



Weyl transformation and regular solutions in a deformed Jackiw–Teitelboim model

Suguru Okumura ^{*}, Kentaroh Yoshida

Department of Physics, Kyoto University, Kitashirakawa Oiwake-cho, Kyoto 606-8502, Japan

Received 14 March 2018; received in revised form 1 June 2018; accepted 6 June 2018

Available online 14 June 2018

Editor: Leonardo Rastelli

Abstract

We revisit a deformed Jackiw–Teitelboim model with a hyperbolic dilaton potential, constructed in the preceding work [arXiv:1701.06340]. Several solutions are discussed in a series of the subsequent papers, but all of them are pathological because of a naked singularity intrinsic to the deformation. In this paper, by employing a Weyl transformation to the original deformed model, we consider a Liouville-type potential with a cosmological constant term. Then regular solutions can be constructed with coupling to a conformal matter by using $SL(2)$ transformations. For a black hole solution, the Bekenstein–Hawking entropy is computed from the area law. It can also be reproduced by evaluating the boundary stress tensor with an appropriate local counter-term (which is essentially provided by a Liouville-type potential).

© 2018 Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Introduction

A recent interest in the study of String Theory is to establish a toy model of the AdS/CFT correspondence [1–3]. In particular, it is significant to understand a quantum mechanical description of black hole with the holographic principle [4,5]. Along this direction, Kitaev proposed a one-dimensional system composed of $N \gg 1$ fermions with a random, all-to-all quartic coupling [6]. This model is a variant of the Sachdev–Ye (SY) model [7], and so it is called the Sachdev–

^{*} Corresponding author.

E-mail addresses: s.okumura@gauge.scphys.kyoto-u.ac.jp (S. Okumura), kyoshida@gauge.scphys.kyoto-u.ac.jp (K. Yoshida).

Ye–Kitaev (SYK) model. A remarkable point is that this model exhibits the maximal Lyapunov exponent in out-of-time-order four-point functions [6,8,9]. Hence the SYK model may have a gravity dual described by an Einstein gravity [10,11].¹ There are a lot of developments. All-point correlation functions have been computed in [14]. The SYK model is extended to multi-flavor cases [15]. Supersymmetric extensions are presented in [16]. A possible relation to 3D bulk dual is argued in [17]. The spectral density of the SYK model is analytically computed in [18]. Some models without the disordered coupling are proposed in [19,20] and the related tensor models are discussed in [21,22].

A simple candidate of the gravity dual for the SYK model is a particular 1 + 1 dimensional dilaton gravity system originally introduced by Jackiw [23] and Teitelboim [24] (called the Jackiw–Teitelboim (JT) model).² This model has been revisited by Almheiri and Polchinski [26] from the point of view of holography. Hence this model is sometimes called the AP model. A lot of efforts have been made to clarify the relation between the JT model and the SYK model, but it would be fair to say that the two models coincide in a low-energy region as the Schwarzian theory [27–30].

In the preceding works [31,32], we have studied a deformation of the JT model by applying the Yang–Baxter deformation technique [33–35]. The deformed model has a hyperbolic dilaton potential. We have shown that solutions in the deformed model can be represented by a couple of Liouville’s solutions and found several solutions such as the general vacuum solutions, shock waves and deformed black holes. Typically, the region near the boundary is deformed to a two-dimensional de Sitter (dS_2) and a new naked singularity appears. These are common features in geometries generated by Yang–Baxter deformations with classical r -matrices of Drinfeld–Jimbo type. However, what these features indicate has not been revealed so far.

In this paper, we study the deformed JT model by employing a proper frame proposed by Frolov and Zelnikov [36]. This frame is realized by performing a dilaton-dependent Weyl transformation and in this frame the deformed model is described as a Liouville gravity model with a cosmological constant. Then we find regular solutions with no singularities intrinsic to the deformations by using $SL(2)$ transformations. Among them, a black hole solution is included. We investigate the thermodynamics properties by examining the Bekenstein–Hawking entropy. This entropy can also be reproduced by evaluating the boundary stress tensor with a local counter-term. Notably, the counter-term in this proper frame is much simpler than the ones utilized in the preceding works [31,32].

