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# On the sheaf of Laplace hyperfunctions in several variables

*Dedicated to Professor Takashi Aoki on his sixtieth birthday*

By

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**Abstract**

In this note we discuss the sheaf of Laplace hyperfunctions in several variables and its fundamental properties. To construct the sheaf we give an edge of the wedge theorem for the sheaf of holomorphic functions of exponential type.

## § 1. Introduction

A Laplace hyperfunction in one variable was first introduced by H. Komatsu ([5]-[10]) to consider the Laplace transform of a hyperfunction, which was effectively employed in giving justification of the Heaviside operational calculus on a wider class of functions. After his success, the authors of this note also constructed the sheaf of one dimensional Laplace hyperfunctions in the paper [1]. Here, in order to localize the notion of Laplace hyperfunctions, we also established the vanishing theorem of global cohomology groups on a Stein domain with values in the sheaf of holomorphic functions of exponential type (see [1]).

The purpose of this note is to construct the sheaf of Laplace hyperfunctions in several variables and establish some fundamental properties. Its construction depends on several vanishing theorems such as that of global cohomology groups on a Stein domain with coefficients in the sheaf $\mathcal{O}_{\mathbb{D}^{2n}}^{\exp}$ of holomorphic functions of exponential type, pure $n$-codimensionality of the radial compactification of the Euclidean space $\mathbb{R}^n$ with respect to $\mathcal{O}_{\mathbb{D}^{2n}}^{\exp}$ which is often called "an edge of the wedge theorem" and so on.

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As the proofs of these vanishing theorems are rather long and technical, in this note, we only announce the main theorems and related results of their importance. For their details and proofs, we refer the reader to the paper [1], [12], [13] and the forthcoming one [2].

## § 2. A vanishing theorem of global cohomology groups on a Stein domain for the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$

In this section, we review the vanishing theorem of global cohomology groups on a Stein domain with coefficients in holomorphic functions of exponential type, which was established in our previous paper [1]. The theorem stated here is the most fundamental one for construction of the sheaf of Laplace hyperfunctions in several variables. The essential idea in the proof of the theorem relies on the theory of $L^2$-estimates for the $\overline{\partial}$ operator obtained by L. Hörmander [3] as in the paper [4], [11].

Let $\mathbb{D}^{2n}$ be the radial compactification of $\mathbb{C}^n$, i.e., the disjoint union of $\mathbb{C}^n$ and the real $(2n-1)$-dimensional unit sphere $S^{2n-1}\infty$ equipped with an appropriate topology. Let $U$ be an open subset in $\mathbb{D}^{2n}$. A holomorphic function $f$ in $U \cap \mathbb{C}^n$ is said to be of exponential type if $f$ satisfies the following condition: For any compact set $K$ in $U$, there exist positive constants $C_K$, $H_K$ such that $|f(z)| \le C_K e^{H_K|z|}$ for $z \in K \cap \mathbb{C}^n$. Let us denote by $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$ the sheaf of holomorphic functions of exponential type on $\mathbb{D}^{2n}$. Note that the restriction of $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$ to $\mathbb{C}^n$ is the sheaf $\mathcal{O}_{\mathbb{C}^n}$ of holomorphic functions.

We introduce the regularity condition at $\infty$ for an open subset in $\mathbb{D}^{2n}$.

**Definition 2.1.** Let $A$ be a subset in $\mathbb{D}^{2n}$. The set $\mathrm{clos}^1_\infty(A) \subset S^{2n-1}\infty$ consists of a point $z \in S^{2n-1}\infty$ which satisfies the condition below:

$$\begin{cases} \text{There exist points } \{z_k\}_k \text{ in } A \cap \mathbb{C}^n \text{ such that} \\ z_k \to z \text{ in } \mathbb{D}^{2n} \text{ and } \frac{|z_{k+1}|}{|z_k|} \to 1 \text{ as } k \to \infty. \end{cases}$$

Set $N^1_\infty(A) := S^{2n-1}\infty \setminus \mathrm{clos}^1_\infty(\mathbb{C}^n \setminus A)$. An open subset $U$ in $\mathbb{D}^{2n}$ is said to be regular at $\infty$ if $N^1_\infty(U) = U \cap S^{2n-1}\infty$ is satisfied.

**Theorem 2.2** ([1], Theorem 3.7). *Let $\Omega$ be an open subset in $\mathbb{D}^{2n}$. If $\Omega \cap \mathbb{C}^n$ is pseudoconvex in $\mathbb{C}^n$ and if $\Omega$ is regular at $\infty$, then we have*

$$H^k(\Omega, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0 \quad \text{for} \quad k \neq 0. \tag{2.1}$$

Note that, if $n = 1$, the above theorem holds for any open subset $\Omega$ without the regularity condition ([1], Theorem 5.2). However, if $n \ge 2$, the theorem does not hold anymore without the regularity condition as the following example shows.

