For each polynomial as above and for every α, β reals with $0 \leq \beta - \alpha \leq 2\pi$, we have*

$$|N(\alpha, \beta) - \frac{\beta - \alpha}{2\pi} n| \leq 16\sqrt{n} \log \|P\|.$$

This theorem make precise the $\mathrm{o}(m_n)$-term in Jentzch-Szëgo result and clarifies the angular distribution of the zeros.

The polynomial sections $P_n(z) = \sum_{k=0}^{n} z^{2^k}, Q_n(z) = \sum_{k=0}^{n} (-1)^k z^{2^k}, n \geq 1$ are very lacunar with $\|P_n\| = \|Q_n\| = n + 1$, $\deg P_n = \deg Q_n = 2^n$. The zeros of P_n and Q_n are thus equi-distributed:

$$\lim_{n\to\infty} \frac{1}{2^n} N_{\frac{1}{2^n}}(\alpha, \beta) = \frac{\beta - \alpha}{2\pi}.$$

We can not say as much about the polynomials $S_n(x) = \sum_{k=1}^{n} s_k x^k$ where s_k is the paperfolding sequence, given by (3.2). But the three family of polynomials $P_n(z), Q_n(z), S_n(z)$ have all their zeros contained in the set $\{0\} \cup \{\frac{1}{2} \le |z| \le 2\}$, as does every polynomial whose coefficients are $0, \pm 1$. Furthermore each $P_n(z)$ have only one real zero z_n which is negative and $z_n \le z_{n+1}$. The first twelve zeros z_n are:

$$\mathbf{z_1} = -1.000000000000000000000000000000000000000000000000000000000000000000000$$

$$\mathbf{z_2} = -0.6823278038280193273694837397110482568911885818979985778037286066398967$$

$$\mathbf{z_3} = -0.6592895449569986283320207591165189714731785041876226862903442110856223$$

$$\mathbf{z_4} = -0.6586275727947914281155763976034397132031395200298308926274658839407222$$

$$\mathbf{z_5} = -0.6586267543014505676645776925986416538401407882505332206733236057119207$$

$$\mathbf{z_6} = -0.6586267543001639224134760102227554371362052786117330805918318056595727$$

$$\mathbf{z_7} = -0.6586267543001639224134728305795016459409327962398551705216177085272378$$

$$\begin{aligned}\mathbf{z_8} = {} & -0.6586267543001639224134728305795016459409327962204365870628047777374586\\ & 8299975130224075993074098734387763223946874982322867952183799081653559185989\\ & 744257402868967105119981765674169220114927141874085437\end{aligned}$$

$$\begin{aligned}\mathbf{z_9} = {} & -0.6586267543001639224134728305795016459409327962204365870628047777374586\\ & 8299975130224075993074026308163445925523062324487795701245654471086611436057\\ & 295820443839306149053769731049528666760996220137777720\end{aligned}$$

$$\begin{aligned}\mathbf{z_{10}} = {} & -0.6586267543001639224134728305795016459409327962204365870628047777374586\\ & 8299975130224075993074026308163445925523062324487795701245654471086611436057\\ & 295820443839306149053769731049528666760895468423506261\end{aligned}$$

$$\begin{aligned}\mathbf{z_{11}} = {} & -0.6586267543001639224134728305795016459409327962204365870628047777374586\\ & 8299975130224075993074026308163445925523062324487795701245654471086611436057\\ & 295820443839306149053769731049528666760895468423506261\end{aligned}$$

$$\begin{aligned}\mathbf{z_{12}} = {} & -0.6586267543001639224134728305795016459409327962204365870628047777374586\\ & 8299975130224075993074026308163445925523062324487795701245654471086611436057\\ & 295820443839306149053769731049528666760895468423506261\end{aligned}$$

The sequence $(\mathbf{z_k})$ converges very rapidly to the unique negative real zero of $\chi_+(z)$.

§ 6. χ_+, χ_- and modular forms

§ 6.1. Elliptic curves

This section is a complement of our work [20]. We give an idea on the relation connecting some lacunary series (like χ_+, χ_-) and elliptic curves and modular forms, taking into account, of course, the existence of a strong relationship between these two last concepts. The reason is that the moduli of elliptic curves are expressible in terms of modular forms of the parameter τ, $\Im\tau > 0$. Moreover, in the 1950's a precise relation between elliptic curves and modular forms was formulated, first by Taniyama, then by Shimura and Weil. We refer to [5] for a thorough presentation. Let E be an elliptic curve defined over Q, say by an equation $y^2 = 4x^3 - ax - b$ with rational integers a, b. For every prime p not dividing the discriminant $\Delta = a^3 - 27b^2$, we get an elliptic curve over the finite field F_p with this equation. We therefore have its zeta function, the numerator of which has the shape $1 - a_p t + pt^2$, with a_p defined by counting the number of solutions to the congruence $y^2 \equiv 4x^3 - ax - b \pmod{p}$,

