<table>
<thead>
<tr>
<th>Title</th>
<th>Exact WKB analysis of confluent hypergeometric differential equations with a large parameter (Exponential Analysis of Differential Equations and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Aoki, Takashi; Takahashi, Toshinori; Tanda, Mika</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2014), B52: 165-174</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/232916">http://hdl.handle.net/2433/232916</a></td>
</tr>
<tr>
<td>Rights</td>
<td>© 2014 by the Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.</td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

京都大学学術情報リポジトリ KURENAI Kyoto University Research Information Repository
Exact WKB analysis of confluent hypergeometric differential equations with a large parameter

By

Takashi AOKI*, Toshinori TAKAHASHI**, and Mika TANDA**

Abstract

Voros coefficients of the confluent hypergeometric differential equations (the Kummer equation or the Whittaker equation) with a large parameter are defined and explicit forms of them are obtained.

Introduction

The aim of this article is to define and to compute explicit forms of the Voros coefficients of the Whittaker equation, or equivalently, the Kummer equation with a large parameter. In [5], the first and the third authors have investigated the Stokes phenomena for WKB solutions with respect to parameters of the Gauss equation with a large parameter. They have defined the Voros coefficients for the Gauss equation and given the explicit forms of them as well as their Borel sums. We shall show that the same definition works and a similar computation can be carried out for confluent hypergeometric equations. In this article, we restrict our discussion to the Voros coefficients and we do not mention the parametric Stokes phenomena for WKB solutions of the Whittaker equation. Those will be discussed in our forthcoming paper with the investigation of the case where the way of putting the large parameter to be more general.

The notion of the Voros coefficients was introduced by Voros [15] for the Weber equation and for the quartic oscillator. It was effectively used for the analysis of the...
Stokes phenomena of WKB solutions with respect to parameters. In [15], Voros computed the explicit form of the Jost function for the Weber equation and obtained a formal power series as its asymptotic expansion. Later, Delabaere, Dillinger and Pham [7] discussed such formal series for general polynomial potentials where two simple turning points are connected by a Stokes curve and called them (or, more precisely, their exponentials) Voros coefficients. They introduced the notion of Voros coefficients as a formal series independently of the Jost functions. Starting from this point of view, Shen and Silverstone [12] and Takei [13] obtained the explicit form of the Voros coefficient directly for the Weber equation. These results were extended by Koike and Takei [10] to a special case of the Whittaker equation. They defined and computed the Voros coefficient for the Whittaker equation in a degenerate case where a simple turning point and the regular singular point at the origin merge. In this article, we are interested in the Whittaker equation in more general case. We employ a similar idea used in [10], [13] to obtain systems of difference equations for the Voros coefficients. Since the number of parameters (besides the large parameter) is two, the same method as in those article, namely, the division for the Borel transforms, cannot be applicable to solve the systems. As in the case of the Gauss equation [5], the systems are solved by using formal differential operators of infinite order. Our results can be obtained by the procedure of confluence from some of the formulas obtained in [5]. This observation is given in [1], where a part of the results of [10] has been recovered by considering degeneration of parameters which induces the merging of a simple turning point and the regular singularity at the origin.

§ 1. The Whittaker equation and the Kummer equation with a large parameter.

Let $\alpha$ and $\gamma$ be complex constants. Let us consider the following Whittaker equation with a large parameter $\eta$:

\begin{equation}
\left( - \frac{d^2}{dx^2} + \eta^2 Q \right) \psi = 0,
\end{equation}

where $Q = Q_0 + \eta^{-2} Q_1$ with

\begin{align*}
Q_0 &= \frac{x^2 + 2(2\alpha - \gamma) x + \gamma^2}{4x^2}, \\
Q_1 &= -\frac{1}{4x^2}.
\end{align*}

Equation (1.1) has a regular singular point at $b_0 := 0$ and an irregular singular point at $b_2 := \infty$. Here the lacking of the subscript index “1” in $b$’s comes from the usual procedure of confluence of regular singularities $b_1 = 1$ and $b_2 = \infty$ in the Gauss equation ([1], [5]). Equation (1.1) is obtained from the following Kummer equation:

\begin{equation}
\frac{x d^2 w}{dx^2} + (\gamma + \eta^{-1} - x) \eta \frac{dw}{dx} - \eta^2 \left( \alpha + \frac{1}{2} \eta^{-1} \right) w = 0
\end{equation}
by introducing a new unknown function $\psi$:

$$w = x^{-\frac{1}{2}-\frac{\gamma}{2}} \exp\left(\frac{\eta x}{2}\right) \psi.$$  