This paper is organized as follows. Section 2 gives a brief review of the deformed JT model. In section 3, we revisit the deformed JT model by employing a particular Weyl transformation. In this new frame, the deformed model is described as a Liouville gravity with a cosmological constant. In section 4, we consider a black hole solution and examine its thermodynamic properties. In particular, the Bekenstein–Hawking entropy is reproduced by evaluating the boundary stress tensor with an appropriate counter-term. Section 5 is devoted to conclusion and discussion. In Appendix A, we show the difference between the black hole solution newly obtained in this paper and the one previously found in the preceding works [31,32].

¹ As another interesting direction, the conformal SYK model is proposed by Gross and Rosenhaus [12,13]. Then, the bulk dual may not be a gravitational theory but scalar field theories on the rigid AdS_2 .

² For a nice review on 2D dilaton-gravity, see [25].

2. A review of a deformed Jackiw–Teitelboim model

In this section, let us give a short review of a deformed Jackiw–Teitelboim (JT) model with a hyperbolic dilaton potential presented in the preceding works [31,32].

2.1. The general vacuum solutions

We work here in the Lorentzian signature and the (1 + 1)-dimensional spacetime is described by the coordinates $x^\mu = (x^0, x^1) = (t, x)$. The basic ingredients of this system are the 2D metric $g_{\mu\nu}$ and the dilaton Φ .

The classical action for $g_{\mu\nu}$ and Φ is given by [31,32]³

$$S_\Phi = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \left[\Phi^2 R + \frac{1}{\eta L^2} \sinh(2\eta \Phi^2) \right] + \frac{1}{8\pi G} \int dt \sqrt{-\gamma_{tt}} \Phi^2 K, \quad (2.1)$$

where G is a two-dimensional Newton constant, R and g are Ricci scalar and determinant of $g_{\mu\nu}$, and L is an AdS radius. The last term is the Gibbons–Hawking term that consists of extrinsic metric γ_{tt} and extrinsic curvature K .

The real constant parameter η measures the deformation. We assume that η is positive. In the $\eta \rightarrow 0$ limit, the classical action (2.1) reduces to the original JT model [23,24] (without matter fields)

$$S_\Phi^{(\eta=0)} = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi^2 \left[R + \frac{2}{L^2} \right] + \frac{1}{8\pi G} \int dt \sqrt{-\gamma_{tt}} \Phi^2 K. \quad (2.2)$$

Thus the model (2.1) can be regarded as a deformation of the JT model.

In the following, we will work with the metric in conformal gauge,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\omega} dx^+ dx^-, \quad (2.3)$$

where the light-cone coordinates x^\pm are defined as

$$x^\pm \equiv t \pm z. \quad (2.4)$$

By taking variations of S_Φ , the equations of motion are obtained as follows:

$$4\partial_+ \partial_- \Phi^2 + \frac{e^{2\omega}}{\eta L^2} \sinh(2\eta \Phi^2) = 0, \quad (2.5)$$

$$4\partial_+ \partial_- \omega + \frac{e^{2\omega}}{L^2} \cosh(2\eta \Phi^2) = 0, \quad (2.6)$$

$$-e^{2\omega} \partial_+ (e^{-2\omega} \partial_+ \Phi^2) = 0, \quad (2.7)$$

$$-e^{2\omega} \partial_- (e^{-2\omega} \partial_- \Phi^2) = 0. \quad (2.8)$$

To solve the equations of motion [(2.5)–(2.8)] systematically, it is very useful to introduce a set of new variables $\tilde{\omega}_1$ and $\tilde{\omega}_2$:

$$\tilde{\omega}_1 \equiv \omega + \eta \Phi^2, \quad \tilde{\omega}_2 \equiv \omega - \eta \Phi^2. \quad (2.9)$$

From (2.5) and (2.6), one can obtain two Liouville equations for $\tilde{\omega}_1$ and $\tilde{\omega}_2$:

³ The sinh-type potential is related to a q -deformed $\mathfrak{sl}(2)$ algebra via (4.1) in the work [37].

$$\begin{aligned}
 4\partial_+\partial_-\tilde{\omega}_1 + \frac{1}{L^2}e^{2\tilde{\omega}_1} &= 0, \\
 4\partial_+\partial_-\tilde{\omega}_2 + \frac{1}{L^2}e^{2\tilde{\omega}_2} &= 0.
 \end{aligned}
 \tag{2.10}$$