$$1 - a_p + p = \sharp(F_p).$$

Note that $\sharp(F_p)$ is actually one more than the number of solutions to the congruence, since E has one point at infinity in the projective plane. Following Hasse, we consider the infinite product

$$L(E, s) = \prod_p (1 - a_p^{p-s} + p^{1-2s})^{-1}.$$

Then Wiles' theorem, conjectured by Taniyama, Shimura and Weil, is [5]

Theorem 6.1 (Modularity theorem)**.** *Let E be an elliptic curve E over $\mathbb{Q}$ with conductor N; there exists $f(\tau)$, a cusp form of weight 2 for $\Gamma_0(N)$, such that $L(f, s) = L(E, s)$.*

In a very simplified way, Wiles' theorem says that if an elliptic curve E, say defined by an equation of the form $f(x, y) = 0$, $f(x, y) \in \mathbb{Z}[x, y]$, and for any prime p not dividing its discriminant, let $E(F_p)$ the number of solutions to the congruence $f(x, y) = 0$ mod p including the point(s) at infinity, written in the form $E(F_p) = 1 + p - a_p(E)$. Then the integer $a_p(E)$ is the $p-$th Fourier coefficient of a cusp form of weight 2, associated to $\Gamma_0(N)$. Here, N is the conductor of E.

§ 6.2. η-products

We recall first some properties of the Dedekind η-function (1.4), $\eta(\tau) = e^{\frac{2i\pi\tau}{24}} \prod_{n=1}^{\infty} \left(1 - e^{2i\pi n\tau}\right)$. It satisfies [18]

$$\eta(\tau+1) = \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

and for all $\begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)(c\tau+d)^{\frac{1}{2}}\eta(\tau)$$

where

$$\epsilon(a,b,c,d) = e^{\frac{ib\pi}{12}} \quad \text{for} \quad c = 0 \quad \text{and} \quad d = 1,$$
$$\epsilon(a,b,c,d) = e^{i\pi\left(\frac{a+d}{12c} - S(d,c) - \frac{1}{4}\right)} \quad \text{for} \quad c > 0$$

and

$$S(h,k) = \sum_{n=1}^{k-1} \frac{n}{k}\left(\frac{hn}{k} - [\frac{hn}{k}] - \frac{1}{2}\right).$$

The interest in the η-function lies in the following theorem [17] p.18:

Theorem 6.2. *Let N a positive integer and let $f(z) = \prod_{d|N} \eta^{r_d}(dz)$ be an η-product, $r_d \in \mathbb{Z}$. If N is such that*

$$\sum_{d|N} dr_d = 0 \pmod{24},$$
$$\sum_{d|N} \frac{Nr_d}{d} = 0 \pmod{24}.$$

Then

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all

$$\begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in \Gamma_0(N) = \left\{ \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \text{ (mod N)} \right\}$$

where $k = \frac{1}{2}\sum_{d|N} r_d$. The character χ is defined by the Legendre-Jacobi symbol

$$\chi(d) = \left(\frac{(-1)^k s}{d}\right), \quad s = \prod_{d|N} d^{r_d}.$$

More interesting for us is the example of the η-product

$$S(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau). \tag{6.1}$$

It is one of the exactly 30 cusp forms, with multiplicative coefficients, of the forms

$$\prod_{k=1}^{s} \eta^{t_k}(a_k\tau),\ a_k, t_k \in \mathbb{N},$$

discovered by D.Dummit, H.Kisilevskii and J.McKay [6]. It is a newform ([5], [17]) of weight 2 whose Mellin transform agrees with the Hasse-Weil L-function of the (isogeny class of the) elliptic curve of equation

$$y^2 + xy + y = x^3 - x.$$

This curve is modular, of conductor 14 and is generally denoted by $X_0(14)$. We will soon meet this curve by another method.