**Example 2.3** ([1], Example 3.17). We consider the following open set $\Omega$ in $\mathbb{D}^4$.

$$\Omega := (\overline{U})^\circ \setminus \{p\infty\},$$
$$U := \{(z_1, z_2) \in \mathbb{C}^2; |\arg(z_1)| < \pi/4, |z_2| < |z_1|\}.$$

Here $p\infty$ is the point $(+1, 0, 0, 0)$ in $S^3\infty$. The closure and the interior of $U$ are taken in $\mathbb{D}^4$. We see that $\Omega \cap \mathbb{C}^2$ is pseudoconvex in $\mathbb{C}^2$, however, $\Omega$ is not regular at $\infty$. In this case, we have $H^1(\Omega, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \neq 0$.

## § 3. An edge of the wedge theorem for the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$

Let us first show the Martineau type theorem for holomorphic functions of exponential type. This theorem plays an essential role in the proof of the edge of the wedge theorem for the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$. All the proofs of results stated here are given in [13] and our forthcoming paper [2].

We denote by $\overline{\mathbb{R}^n}(n \geq 1)$ the closure of $\mathbb{R}^n$ in $\mathbb{D}^{2n}$.

**Theorem 3.1.** *Let $[a, \infty](a \in \mathbb{R})$ be a closed subset in $\overline{\mathbb{R}}$, $S \subset K$ closed polydiscs in $\mathbb{C}^{n-1}$ and $V \subset U$ non-empty connected Stein open subsets in $\mathbb{C}^m(m \geq 0)$. Then the following canonical morphism is injective.*

$$(3.1)\quad H^n_{[a,\infty]\times S\times U}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times U, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times U}) \to H^n_{[a,\infty]\times K\times V}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times V, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times V}).$$

By making use of Oka's embedding and the above theorem, we get the following result.

**Theorem 3.2.** *Let $[a, \infty](a \in \mathbb{R})$ be a closed subset in $\overline{\mathbb{R}}$. Let $S$ and $K$ be two compact analytic polyhedra in $\mathbb{C}^{n-1}$, and let $U$ be a Stein domain in $\mathbb{C}^m$. Then*

$$(3.2)\qquad H^k_{[a,\infty]\times(K\setminus S)\times U}(\mathbb{D}^2 \times \mathbb{C}^{n-1} \times U, \mathcal{O}^{\exp}_{\mathbb{D}^2\times\mathbb{C}^{n-1}\times U}) = 0 \quad \text{for} \quad 0 \leq k \leq n-1.$$

From this theorem, the following main theorem easily follows:

**Theorem 3.3.** *The closed set $\overline{\mathbb{R}^n} \subset \mathbb{D}^{2n}$ is purely n-codimensional relative to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$, i.e.,*

$$(3.3)\qquad \mathscr{H}^k_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0 \quad \text{for} \quad k \neq n.$$

According to the edge of the wedge theorem for the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$, the global sections of the sheaf $\mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})$ can be written in terms of cohomology groups. For an open subset $\Omega$ in $\overline{\mathbb{R}^n}$, we have

$$\Gamma(\Omega, \mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}})) = H^n_\Omega(U, \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}),$$

where $U$ is an arbitrary neighborhood of $\Omega$ in $\mathbb{D}^{2n}$. Moreover we have the following result.

**Theorem 3.4.** *The boundary $\partial\mathbb{R}^n$ of $\mathbb{R}^n$ in $\overline{\mathbb{R}^n}$ is purely n-codimensional relative to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}$, i.e.,*

$$\mathscr{H}^k_{\partial\mathbb{R}^n}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) = 0 \quad \text{for} \quad k \neq n. \tag{3.4}$$

## § 4. The sheaf of Laplace hyperfunctions on $\overline{\mathbb{R}^n}$

In this section, we define the sheaf of Laplace hyperfunctions on $\overline{\mathbb{R}^n}$ and show several properties of this sheaf.

Let $\mathbb{Z}_{\overline{\mathbb{R}^n}}$ be the constant sheaf on $\overline{\mathbb{R}^n}$ having stalk $\mathbb{Z}$.