It is well known that the general solutions to the Liouville equation are given by arbitrary holomorphic functions $X_i^+(x^+)$ ($i = 1, 2$) and anti-holomorphic functions $X_j^-(x^-)$ ($j = 1, 2$):

$$\begin{aligned}
 e^{2\tilde{\omega}_1} &= \frac{4L^2\partial_+X_1^+\partial_-X_1^-}{(X_1^+ - X_1^-)^2}, \\
 e^{2\tilde{\omega}_2} &= \frac{4L^2\partial_+X_2^+\partial_-X_2^-}{(X_2^+ - X_2^-)^2}.
 \end{aligned}
 \tag{2.11}$$

By using X_1^\pm and X_2^\pm , the constraint conditions (2.7) and (2.8) can be simplified into

$$\text{Sch}\{X_1^\pm, x^\pm\} - \text{Sch}\{X_2^\pm, x^\pm\} = 0.
 \tag{2.12}$$

Here $\text{Sch}\{X, x\}$ is the Schwarzian derivative defined as

$$\text{Sch}\{X, x\} \equiv \frac{X'''}{X'} - \frac{3}{2}\left(\frac{X''}{X'}\right)^2.
 \tag{2.13}$$

As a result, the metric $e^{2\omega}$ and the dilaton Φ^2 are represented by two solutions to the Liouville equations:

$$e^{2\omega} = \sqrt{e^{2\tilde{\omega}_1}e^{2\tilde{\omega}_2}} = 4L^2 \sqrt{\frac{\partial_+X_1^+\partial_-X_1^-}{(X_1^+ - X_1^-)^2} \frac{\partial_+X_2^+\partial_-X_2^-}{(X_2^+ - X_2^-)^2}},
 \tag{2.14}$$

$$\Phi^2 = \frac{\tilde{\omega}_1 - \tilde{\omega}_2}{2\eta} = \frac{1}{4\eta} \log \left| \frac{\partial_+X_1^+\partial_-X_1^-}{(X_1^+ - X_1^-)^2} \frac{(X_2^+ - X_2^-)^2}{\partial_+X_2^+\partial_-X_2^-} \right|.
 \tag{2.15}$$

Notably, the general solutions (2.1) are discussed as Yang–Baxter deformations of AdS_2 [32].

2.2. Deformed black hole with conformal matter

Let us next consider to add a conformal matter χ , which couples to the Ricci scalar and the dilaton. The classical action is given by

$$S_\chi = -\frac{N}{24\pi} \int d^2x \sqrt{-g} \left[\chi(R - 2\eta\nabla^2\Phi^2) + (\nabla\chi)^2 \right] - \frac{N}{12\pi} \int dt \sqrt{-\gamma_{tt}} \chi K.
 \tag{2.16}$$

Here N denotes the central charge of χ .

Similarly to the derivation of (2.10), the equations of motion can be rewritten in terms of $\tilde{\omega}_1$ and $\tilde{\omega}_2$. After all, the equations of motion are given by

$$\begin{aligned}
 \partial_+\partial_-(\tilde{\omega}_1 + \chi) &= 0, \\
 4\partial_+\partial_-\tilde{\omega}_1 + e^{2\tilde{\omega}_1} &= \frac{16}{3}GN\eta\partial_+\partial_-\chi, \\
 4\partial_+\partial_-\tilde{\omega}_2 + e^{2\tilde{\omega}_2} &= 0, \\
 e^{\tilde{\omega}_1}\partial_\pm\partial_\pm e^{-\tilde{\omega}_1} - e^{\tilde{\omega}_2}\partial_\pm\partial_\pm e^{-\tilde{\omega}_2} &= \frac{2}{3}GN(-\partial_\pm\partial_\pm\chi + \partial_\pm\chi\partial_\pm\chi + 2\partial_\pm\chi\partial_\pm\tilde{\omega}_1).
 \end{aligned}
 \tag{2.17}$$

Note here that the third equation is still the Liouville equation, while the second one acquires the source term due to the presence of the conformal matter. The last one gives rise to the constraint conditions for the solutions.