§ 6.3. Main results

We continue to use the previous notations except that z is replaced by q, according to a current usage.

$$\chi_+(q) = \sum_{n\geq 0} q^{2^n}, \quad \chi_-(q) = \sum_{n\geq 0}(-1)^n q^{2^n},\ |q| < 1.$$

We fix an odd positive integer l. let f be a modular form in $M_2(2l)$, that is of weight 2 on the group $\Gamma_0(2l)$. We assume that f is an eigenform of the Hecke operator $T(2)$, which is defined by

$$T(2) : f = \sum_{n\geq 0} a_f(n)q^n \to \sum_{n\geq 0} a_f(2n)q^n.$$

It is known that the only eigenvalues of $T(2)$ on newforms in $M_2(2n)$ are ± 1 [22]. Let ϵ be the eigenvalue of f, and assume that it equals ± 1. Then for all n

$$a_f(2n) = \epsilon a_f(n)$$

In other words, we have

$$f = a_f(0) + \sum_{n \text{ odd}} a_f(n)\chi_\epsilon(q^n).$$

Now for an arithmetical function $\alpha(n)$ with $\alpha(1) = 1$ we define $\hat{\alpha}(n)$ by

$$\sum_{n\geq 1} \frac{\hat{\alpha}(n)}{n^s} = \left(\sum_{n\geq 1} \frac{\alpha(n)}{n^s}\right)^{-1}.$$

We have the following lemma, easily proved by taking Mellin transforms.

Lemma 6.3. *Let F, G be two formal power series in q, and let $\alpha(n)$ be arithmetical function with $\alpha(1) = 1$, then*

$$G(q) = \sum_{n \geq 1} \alpha(n) F(q^n)$$

if and only if

$$F(q) = \sum_{n \geq 1} \hat{\alpha}(n) G(q^n).$$

We now assume that f is normalized, i.e. $a_f(1) = 1$. Then we can apply the lemma to deduce

$$\chi_\epsilon(q) = \sum_{n \geq 1,\, n \,\mathrm{odd}} \hat{a}_f(n) \left(f(nz) - a_f(0) \right).$$

We look at some examples. The first nonzero space is $M_2(2)$. It is 1-dimensional and $T(2)$ is the identity. Let E be the unique normalized modular form in $M_2(2)$.

$$E(z) = \frac{1}{24} + \sum_{n \geq 0} \left(\sum_{d | n,\, d \,\mathrm{odd}} d \right) q^n = \frac{1}{24} \left(\theta_0^4 + \theta_1^4 \right),$$

where, with a classical notation for two other Jacobi theta functions

$$\theta_0(q) = \sum_{n \equiv 0 \,(\mathrm{mod} 2)} q^{\frac{n^2}{4}}, \qquad \theta_1(q) = \sum_{n \equiv 1 \,(\mathrm{mod} 2)} q^{\frac{n^2}{4}}.$$

In particular, we have

$$\hat{\alpha}_E(n) = \sum_{d | n,\, d \,\mathrm{odd}} \mu(d) \mu(\frac{n}{d}) d.$$

Summarizing we find (Compare with [15])

Theorem 6.4.

$$\chi_+(q) = \frac{1}{24} \sum_{n \geq 1, n \,\mathrm{odd}} \hat{\alpha}_E(n) \left(\theta_0^4(nz) + \theta_1^4(nz) - 1 \right).$$

Remark 6.1. In spite of this relation, the functions χ_+ and theta functions present different behavior at 1:

$$\sum_{n=0}^{\infty} x^{n^2} \sim \frac{\sqrt{\pi}}{2\sqrt{1-x}} \sim \frac{\sqrt{\pi}}{2\sqrt{-\log x}}, \qquad \sum_{n=0}^{\infty} x^{2^n} \sim -\frac{\log(1-x)}{\log 2}.$$

We now consider the function χ_-: the first even level where $T(2)$ has eigenvalue -1 is $l = 14$. The space $M_2(14)$ is 4-dimensional. It contains a unique cusp form S, which is also the unique eigenform of $T(2)$ with eigenvalue -1. We can normalize it so that $a_S(1) = 1$. Its Fourier expansion is, with $q = e^{2i\pi\tau}, \Im\tau > 0$:

$$S(\tau) = q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 - 2q^{12} - 4q^{13} - q^{14} + q^{16} + 6q^{17} - q^{18} \\ +2q^{19} - 2q^{21} + 2q^{24} - 5q^{25} + 4q^{26} + 4q^{27} + q^{28} - 6q^{29} - 4q^{31} - q^{32} - 6q^{34} + q^{36} \\ +2q^{37} - 2q^{38} + 8q^{39} + 6q^{41} + \cdots$$

It can be described in closed form using the Dedekind eta function by

$$S(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$$

already met in (6.1). This is a concrete example of the Modularity Theorem (6.1). The L-series of S is the L-function of the elliptic curve $X_0(14)$:

$$E : y^2 + xy + y = x^3 - x$$

given by

$$\sum_{n\geq 1} \frac{a_S(n)}{n^s} = \prod_p \frac{1}{1 - [p - \sharp E(\mathbb{F}_p)]p^{-s} + p^{1-2s}}$$

where

$$\sharp E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p : y^2 + xy + y = x^3 - x\}.$$