The parameters $\alpha$ and $\gamma$ correspond to those in the Gauss equation investigated in [2], [5], [14]. The Whittaker equation is normally written as (cf. [11])

$$\frac{d^2W}{dz^2} + \left( -\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) W = 0.$$  

If we set $\mu = \frac{1}{2} \gamma \eta$, $\kappa = \left( \frac{1}{2} \gamma - \alpha \right) \eta$ and $z = \eta x$ in (1.4), then we have (1.1). Our potential $Q$ is invariant under the involution $\iota$ on the space of parameters defined by

$$\iota : (\alpha, \gamma) \mapsto (\alpha - \gamma, -\gamma).$$  

We assume the following condition:

$$\alpha \gamma (\alpha - \gamma) \neq 0.$$  

This implies that there are two distinct simple turning points $a_0$ and $a_1$, namely, simple zeros of $Q_0 dx^2$ and that $a_0, a_1 \neq b_0, b_2$.

Remark. If we consider the limit as $\gamma$ tends to 0 in (1.1), we have a special case of the Whittaker equation treated in [10]. See [1] for details.

§ 2. The Voros coefficients of the Whittaker equation

Let

$$S = \sum_{j=-1}^{\infty} \eta^{-j} S_j$$

be a formal solution of the Riccati equation associated with (1.1)

$$\frac{dS}{dx} + S^2 = \eta^2 Q$$

with the leading term $S_{-1} = \sqrt{Q_0}$. Here we fix a branch of $\sqrt{Q_0}$ suitably by taking a segment or an arc connecting $a_0$ and $a_1$ in $\mathbb{C} - \{0\}$ as a branch cut. Then the higher-order terms $S_j$ ($j \geq 0$) are uniquely determined. Let $S_{\text{odd}}$ be the odd-order part of $S$ with respect to $\eta$. Then

$$\psi_{\pm} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \pm \int_{x_0}^{x} S_{\text{odd}} dx \right)$$
are formal solutions to (1.1), which are called WKB solutions (see, for example, [9] and the references cited therein). Here \( x_0 \) is a fixed point in \( \mathbb{C} - \{0\} \). When we choose \( x_0 = a_j \) \((j = 0 \text{ or } 1)\) (see [9] for the meaning of the integration), the solutions are said to be normalized at the turning point \( a_j \). Let \( C_j \) denote a contour starting from \( b_j \), going to \( a_0 \) through the first sheet of the Riemann surface of \( \sqrt{Q_0} \), stopping just before arriving at \( a_0 \) and turning once around \( a_0 \) counterclockwise and finally going back to \( b_j \) through the second sheet \((j = 0, 2)\). The branches of \( \sqrt{Q_0} \) on \( C_j \) are chosen so that at the starting point \( b_j \) we have

\[
\sqrt{Q_0} \sim \frac{\gamma}{2x} \quad \text{at} \quad x = b_0,
\]

\[
\sqrt{Q_0} \sim \frac{1}{2} \quad \text{at} \quad x = b_2.
\]

Since \( b_0 = 0 \) is a regular singular point of (1.1), \( S_{\text{odd}}dx \) and \( S_{-1}dx \) have a simple pole there and their residues at \( b_0 \) are the same (Cf. [9]). On the other hand, \( b_2 = \infty \) is an irregular singularity of rank 1 of (1.1). As in the case of regular singularities, we can see that the principal part of \( S_{\text{odd}}dx \) coincides with that of \( S_{-1}dx \) and that it is

\[-\eta \left( \frac{1}{2y^2} + \frac{2\alpha - \gamma}{2y} \right) dy\]

in the local coordinate \( y = 1/x \) at the infinity. Hence \( (S_{\text{odd}} - S_{-1})dx \) is integrable on \( C_j \) for \( j = 0, 2 \).