A black hole solution

Here, let us derive a black hole solution. Suppose that the solution is static. Then, by solving the first equation in the set of equations of motion (2.17), χ can be expressed as⁴

$$\chi = -\tilde{\omega}_1 - \sqrt{\mu}(x^+ - x^-). \tag{2.18}$$

By eliminating χ from the other equations, one can derive a couple of Liouville equations and the constraint conditions:

$$\begin{aligned} 4\left(1 + \frac{4}{3}GN\eta\right)\partial_+\partial_-\tilde{\omega}_1 + e^{2\tilde{\omega}_1} &= 0, \\ 4\partial_+\partial_-\tilde{\omega}_2 + e^{2\tilde{\omega}_2} &= 0, \\ \left(1 + \frac{2}{3}GN\right)e^{\tilde{\omega}_1}\partial_\pm\partial_\pm e^{-\tilde{\omega}_1} - e^{\tilde{\omega}_2}\partial_\pm\partial_\pm e^{-\tilde{\omega}_2} &= \frac{2}{3}GN\mu. \end{aligned} \tag{2.19}$$

Now $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are the general solutions to the Liouville equation:

$$\begin{aligned} e^{2\tilde{\omega}_1} &= \frac{4L^2(1 + \frac{4}{3}GN\eta)}{(X_1^+ - X_1^-)^2} \partial_+X_1^+\partial_-X_1^-, \\ e^{2\tilde{\omega}_2} &= \frac{4L^2}{(X_2^+ - X_2^-)^2} \partial_+X_2^+\partial_-X_2^-. \end{aligned} \tag{2.20}$$

By using X_i^\pm and the Schwarzian derivative, the constraint conditions can be rewritten as

$$\begin{aligned} \left(1 + \frac{2}{3}GN\right)\text{Sch}\{X_1^+, x^+\} - \text{Sch}\{X_2^+, x^+\} &= -\frac{4}{3}GN\mu, \\ \left(1 + \frac{2}{3}GN\right)\text{Sch}\{X_1^-, x^-\} - \text{Sch}\{X_2^-, x^-\} &= -\frac{4}{3}GN\mu. \end{aligned} \tag{2.21}$$

It is an easy task to see that

$$X_{1,2}^\pm = \tanh(\sqrt{\mu}x^\pm)$$

satisfy the constraint conditions. In general, the linear fractional transformations of them,

$$X_{1,2}^\pm = \frac{a \tanh(\sqrt{\mu}x^\pm) + b}{c \tanh(\sqrt{\mu}x^\pm) + d}, \tag{2.22}$$

also satisfy the constraint conditions, thanks to a property of the Schwarzian derivative.

⁴ The general solution is given by $\chi + \tilde{\omega}_1 = c_1(x^+ - x^-) + c_0$, where c_1 and c_0 are arbitrary real constants. We have set that $c_0 = 0$ for simplicity and $c_1 = \sqrt{\mu}$ for later convenience, where μ is a real positive constant.

By taking certain parameters, a deformed black hole solution [31,32] is given by

$$e^{2\omega} = \frac{4\mu L^2(1 - \eta^2\mu)\sqrt{1 + \frac{4}{3}GN\eta}}{\sinh^2(2\sqrt{\mu}Z) - \eta^2\mu \cosh^2(2\sqrt{\mu}Z)}, \tag{2.23}$$

$$\Phi^2 = \frac{1}{2\eta} \log \left| \frac{1 + \eta\sqrt{\mu} \coth(2\sqrt{\mu}Z)}{1 - \eta\sqrt{\mu} \coth(2\sqrt{\mu}Z)} \right| + \frac{1}{4\eta} \log \left(1 + \frac{4}{3}GN\eta \right), \tag{2.24}$$

where the range of $\sqrt{\mu}$ is restricted as

$$0 \leq \sqrt{\mu} \leq \frac{1}{\eta}$$

so as to ensure the positivity of the exponential in (2.23). Notably, the solutions have a naked singularity intrinsic to the deformation. For the detail, see [31,32].

3. Moving to a proper frame: Weyl transformation

In this section, we shall introduce a new proper frame, which was originally utilized by Frolov and Zelnikov [36]. One can see that in this proper frame, the deformed JT model can be recaptured as a Liouville dilaton gravity model with a cosmological constant term, while solutions are still given by $\tilde{\omega}_1$ and $\tilde{\omega}_2$.