Now the series χ_- can be given by η-function as follows

Theorem 6.5. *With $q = e^{2i\pi\tau}, \Im\tau > 0$ and $\chi_-(q) = \sum_{n\geq 0}(-1)^n q^{2^n}$, we have*

$$\chi_-(q) = \sum_{n\geq 1,\, n \,\mathrm{odd}} \hat{\alpha}_S(n)\eta(n\tau)\eta(2n\tau)\eta(7n\tau)\eta(14n\tau).$$

Like in theorem (6.4), we can rewrite the identity in the last theorem by using four theta functions and the identity (1.5).

Definition 6.1. For each $a \in \mathbb{N}$, we define the theta series $\theta_{6,a}$ by

$$\theta_{6,a}(q) = \sum_{n\in\mathbb{Z},\, n\,\equiv a \,(\mathrm{mod}\, 12)} q^{\frac{n^2}{24}}.$$

According to (1.5), $\eta = \theta_{6,1} - \theta_{6,5}$, hence

Corollary 6.6.

$$\chi_-(q) = \sum_{k\geq 0}(-1)^k q^{2^k} = \sum_{n\geq 1,\, n\,\mathrm{odd}} \hat{\alpha}_S(n) \\ \sum_{a_1,a_2,a_3,a_4 \equiv 1,5\,(\mathrm{mod}\, 12)} \left\{\frac{a_1a_2a_3a_4}{3}\right\} \theta_{6,a_1}(n\tau)\theta_{6,a_2}(2n\tau)\theta_{6,a_3}(7n\tau)\theta_{6,a_4}(14n\tau).$$

Here $\left\{\dfrac{n}{3}\right\} = \epsilon$, *where* $\epsilon \equiv n \pmod 3$ with $-1 \leq \epsilon \leq +1$.

According to the fact that the η-function is also a θ-function, we are tempted to think that there must be some hidden relationship between χ_+ and χ_-. This relation should be highly transcendental. Below we sketch that, at least in one respect, there is no hope for "relationships" of algebraic nature. This observation and the accompanied proof were communicated to us by Peter Bundschuh [3]:

Let $d \geq 2$ be an integer and consider the two power series

$$\chi_\pm(z;d) := \sum_{n=0}^{\infty} (\pm 1)^n z^{d^n}, \quad |z| < 1.$$

They are Mahler's functions:

$$\chi_\pm(z;d) = \pm\chi_\pm(z^d;d) + z$$

with $\chi_+(z;2) = \chi_+(z)$, $\chi_-(z;2) = \chi_-(z)$. Using solely this fact, the Theorem 3.2.2 in [16] leads to the result that, for any $d \geq 2$, the two functions $\chi_+(z;d)$ and $\chi_-(z;d)$ are algebraically independent over the rational function field $\mathbb{C}(z)$. Using this result, the Theorem 4.2.1, combined with the Theorem 4.5.1 of [16], implies that, for any integer $d \geq 2$ and for any complex number a with $0 < |a| < 1$, we have for the transcendence degree

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{Q}\,(a, \chi_+(a;d), \chi_-(a;d)) \geq 2.$$

In particular (and this is the message of theorem 4.2.1 in [16] alone) if a as above is algebraic, then $\chi_+(a;d), \chi_-(a;d)$ are algebraically independent over the rational field $\mathbb{Q}$.

Remark 6.2. This latter conclusion can also be deduced from the Theorem 3.4.3 of [16] by choosing

$$n = 2,\ (b_k^{(1)})_k = (1,1,1,1,\cdots), \quad (b_k^{(2)})_k = (1,-1,1,-1,\cdots).$$

§ 7. More examples of Mahler's functions

§ 7.1. Definition and examples

The two fundamental functions in this work

$$\chi_+(z) = \sum_{n\geq 0} z^{2^n}, \quad \chi_-(z) = \sum_{n\geq 0} (-1)^n z^{2^n}, \quad |z| < 1,$$

are special cases of what we called a Mahler's function according to [12, 13, 14]. Mahler developped a method to prove transcendence and algebraic independence of values at algebraic points of locally analytic functions satisfying a more general functional equations of the form

$$f(z^k) = R(z, f(z))$$

where $R(x,y)$ denotes a rational function with coefficients in a number field.