**Definition 2.1.** Let \( V_j = V_j(\alpha, \gamma, \eta) \) \((j = 0, 2)\) denote the formal power series in \( \eta^{-1} \) defined by

\[
\frac{1}{2} \int_{C_j} (S_{\text{odd}} - S_{-1})dx.
\]

We call \( V_j \) the Voros coefficient of (1.1) or of (1.2) with respect to \( b_j \) for \( j = 0, 2 \).

The explicit forms of \( V_j \) \((j = 0, 2)\) are given by the following theorem which has been announced in [1].

**Theorem 2.2.** The Voros coefficients \( V_j \) \((j = 0, 2)\) have the following forms:

\[
V_0 = \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} \left\{ (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} + \frac{1}{(\gamma - \alpha)^{n-1}} \right) + \frac{2}{\gamma^{n-1}} \right\},
\]

\[
V_2 = \frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n \eta^{1-n}}{n(n-1)} (1 - 2^{1-n}) \left( \frac{1}{\alpha^{n-1}} - \frac{1}{(\gamma - \alpha)^{n-1}} \right).
\]

Here \( B_n \) denote the Bernoulli numbers defined by

\[
\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.
\]
To prove the theorem, we first find systems of difference equations which characterize $V_0$ and $V_2$. As in the cases of the Weber equation ([13]), of the degenerate Whittaker equation ([10]) and of the Gauss equation ([2], [5]), the ladder operators yield the systems. Let $S(\alpha, \gamma)$ be the space of all formal solutions of (1.2). As is well known, the operators

\begin{equation}
H_1 := x \frac{d}{dx} + \eta \alpha + \frac{1}{2}
\end{equation}

and

\begin{equation}
H_2 := \frac{d}{dx} - \eta
\end{equation}

induce linear isomorphisms

\begin{equation}
H_1 : S(\alpha, \gamma) \rightarrow S(\alpha + \eta^{-1}, \gamma)
\end{equation}

and

\begin{equation}
H_2 : S(\alpha, \gamma) \rightarrow S(\alpha, \gamma + \eta^{-1}),
\end{equation}

respectively.

**Lemma 2.3.** The formal solution $S = S(\alpha, \gamma; x, \eta)$ of (2.2) satisfies the following relations:

\begin{equation}
S(\alpha + \eta^{-1}, \gamma; x, \eta) - S(\alpha, \gamma; x, \eta)
= \frac{d}{dx} \log \left( -\frac{\eta \gamma + 1}{2} + x \left( \frac{\eta}{2} + S(\alpha, \gamma; x, \eta) \right) + \eta \alpha + \frac{1}{2} \right),
\end{equation}

\begin{equation}
S(\alpha, \gamma + \eta^{-1}; x, \eta) - S(\alpha, \gamma; x, \eta)
= \frac{d}{dx} \log \left( -\frac{\eta \gamma + 1}{2x} - \frac{\eta}{2} + S(\alpha, \gamma; x, \eta) \right) + \frac{1}{2x}.
\end{equation}

**Proof.** Firstly we note that

\[
\hat{w} := H_1 \left( x^{-\frac{1}{2} - \frac{\eta \gamma}{2}} \exp \left( \frac{\eta x}{2} \right) \exp \left( \int^x S(\alpha, \gamma; x, \eta) dx \right) \right)
\]

belongs to $S(\alpha + \eta^{-1}, \gamma)$. Let $\hat{S}$ denote the logarithmic derivative of $x^{\frac{1}{2} + \frac{\eta \gamma}{2}} \exp \left( -\frac{\eta x}{2} \right) \hat{w}$. Then we can see that $\hat{S}$ satisfies the equation obtained from (2.2) by replacing $\alpha$ by $\alpha + \eta^{-1}$. Since the leading term of $\hat{S}$ coincides with that of $S(\alpha + \eta^{-1}, \gamma; x, \eta)$, we conclude that

\begin{equation}
\hat{S} = S(\alpha + \eta^{-1}, \gamma; x, \eta).
\end{equation}

Hence we have (2.13). Similarly, (2.14) can be proved. 