**Definition 4.1.** Let $\omega_{\overline{\mathbb{R}^n}}$ be the orientation sheaf $\mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathbb{Z}_{\mathbb{D}^{2n}})$ on $\overline{\mathbb{R}^n}$. The sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ of Laplace hyperfunctions on $\overline{\mathbb{R}^n}$ is defined as

$$\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} := \mathscr{H}^n_{\overline{\mathbb{R}^n}}(\mathcal{O}^{\exp}_{\mathbb{D}^{2n}}) \underset{\mathbb{Z}_{\overline{\mathbb{R}^n}}}{\otimes} \omega_{\overline{\mathbb{R}^n}}. \tag{4.1}$$

It is obvious that the restriction of the sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ on $\mathbb{R}^n$ is isomorphic to the sheaf $\mathcal{B}_{\mathbb{R}^n}$ of hyperfunctions. By Theorem 3.4, we have the relation between $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ and $\mathcal{B}_{\mathbb{R}^n}$ as follows. Let $j : \mathbb{R}^n \hookrightarrow \overline{\mathbb{R}^n}$ be the natural embedding.

**Proposition 4.2.** *The natural morphism $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}} \to j_*\mathcal{B}_{\mathbb{R}^n}$ is surjective.*

This proposition asserts that every hyperfunction can be extended to a Laplace hyperfunction.

Let $i : \overline{\mathbb{R}^n} \hookrightarrow \mathbb{D}^{2n}$ be the canonical closed embedding. The sheaf $\mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}}$ of real analytic functions of exponential type on $\overline{\mathbb{R}^n}$ is defined by

$$\mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}} := i^{-1}\mathcal{O}^{\exp}_{\mathbb{D}^{2n}} = \mathcal{O}^{\exp}_{\mathbb{D}^{2n}}|_{\overline{\mathbb{R}^n}}. \tag{4.2}$$

**Proposition 4.3.** *There exists the canonical morphism $\mathcal{A}^{\exp}_{\overline{\mathbb{R}^n}} \to \mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ which is injective.*

Hence, through the morphism, a real analytic function of exponential type is naturally regarded as a Laplace hyperfunction. Furthermore the sheaf of Laplace hyperfunctions enjoys the following good property:

**Theorem 4.4.** *The sheaf $\mathcal{B}^{\exp}_{\overline{\mathbb{R}^n}}$ of Laplace hyperfunctions is soft on $\overline{\mathbb{R}^n}$.*

## References

[1] N. Honda and K. Umeta, On the sheaf of Laplace hyperfunctions with holomorphic parameters, *Journal of Mathematical Sciences the University of Tokyo*, **19** (2012), 559-586.

[2] N. Honda and K. Umeta, in preparation.

[3] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd edition, North-Holland, Amsterdam (1990), 89.

[4] T. Kawai, On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, J. Fac. Sci. Univ. Tokyo, **17** (1970), 467-517.

[5] H. Komatsu, Laplace transforms of hyperfunctions: a new foundation of the Heaviside calculus, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 805-820.

[6] ____, Laplace transforms of hyperfunctions: another foundation of the Heaviside operational calculus, Generalized Functions, Convergence Structures, and Their Applications (Proc. Internat. Conf., Dubrovnik, 1987; B. Stanković, editor), Plenum Press, New York (1988), 57-70.

[7] ____, Operational calculus, hyperfunctions and ultradistributions, Algebraic Analysis (M. Sato Sixtieth Birthday Vols.), Vol. I, Academic Press, New York (1988), 357-372.

[8] ____, Operational calculus and semi-groups of operators, Functional Analysis and Related Topics (Proc. Internat. Conf. in Memory of K. Yoshida, Kyoto, 1991), Lecture Notes in Math., vol. 1540, Springer-Verlag, Berlin (1993), 213-234.

[9] ____, Multipliers for Laplace hyperfunctions - A Justification of Heaviside's rules, Proceedings of the Steklov Institute of Mathematics, **203** (1994), 323-333.

[10] ____, Solution of differential equations by means of Laplace hyperfunctions, in Structure of Solutions of Differential Equations, World Scientific, Singapore (1996), 227-252.

[11] Y. Saburi, Vanishing theorems of cohomology groups with values in the sheaves $\mathcal{O}_{\mathrm{inc},\,\phi}$ and $\mathcal{O}_{\mathrm{dec}}$, Tokyo J. Math. vol. **5** (1982), 225-248.

[12] K. Umeta, Vanishing theorem for holomorphic functions of exponential type and Laplace hyperfunctions, RIMS Kôkyûroku Bessatsu B40 (2013), 137-145.

[13] K. Umeta, The edge of the wedge theorem for the sheaf of holomorphic functions of exponential type and Laplace hyperfunctions, Doctor Thesis, Hokkaido University (2014).