We would like to give two other examples, related to the present work, with their corresponding expansions. For the first one we consider the formal series:

$$f(z) = \prod_{n=0}^{\infty}(1 - z^{2^n}) = \sum_{n=0}^{\infty}(-1)^{t_n} x^n, \ |z| < 1,$$

$(t_n)_{n\geq 0}$ is the Thue-Morse sequence defined by $t_0 = 0$ and for $n \in \mathbb{N}^{\times}$

$$\begin{cases} t_{2n} &= t_n \\ t_{2n+1} &= 1 - t_n. \end{cases}$$

The first terms are

$$0110100110010110\cdots$$

and we can see by elementary manipulations that f and χ_+ are related by

$$f(z) = \prod_{n=0}^{\infty}(1 - z^{2^n}) = \prod_{k=1}^{\infty} e^{-\frac{1}{k}\chi_+(z^k)}.$$

We clearly have

$$f(z^2) = \frac{f(z)}{1 - z^2}$$

so $f(z)$ is a Mahler's function. Our other example of a Mahler's function will be the main topic of the next subsection.

§ 7.2. Overview on L-series

To consider the other example, we need to recall some facts on Γ-function ([18], p.30) and Dirichlet L-series. At the origin, the Γ-function has the expansion, γ being the classical Euler constant:

$$\Gamma(s) = \frac{1}{s} - \gamma + \frac{1}{6}\left(3\gamma^2 + \frac{\pi^2}{2}\right) s + \mathrm{O}(s). \tag{7.1}$$

For our need, it is sufficient to use, instead of the Stirling formula, only ([18], p.38)

$$\Gamma(s) = \mathrm{O}\left(e^{-\frac{\pi}{2}|t|}\, |t|^{\sigma - \frac{1}{2}}\right), \quad s = \sigma + it. \tag{7.2}$$

Let $q \in \mathbb{N}, q \neq 0$ and let χ be a primitive character modulo q. The L-series with character χ is defined by $L(s,\chi) = \displaystyle\sum_{n\geq 1}\frac{\chi(n)}{n^s}$. We define $\delta(\chi) = 0$ if $q \neq 1$ and $\delta(\chi) = 1$

if $q = 1$, in which case $\chi = 1$ and $L(s,\chi) = \zeta(s)$, the Riemann zeta function. We also define $\kappa = \frac{1}{2}(1-\chi(-1))$, so $\kappa = 0$ if χ is even and $\kappa = 1$ if χ is odd. The L-series $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ extends to a meromorphic function on $\mathbb{C}$ which is entire if $\chi \neq 1$ and otherwise admits a unique simple pole with residue 1 at $s = 1$. The completed L-function

$$\Lambda(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s+\kappa}{2}\right) L(s,\chi)$$

is entire if $q \neq 1$ and has simple poles with residues 1 at $s = 0$ and $s = 1$ otherwise. Furthermore it satisfies the functional equation

$$\Lambda(s,\chi) = \epsilon(\chi)\Lambda(1-s,\bar{\chi}) \tag{7.3}$$

where

$$\epsilon(\chi) = i^{-\kappa}\frac{\tau(\chi)}{\sqrt{q}}$$

and the Gauss sum associated to χ:

$$\tau(\chi) = \sum_{x \,(\mathrm{mod}\, q)} \chi(x) e^{2i\pi\frac{x}{q}}.$$

The Dirichlet L-series $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ is bounded in every right plane $\{\Re s \geq 1+\epsilon\}$. According to the Phragmén-Lindelöf principle ([18] p.66, 68), for each $0 \leq c \leq 1$ and for each $\Re s \geq \frac{1-c}{2}$ we have

$$\left|\frac{\Gamma(s+c)}{\Gamma(s)}\right| \leq |s|^c.$$

By the Phragmén-Lindelöf principle again and the functional equation (7.3) this implies the

Lemma 7.1. *$L(s,\chi)$ is of polynomial growth in each vertical strips*

$$\{s = \sigma + it,\ a \leq \sigma \leq b,\ |t| \geq 1\}.$$

The Dirichlet L-series associated to the character $\left(\frac{D}{n}\right)$ is

$$L_D(s) = \sum_{n\geq 1} \left(\frac{D}{n}\right) \frac{1}{n^s}, \quad \Re s > 1.$$

It can be written as an infinite product over primes

$$L_D(s) = \prod_p \left(1 - \left(\frac{D}{p}\right) p^{-s}\right)^{-1}, \quad \Re s > 1.$$