Integrating the above relations with the difference of $S_{-1}$ on the contours $C_j$ ($j = 0, 2$), we have
Proposition 2.4. The Voros coefficients \( V_j \) \((j = 0, 2)\) satisfies the following systems of difference equations:

\[
(2.16) \quad V_0(\alpha + \eta^{-1}, \gamma, \eta) - V_0(\alpha, \gamma, \eta) = \frac{1}{2} \log \frac{\gamma - \alpha - \frac{1}{2}\eta^{-1}}{\alpha + \frac{1}{2}\eta^{-1}} + \frac{1}{2} \eta((\alpha + \eta^{-1}) \log(\alpha + \eta^{-1}) - \alpha \log \alpha) \\
+ \frac{\eta}{2}((\gamma - \alpha - \eta^{-1}) \log(\gamma - \alpha - \eta^{-1}) - (\gamma - \alpha) \log(\gamma - \alpha)),
\]

\[
(2.17) \quad V_0(\alpha, \gamma + \eta^{-1}, \eta) - V_0(\alpha, \gamma, \eta) = \frac{1}{2} \log \frac{\gamma(\gamma + \eta^{-1})}{\gamma - \alpha + \frac{1}{2}\eta^{-1}} - \eta((\gamma + \eta^{-1}) \log(\gamma + \eta^{-1}) - \gamma \log \gamma) + \frac{1}{2}((\gamma + \eta^{-1}) - \gamma)) \\
+ \frac{1}{2} \eta((\gamma + \eta^{-1} - \alpha) \log(\gamma + \eta^{-1} - \alpha) - (\gamma - \alpha) \log(\gamma - \alpha)),
\]

\[
(2.18) \quad V_2(\alpha + \eta^{-1}, \gamma, \eta) - V_2(\alpha, \gamma, \eta) = \frac{1}{2} \log \left( \left( \alpha + \frac{1}{2}\eta^{-1} \right) \left( \gamma - \alpha - \frac{1}{2}\eta^{-1} \right) \right) + \eta((\alpha + \eta^{-1}) - \alpha) \\
+ \frac{1}{2} \eta((\gamma - \alpha - \eta^{-1}) \log(\gamma - \alpha - \eta^{-1}) - (\gamma - \alpha) \log(\gamma - \alpha)) \\
- \frac{1}{2} \eta((\alpha + \eta^{-1}) \log(\alpha + \eta^{-1}) - \alpha \log \alpha),
\]

\[
(2.19) \quad V_2(\alpha, \gamma + \eta^{-1}, \eta) - V_2(\alpha, \gamma, \eta) = -\frac{1}{2} \log \left( \gamma - \alpha + \frac{1}{2}\eta^{-1} \right) - \frac{1}{2} \eta((\gamma + \eta^{-1}) - \gamma) \\
+ \frac{1}{2} \eta((\gamma + \eta^{-1} - \alpha) \log(\gamma + \eta^{-1} - \alpha) - (\gamma - \alpha) \log(\gamma - \alpha)).
\]

Moreover, \( V_j \) are characterized by those systems as the formal power series solutions in \( \eta^{-1} \) which are homogeneous of degree 0 in \((\alpha, \gamma, \eta^{-1})\) and which do not have constant terms.

Proof. We only give the proof of (2.18), since the others can be shown similarly. As is shown in [13], \( \text{Res}_{x=a_0} S = \text{Res}_{x=a_0} S_0 = -1/4 \) holds also in this case. Hence we have

\[
(2.20) \quad \frac{1}{2} \int_{C_2} (S_{\text{odd}} - S_{-1}) dx = \frac{1}{2} \int_{C_2} (S - \eta S_{-1} - S_0) dx.
\]

Let \( x_0 \) be a complex number and let \( C_{x_0} \) be a contour starting at \( x_0 \) on the first sheet of the Riemann surface of \( \sqrt{Q_0} \), going to \( a_0 \) along a straight line connecting \( x_0 \) and \( a_0 \) just
before arriving at $a_0$, turning $a_0$ once counterclockwise and going back to $x_0$. Modifying the contour slightly if necessary, we may assume that $C_{x_0}$ does not hit another turning point. Let $\hat{S}$ and $\hat{S}_{-1}$ be formal series obtained by replacing $\alpha$ by $\alpha + \eta^{-1}$ in $S$ and $S_{-1}$, respectively. To obtain $V_2(\alpha + \eta^{-1}, \gamma, \eta) - V_2(\alpha, \gamma, \eta)$, it suffices to compute

$$\lim_{x_0 \to \infty} \frac{1}{2} \left( \int_{C_{x_0}} (\hat{S} - S) \, dx - \eta \int_{C_{x_0}} (\hat{S}_{-1} - S_{-1}) \, dx \right).$$