The proper frame is introduced through a dilaton-dependent Weyl transformation:

$$g_{\mu\nu} = e^{-2\eta\Phi^2} \tilde{g}_{\mu\nu}. \tag{3.1}$$

In conformal gauge, $\tilde{\omega}_1$ plays the role of the conformal factor in front of the metric:

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -e^{2\tilde{\omega}_1} dx^+ dx^-. \tag{3.2}$$

In terms of the new metric $\tilde{g}_{\mu\nu}$, the classical action of the deformed JT model (2.1) can be rewritten into the following form:

$$\tilde{S}_\Phi = \frac{1}{16\pi G} \int d^2x \sqrt{-\tilde{g}} \left[\Phi^2 \tilde{R} - 2\eta(\tilde{\nabla}\Phi^2)^2 - \frac{1}{2\eta L^2} (e^{-4\eta\Phi^2} - 1) \right]. \tag{3.3}$$

Note here that the kinematic term of Φ^2 is well-defined because we have assumed that η is a positive real constant. The potential is now bounded from below, but it is a run-away type potential. It is also remarkable that Φ^2 (instead of Φ) appears in the classical action (3.3) and Φ^2 should be definitely positive. Hence this is not the usual Liouville gravity but rather a *constrained* Liouville gravity. Interestingly, this constrained system can also be derived from Einstein–Hilbert action with a cosmological constant [38].

It is remarkable that the equations of motion for ω and Φ^2 are equivalent to the equations for $\tilde{\omega}_1$ and $\tilde{\omega}_2$. Thus, the solutions to (3.3) are obtained by $\tilde{\omega}_1$ and $\tilde{\omega}_2$ as in (2.11). From $\tilde{\omega}_1$ and $\tilde{\omega}_2$, the dilation is determined by (2.15) again. However, in the proper frame, the metric is given by $\tilde{\omega}_1$ only.

In summary, the general vacuum solutions to (3.3) are given by

$$e^{2\tilde{\omega}_1} = \frac{4L^2 \partial_+ X_1^+ \partial_- X_1^-}{(X_1^+ - X_1^-)^2},$$

$$\Phi^2 = \frac{1}{4\eta} \log \left| \frac{\partial_+ X_1^+ \partial_- X_1^-}{(X_1^+ - X_1^-)^2} \frac{(X_2^+ - X_2^-)^2}{\partial_+ X_2^+ \partial_- X_2^-} \right|. \tag{3.4}$$

Note here that the metric is given by the general solution to Liouville equation and the rigid AdS₂ geometry is preserved in the new frame (i.e., proper frame). This result indicates that the Weyl transformation carried out here has undone the Yang–Baxter deformation of the metric, while the deformation effect is now encoded into only the dilaton part. It should be remarked that this is a rather natural result, noticing that the Yang–Baxter deformation effect can be factored out as the overall factor of the metric, as shown in [31,32].

4. A new black hole solution and its thermodynamics

In this section, we present a new black hole solution with a conformal matter.⁵ The proper frame (3.1) enables us to construct an AdS₂ black hole solution (i.e., the metric is the same as the undeformed case [26]).⁶ Then, we compute the entropy of the black hole solution in two manners: 1) the Bekenstein–Hawking entropy and 2) the boundary stress tensor with a certain counter-term. Both results are consistent.

A new black hole solution

In the proper frame, the classical action of the matter (2.16) is given by

$$\tilde{S}_\chi = -\frac{N}{24\pi} \int d^2x \sqrt{-\tilde{g}} \left[\chi \tilde{R} + (\tilde{\nabla}\chi)^2 \right] - \frac{N}{12\pi} \int dt \sqrt{-\tilde{\gamma}_{tt}} \chi K. \tag{4.1}$$

Note here that in the proper frame, χ couples to only the Ricci scalar, while in the old frame (2.16), χ coupled to both Ricci scalar and dilaton. This point is the same as in the undeformed case [26].