If $D = 1$ then $L_1(s) = \zeta(s)$, the Riemann zeta-function, which admits a meromorphic continuation to all the plane with a simple pole at $s = 1$. For all other value of $D, L_D(s)$ is an entire function with

$$L_D(1) = \begin{cases} \frac{\pi}{3\sqrt{3}} & if \quad D = -3 \\ \frac{\pi}{4} & if \quad D = -4 \\ \frac{\pi h(D)}{\sqrt{-D}} & if \quad D < -4 \\ \frac{2h(D)\log \epsilon}{\sqrt{D}} & if \quad D > 1 \end{cases}$$

where $h(D)$ is the ideal class number of the quadratic field $\mathbb{Q}\left(\sqrt{D}\right)$ and ϵ is the fundamental unit of the integer subring $\mathbb{Z}+\frac{D+\sqrt{D}}{2}\mathbb{Z}$. The L-series $L_D(s)$ satisfies a functional equation: if we introduce

$$L_D^*(s) = \begin{cases} (-D)^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma(\frac{s+1}{2})L_D(s) & if \quad D < 0 \\ D^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})L_D(s) & if \quad D > 0, \end{cases}$$

then the functional equation reads

$$L_D^*(s) = L_D^*(1-s).$$

The constant $G = L_{-4}(2) = L\left(2, \left(\frac{-4}{\bullet}\right)\right) = 0.9159655941\cdots$ is the Catalan constant. Given a fundamental discriminant D, we define the Dedekind zeta function of the field $\mathbb{Q}\left(\sqrt{D}\right)$ by

$$\zeta_D(s) = \zeta(s)L_D(s) = \prod_{\left(\frac{D}{p}\right)=1} \left(1-p^{-s}\right)^{-2} \prod_{\left(\frac{D}{p}\right)=-1} \left(1-p^{-2s}\right)^{-1} \prod_{\left(\frac{D}{p}\right)=0} \left(1-p^{-s}\right)^{-1}.$$

In particular for $D = -4$, we recover the Dirichlet β-function

$$\begin{aligned} \zeta_4(s) &= \zeta(s)L_D(s) \\ &= \prod_{p\equiv 1(4)} \left(1-p^{-s}\right)^{-2} \prod_{p\equiv 3(4)} \left(1-p^{-2s}\right)^{-1} (1-2^{-s})^{-1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}. \end{aligned}$$

Since $\zeta_D(s)$ has a simple pole at $s = 1$, its Laurent expansion at $s = 1$ is

$$\zeta_D(s) = c_{-1}(s-1)^{-1} + c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots, \quad c_{-1} \neq 0.$$

The Euler-Kronecker constant $\mathbb{Q}\left(\sqrt{D}\right)$ is defined by

$$\gamma_D = \frac{c_0}{c_1} = \gamma + \frac{L'_D(1)}{L_D(1)}.$$

If $D = 1$, we merely have $\zeta_D(z) = \zeta(z)$, $c_{-1} = 1, c_0 = 1$ and $c_0 = \gamma$. We will need the precise value of the Sierpinski constant [7]:

$$\begin{aligned}\gamma_{-4} &= \log\left(2\pi e^{2\gamma}\frac{\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\right)\\ &= \frac{\pi}{3} - \log 4 + 2\gamma - 4\sum_{k=1}^{\infty}\log\left(1 - e^{-2\pi k}\right)\\ &= 0.8228252496\cdots\end{aligned}$$

The other example of Mahler's functions is given by the study of the following functional equation

$$F(z^2) = F(z) - \frac{z}{1+z^2}, \tag{7.4}$$

which is as we will see, highly related to the Dirichlet β-function. We introduce the sequence $a_n = \left(\dfrac{-4}{2n_0+1}\right)$ if $n = 2^k(2n_0+1)$ and we have

Lemma 7.2. *Let $(s_n)_{n\geq 0}$ be the sequence of the paperfolding, then*

$$s_n = a_n, \; n \geq 0.$$

In fact if $n = 2^k(2n_0+1), s_n = (-1)^{n_0}$. But $a_n = \left(\dfrac{-4}{2n_0+1}\right) = \left(\dfrac{-1}{2n_0+1}\right) = (-1)^{n_0}$, according to a classical result (Gauss, Euler).