Integrating (2.13), we have

$$\frac{1}{2} \int_{C_{x_0}} (\hat{S} - S) \, dx = \frac{1}{2} \log \frac{-\frac{1}{2}(\eta \gamma + 1) + \eta \alpha + \frac{1}{2} + x_0 \left( \frac{\eta}{2} + S(\alpha, \gamma; \tilde{x}_0, \eta) \right)}{-\frac{1}{2}(\eta \gamma + 1) + \eta \alpha + \frac{1}{2} + x_0 \left( \frac{\eta}{2} + S(\alpha, \gamma; \tilde{x}_0, \eta) \right)},$$

where $\tilde{x}_0$ denotes the point on the second sheet corresponding to $x_0$. It follows from (2.2) and (2.5) that

$$S(\alpha, \gamma, \tilde{x}_0, \eta) = -\frac{\eta}{2} - \eta \frac{2\alpha - \gamma}{2x_0} + \eta \left( \alpha - \gamma + \frac{\eta^{-1}}{2} \right) \left( \alpha + \frac{\eta^{-1}}{2} \right) \frac{1}{x_0^2} + O\left( \frac{1}{x_0^3} \right)$$

and

$$S(\alpha, \gamma, x_0, \eta) = \frac{\eta}{2} + \eta \frac{2\alpha - \gamma}{2x_0} + \eta \left( \alpha - \gamma - \frac{\eta^{-1}}{2} \right) \left( \alpha - \frac{\eta^{-1}}{2} \right) \frac{1}{x_0^2} + O\left( \frac{1}{x_0^3} \right)$$

hold. Thus (2.21) has the following leading terms:

$$\frac{1}{2} \log \left( \left( \frac{\gamma - \alpha - \eta^{-1}}{2} \right) \left( \alpha + \frac{\eta^{-1}}{2} \right) \right) + \frac{\pi i}{2} - \log x_0 + O\left( \frac{1}{x_0} \right).$$

Here we use the convention of the argument $\alpha - \gamma = e^{\pi i}(\gamma - \alpha)$, etc. On the other hand, $S_{-1}$ can be integrated easily and we have

$$\frac{\eta}{2} \int_{C_{x_0}} (\hat{S}_{-1} - S_{-1}) \, dx = -1 - \log x_0 + \frac{\eta}{2} \left( (\alpha + \eta^{-1}) \log(\alpha + \eta^{-1}) - \alpha \log \alpha \right)$$

$$- \frac{\eta}{2} \left( (\gamma - \alpha - \eta^{-1}) \log(\gamma - \alpha - \eta^{-1}) - (\gamma - \alpha) \log(\gamma - \alpha) \right) + \frac{\pi i}{2} + O\left( \frac{1}{x_0} \right)$$

Subtracting (2.25) from (2.24) and taking the limit $x_0$ to the infinity, we obtain (2.18). Uniqueness of the homogeneous formal solutions follows from a similar argument in the proof of Proposition 2.8 in [5].

We can solve the systems (2.16)–(2.19) in a similar manner as in [5] by using formal differential operators of infinite order ([6])

$$\eta^{-1} \partial_\alpha (\exp(\eta^{-1} \partial_\alpha) - 1)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \eta^{-n} \partial_\alpha^n$$
\[ (2.27) \quad \eta^{-1}\partial_\gamma (\exp(\eta^{-1}\partial_\gamma) - 1)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n B_n}{n!} \eta^{-n} \partial_\gamma^n. \]

Here \( \partial_\alpha \) stands for \( \partial/\partial \alpha \), etc. We omit the computation of solving the systems because it is almost the same as a part of the proof of Theorem 2.3 in [5].