In order to find out a black hole solution of the system (4.1), let us take black hole coordinates for X_1^\pm as

$$X_1^\pm(x^\pm) = \frac{1}{\sqrt{\mu}} \tanh(\sqrt{\mu}x^\pm), \tag{4.2}$$

by following [26]. Then, for X_2^\pm , we take the following linear fractional transformation:

$$\begin{aligned} X_2^+(x^+) &= \frac{\tanh(\sqrt{\mu}x^+) + 2\eta\sqrt{\mu}}{2\eta\mu \tanh(\sqrt{\mu}x^+) + \sqrt{\mu}}, \\ X_2^-(x^-) &= \frac{1}{\sqrt{\mu}} \tanh(\sqrt{\mu}x^-). \end{aligned} \tag{4.3}$$

Thus we obtain the static solutions for $\tilde{\omega}_1$ and $\tilde{\omega}_2$:

$$e^{2\tilde{\omega}_1} = \frac{4\mu(1 + \frac{4}{3}GN\eta)L^2}{\sinh^2(2\sqrt{\mu}Z)}, \tag{4.4}$$

$$e^{2\tilde{\omega}_2} = \frac{4\mu(1 - 4\eta^2\mu)L^2}{(\sinh(2\sqrt{\mu}Z) + 2\eta\sqrt{\mu} \cosh(2\sqrt{\mu}Z))^2}. \tag{4.5}$$

⁵ This solution is different from the one introduced in Sec. 2.2. For the detail, see Appendix A.

⁶ By performing the inverse Weyl transformation for the new solution, a naked singularity appears again. In some sense, this Weyl transformation is similar to a coordinate transformation, which was proposed in [39], in the case of the η -deformed AdS₅ [40].

Hence a black hole solution with a conformal matter is

$$e^{2\tilde{\omega}_1} = \frac{4\mu(1 + \frac{4}{3}GN\eta)L^2}{\sinh^2(2\sqrt{\mu}Z)}, \tag{4.6}$$

$$\Phi^2 = \frac{1}{2\eta} \log \left| 1 + 2\eta\sqrt{\mu} \coth(\sqrt{\mu}Z) \right| + \Phi_0^2. \tag{4.7}$$

Here Φ_0^2 is the constant part of the dilaton given by

$$\Phi_0^2 \equiv \frac{1}{4\eta} \log \left(\frac{1 + \frac{4}{3}GN\eta}{1 - 4\eta^2\mu} \right). \tag{4.8}$$

You see that the matter contribution just rescales the metric and shifts the dilaton by a constant. Note here that the allowed region of μ is restricted like

$$0 \leq \sqrt{\mu} \leq \frac{1}{2\eta} \tag{4.9}$$

so as to make the value of Φ_0^2 well-defined and preserve the positivity of (4.5).

It should be remarked that this solution is different from the previous black hole solution with (2.23) and (2.24), though the two solutions are quite similar but the μ -dependence of the metric and the range of $\sqrt{\mu}$ are different as explicitly shown in Appendix A.

By taking the undeformed limit $\eta \rightarrow 0$, this solution reduces to the black hole solution with a conformal matter in the undeformed case [26]:

$$e^{2\omega} = \frac{4\mu L^2}{\sinh^2(2\sqrt{\mu}Z)}, \tag{4.10}$$

$$\Phi^2 = \sqrt{\mu} \coth(\sqrt{\mu}Z) + \frac{1}{3}GN. \tag{4.11}$$

In the following, let us evaluate the black hole entropy associated with (4.6) and (4.7) in two manners.

1) *Bekenstein–Hawking entropy*

Let us first compute the Bekenstein–Hawking entropy. From the black hole metric (4.6), the Hawking temperature is computed as

$$T_H = \frac{\sqrt{\mu}}{\pi}. \tag{4.12}$$

This is the same as in the undeformed case [26]. From the classical action, one can read off the effective Newton constant G_{eff} as

$$\frac{1}{G_{\text{eff}}} = \frac{\Phi^2}{G} - \frac{2N\chi}{3}. \tag{4.13}$$

Given that the horizon area A is 1, the Bekenstein–Hawking entropy S_{BH} is evaluated as

$$\begin{aligned} S_{\text{BH}} &= \frac{A}{4G_{\text{eff}}} \Big|_{Z \rightarrow \infty} \\ &= \frac{\text{arctanh}(2\pi T_H \eta)}{8G\eta} + \frac{N}{6} \log(T_H) + \text{constant}. \end{aligned} \tag{4.14}$$

The last term is a constant term independent of the Hawking temperature. Note here that the argument of arctanh should be less than 1. This means that

$$0 \leq T_H \leq \frac{1}{2\pi\eta}.$$

This range agrees with the possible values of $\sqrt{\mu}$ given in (4.9).