Theorem 7.3. *The generating function of the paperfolding sequence*

$$\xi(z) = \sum_{n\geq 1} s_n z^n, \; |z| < 1$$

is a Mahler's function. It satisfies the functional equation

$$\xi(z^2) = \xi(z) - \frac{z}{1+z^2}.$$

This functional equation for $\xi(z)$ results immediatly from the lemma (7.2). We give here another proof which highlights how $\xi(z)$ is linked to $\chi_+(z)$. More precisely

$$\xi(z) = \sum_{n\geq 1} \left(\frac{-4}{2n_0+1}\right) z^n = \sum_{n_0\geq 0,\, k\geq 0} \left(\frac{-4}{2n_0+1}\right) z^{2^k(2n_0+1)}$$
$$= \sum_{n_0\geq 0} \left(\frac{-4}{2n_0+1}\right) \sum_{k\geq 0} z^{2^k(2n_0+1)} = \sum_{n_0\geq 0} \left(\frac{-4}{2n_0+1}\right) \chi_+\left(z^{2n_0+1}\right).$$

On the other hand from the functional equation $\chi_+(z^2) = \chi(z) - z$ we have

$$\begin{aligned}\xi(z^2) &= \sum_{n_0\geq 0} \left(\frac{-4}{2n_0+1}\right) \chi_+\left(\left(z^{2n_0+1}\right)^2\right)\\ &= \sum_{n_0\geq 0} \left(\frac{-4}{2n_0+1}\right) \left(\chi_+\left(z^{2n_0+1}\right) - z^{2n_0+1}\right).\end{aligned}$$

But $\left(\frac{-4}{n}\right) = 0$ if n is even, so

$$\begin{aligned}\sum_{n_0\geq 0} \left(\frac{-4}{2n_0+1}\right) z^{2n_0+1} &= \sum_{n\geq 1} \left(\frac{-4}{n}\right) z^n\\ &= \sum_{n\geq 0} z^{4n+1} - \sum_{n\geq 0} z^{4n+3} = \frac{z}{1+z^2}\end{aligned}$$

that is

$$\xi(z^2) = \xi(z) - \frac{z}{1+z^2}.$$

The Dirichlet series $L(s,\chi_{-4}) = \sum_{n\geq 1} \frac{\chi_{-4}(n)}{n^s}$ has the following two properties

(1) $L(s,\chi_{-4}) = \prod_{p\geq 2,\, \text{p prime}} \frac{1}{1-(\frac{-4}{p})p^{-s}}$,

(2) $L(s,\chi_{-4}) = \sum_{n\geq 0} \frac{1}{(4n+1)^s} - \sum_{n\geq 0} \frac{1}{(4n+3)^s} = 4^{-s}\left(\zeta(s,\frac{1}{4}) - \zeta(s,\frac{3}{4})\right)$

where $\zeta(s,\alpha) = \sum_{n\geq 0} \frac{1}{(n+\alpha)^s}$ is the Hurwitz zeta function wich reduces to Riemann zeta function for $\alpha = 1$. The Hurwitz zeta function has a meromorphic continuation to all the plane with a simple pole at $s = 1$. The local analysis at the pole is given by Lerch formulas:

$$\lim_{s\to 1}\left(\zeta(s,\alpha) - \frac{1}{s-1}\right) = -\log\alpha + \frac{1}{2\alpha} + 2\int_0^\infty \frac{t}{\alpha^2+t^2}\,\frac{dt}{e^{2\pi t}-1} = -\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

and

$$\zeta_s'(0,\alpha) = \left(\alpha - \frac{1}{2}\right)\log\alpha - \alpha + 2\int_0^\infty \arctan\frac{t}{\alpha}\,\frac{dt}{e^{2\pi t}-1} = \log\frac{\Gamma(\alpha)}{\sqrt{2\pi}}.$$

For $\alpha = 1$ we recover the classical Riemann zeta function formulas

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \frac{1}{2} + 2\int_0^\infty \frac{t}{1+t^2}\,\frac{dt}{e^{2\pi t}-1} = \gamma, \quad \zeta'(0) = -\log\sqrt{2\pi}.$$

The conclusion is that the Dirichlet series $L(s,\chi_{-4}) = \sum_{n\geq 1}\frac{\chi_{-4}(n)}{n^s} = \sum_{n\geq 1}\left(\frac{-4}{n}\right)\frac{1}{n^s}$ has an analytic continuation to all $\mathbb{C}$ as do all the Dirichlet series associated with non principal character. From general facts, we have $L\,(1,\chi_{-4}) \neq 0$. Actually $L(s,\chi_{-4})$ is the classical Dirichlet β-function and $L(1,\chi_{-4}) = \frac{\pi}{4}$.