\section{Borel sums of the Voros coefficients}

To consider the Borel sums of \( V_j \) \((j = 0, 2)\), we must specify the “Stokes regions” for the parameters, where \( V_j \)'s are Borel summable. These regions are described by using the following sets:

\begin{align*}
\omega_1 &= \{ (\alpha, \gamma) \in \mathbb{C}^2 | 0 < \text{Re} \alpha < \text{Re} \gamma \}, \\
\omega_3 &= \{ (\alpha, \gamma) \in \mathbb{C}^2 | 0 < \text{Re} \gamma < \text{Re} \alpha \}, \\
\omega_4 &= \{ (\alpha, \gamma) \in \mathbb{C}^2 | \text{Re} \alpha < 0 < \text{Re} \gamma \}.
\end{align*}

Regarding the relation between these sets and configurations of the Stokes curves of (1.1), we refer the readers to [1]. Those regions and the images of them by the involution \( \iota \) (cf. (1.5)) cover almost all \( \mathbb{C}^2 \):

\[ \bigcup_{j=1,3,4} (\omega_j \cup \iota(\omega_j)) = \{ (\alpha, \gamma) \in \mathbb{C}^2 | \text{Re} \alpha \text{ Re} \gamma \text{ Re} (\gamma - \alpha) \neq 0 \}. \]

If \( (\alpha, \gamma) \) belongs to the above set, there is no Stokes curves of (1.1) which connect turning point(s).

\textbf{Theorem 3.1.} The Voros coefficients \( V_j \) \((j = 0, 2)\) are Borel summable in \( \omega_k \) and in \( \iota(\omega_k) \) for \( k = 1, 3, 4 \). Here \( \iota \) is the involution in the space of parameters defined
by (1.5). The Borel sums $V_j^k$ of $V_j$ in $\omega_k$ have the following forms:

\begin{align}
(3.5) & \quad V_0^1 = \frac{1}{2} \log \frac{\Gamma(\gamma \eta)^2 \alpha^{\alpha \eta}(\gamma-\alpha)^{(\gamma-\alpha)\eta^{1-\gamma}}}{\Gamma\left(\frac{1}{2} + \alpha \eta\right) \Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right) \gamma^{2\gamma \eta-1}} + \frac{1}{2} \gamma \eta, \\
(3.6) & \quad V_0^3 = \frac{1}{2} \log \frac{\Gamma(\gamma \eta)^2 \Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right) \alpha^{\alpha \eta}}{2\pi \Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right)(\alpha-\gamma)^{(\alpha-\gamma)\eta}\gamma^{2\gamma \eta-1}} + \frac{1}{2} \gamma \eta, \\
(3.7) & \quad V_0^4 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} - \alpha \eta\right) \Gamma(\gamma \eta)^2}{\Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right)(\gamma-\alpha)^{(\gamma-\alpha)\eta}\gamma^{2\gamma \eta-1}} + \frac{1}{2} \gamma \eta, \\
(3.8) & \quad V_2^1 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right) \alpha^{\alpha \eta}}{\Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right)(\gamma-\alpha)^{(\gamma-\alpha)\eta}-\frac{1}{2}(2\alpha-\gamma)\eta}, \\
(3.9) & \quad V_2^3 = \frac{1}{2} \log \frac{2\pi \alpha^{\alpha \eta}(\gamma-\alpha)^{(\alpha-\gamma)\eta^{1-\gamma}}}{\Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right)(\gamma-\alpha)^{(\gamma-\alpha)\eta}} - \frac{1}{2}(2\alpha-\gamma)\eta, \\
(3.10) & \quad V_2^4 = \frac{1}{2} \log \frac{\Gamma\left(\frac{1}{2} - \alpha \eta\right) \Gamma(\gamma \eta)^2}{\Gamma\left(\frac{1}{2} + (\gamma-\alpha)\eta\right)(\gamma-\alpha)^{(\gamma-\alpha)\eta}} - \frac{1}{2}(2\alpha-\gamma)\eta.
\end{align}

Remark. The Borel sum of $V_j$ in $i(\omega_k)$ coincides with $V_j^k$ up to signature, which comes from the choice of the branch of $\sqrt{Q_0}$.

Theorem 3.1 is proved by computing the Borel sums directly from the expressions given in Theorem 2.2. We can use a method developed in [13], where the Borel sum of the series

\begin{equation}
\sum_{n=2}^{\infty} \frac{(1-2^{1-n})B_n}{n(n-1)}(iE \eta)^{1-n}
\end{equation}

is given. Hence we do not repeat the computation here. See the proof of Theorem 3.1 in [5] also.

References