Our main result concerning this function (which is not a lacunary series) is the following decomposition formula, similar to Hardy's (4.1):

Theorem 7.4. *Let $(a_n) = (s_n)$ the sequence of the paperfolding, then*

$$\xi(x) = \frac{1}{\log 2}\left(\frac{\log 2}{2}L(0,\chi_{-4}) - \gamma L(0,\chi_{-4}) + L'(0,\chi_{-4}) - L(0,\chi_{-4})\log\log\frac{1}{x}\right)$$
$$+\sum_{m\geq 1}\frac{(-1)^m}{m!}\frac{L(-m,\chi_{-4})}{1-2^m}\left(\log\frac{1}{x}\right)^m + \sum_{k\neq 0}\frac{\Gamma\left(\frac{2i\pi k}{\log 2}\right)L\left(\frac{2i\pi k}{\log 2},\chi_{-4}\right)}{\log 2}\left(\log\frac{1}{x}\right)^{-\frac{2i\pi k}{\log 2}}.$$

In this identity the values of the $L(s,\chi_{-4})$ at negative integers appear. These are related to Euler numbers by

$$L(-m,\chi_{-4}) = \frac{E_m}{2},\ m \geq 0$$

which are given by the generating function

$$\frac{1}{\cosh t} = \sum_{n\geq 0}\frac{E_n}{n!}t^n.$$

In particular $L(0,\chi_{-4}) = \frac{1}{2}$. It is also known [7] that:

$$L'(0,\chi_{-4}) = \log\frac{\Gamma^2(\frac{1}{4})}{2\pi\sqrt{2}} = 0.391594392\cdots$$

Furthermore, the Euler numbers possess the following asymptotic estimate

$$|E_{2n}| \sim \frac{2^{2n+2}(2n)!}{\pi^{2n+1}} \sim \frac{8\sqrt{n}}{\sqrt{\pi}}\left(\frac{4n}{\pi e}\right)^{2n}, \quad (n\to+\infty).$$

The proof of Theorem (7.4) depend upon Mellin formulas:

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1}\, du,\ \Re s > 0; \qquad e^{-u} = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma(z) u^{-z}\, dz,\ a > 0,\ u > 0.$$

and a shift of the integral path. For $0 < x < 1$ and $n = 2^k(2n_0+1)$

$$x^n = x^{2^k(2n_0+1)} = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma(z) 2^{-kz} (2n_0+1)^{-z} \left(\log \frac{1}{x}\right)^{-z} dz$$

and

$$\xi(x) = \sum_{n \geq 1} s_n x^n = \sum_{k, n_0 \geq 0} \left(\frac{-4}{2n_0+1}\right) x^{2^k(2n_0+1)}$$

or

$$\xi(x) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} \Gamma(z) \left(\log \frac{1}{x}\right)^{-z} \frac{L(z, \chi_{-4})}{1-2^{-z}}\, dz \tag{7.5}$$

The poles of $\Phi(z) = \Gamma(z) \left(\log \frac{1}{x}\right)^{-z} \frac{L(z,\chi_{-4})}{1-2^{-z}}$ are of two kinds:

(1) Those of Γ: $-m$, $m \in \mathbb{Z}_+$, the corresponding residues are:

$$Res(\Phi, -m) = \frac{(-1)^m}{m!} \left(\log \frac{1}{x}\right)^m \frac{L(-m, \chi_{-4})}{1-2^m}$$

and the double pole 0, being also a pole of $\frac{1}{1-2^{-z}}$. Its corresponding residue is the first of the three terms on the right hand of the formula of the theorem.

(2) The other poles of $\frac{1}{1-2^{-z}}$ which are $\frac{2ik\pi}{\log 2}$, $k \in \mathbb{Z}^\times$ and are simple. The corresponding residues are $\frac{1}{\log 2} \Gamma\left(\frac{2ik\pi}{\log 2}\right) \left(\log \frac{1}{x}\right)^{-\frac{2ik\pi}{\log 2}} \frac{L(\frac{2ik\pi}{\log 2}, \chi_{-4})}{2^{-\frac{2ik\pi}{\log 2}}}$.

To complete the proof of Theorem (7.4), we use the Lemma (7.1), the estimates (7.2) and the values of the different residues in the shift to the left of the integration contour in (7.5).

We end this section by some remarks concerning the impossibility to analytically extend $\xi(z)$. The Taylor coefficients of $\xi(z)$ are ± 1 and do not form a periodic sequence. Its radius of convergence is $R = 1$. Moreover if $p(z)$ is a general polynomial of degree $\leq k-1$, then the Taylor coefficients of $\frac{p(z)}{1-z^k}$ form, from some coefficient on, a periodic sequence. This shows that $\xi(z)$ has the unit circle as a natural boundary, according to the following beautiful theorem of G. Szëgo [19](p.260):

Theorem 7.5. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a power series with only finitely many disctinct coefficients and of radius 1. Then either the unit circle is a natural boundary of $f(z)$ or $f(z)$ can be extended analytically to a rational function $\hat{f}(z) = \dfrac{p(z)}{1-z^k}$, $p(z) \in \mathbb{C}[z]$, $k \in \mathbb{N}$.*

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