2) *Boundary stress tensor*

In conformal gauge, the total action, including the Gibbons–Hawking term, is given by

$$\begin{aligned} \tilde{S}_\Phi &= \frac{1}{8\pi G} \int d^2x \left[-4\partial_{(+)}\Phi^2\partial_{(-)}\tilde{\omega}_1 + 4\eta\partial_+\Phi^2\partial_-\Phi^2 - \frac{1}{4\eta L^2}e^{2\tilde{\omega}_1}(e^{-4\eta\Phi^2} - 1) \right], \\ \tilde{S}_\chi &= \frac{N}{6\pi} \int d^2x \left[\partial_+\chi\partial_-\chi + 2\partial_{(+}\chi\partial_-\tilde{\omega}_1) \right]. \end{aligned} \tag{4.15}$$

By using the explicit expression of the black hole solution, the on-shell bulk action can be evaluated on the boundary.

The on-shell action diverges at the boundary $Z = 0$, hence one needs to introduce a cut-off as $Z = \epsilon (> 0)$, where ϵ is an infinitesimal quantity. Then the on-shell action can be expanded with respect to ϵ like

$$\tilde{S}_\Phi + \tilde{S}_\chi = \int dt \left[\frac{1 + \frac{4}{3}GN\eta}{16\pi G\eta\epsilon} + O(\epsilon^1) \right]. \tag{4.16}$$

Here we have ignored the terms which vanish in the $\epsilon \rightarrow 0$ limit, and only the divergent term has explicitly been written down. To cancel out the divergence, it is necessary to add an appropriate counter-term.

Our proposal for the counter-term is the following⁷:

$$\tilde{S}_{ct} = \int dt \frac{\sqrt{-\tilde{\gamma}_{tt}}}{L'} \left[\frac{-1}{16\pi G\eta} \left(1 + (2\eta\Phi_0^2 - 1)e^{-2\eta(\Phi^2 - \Phi_0^2)} \right) - \frac{N}{24\pi} \right]. \tag{4.17}$$

Here Φ_0 is the constant defined in (4.8) and L' is the rescaled AdS radius defined as

$$L'^2 \equiv L^2 \left(1 + \frac{4}{3}GN\eta \right). \tag{4.18}$$

Then the extrinsic metric $\tilde{\gamma}_{tt}$ on the boundary is defined as

$$\tilde{\gamma}_{tt} \equiv -e^{2\tilde{\omega}_1} \Big|_{Z=\epsilon}.$$

Note here that the counter-term (4.17) is local and is represented basically by the Liouville potential. In the undeformed limit $\eta \rightarrow 0$, the counter-term (4.17) reduces to

$$\tilde{S}_{ct}^{(\eta=0)} = \int dt \frac{\sqrt{-\gamma_{tt}}}{L} \left(-\frac{\Phi^2}{8\pi G} - \frac{N}{24\pi} \right). \tag{4.19}$$

This is nothing but the counter-term utilized in the undeformed case [26].

⁷ The uniqueness of the counter-term has not been confirmed. It is significant to revisit it by following the works [41–43].

It is straightforward to check that the sum $\tilde{S} = \tilde{S}_\Phi + \tilde{S}_\chi + \tilde{S}_{ct}$ becomes finite on the boundary by using the expanded form of the counter-term (4.17):

$$\tilde{S}_{ct} = \int dt \left[-\frac{1 + \frac{4}{3}GN\eta}{16\pi G\eta\epsilon} + \frac{1 - 2\eta\Phi_0^2}{16\pi G\eta^2} + O(\epsilon) \right].$$

In a region near the boundary, the warped factor of the metric can be expanded as

$$e^{2\tilde{\omega}_1} = \frac{L'^2}{\epsilon^2} + O(\epsilon^0). \tag{4.20}$$

Hence, by normalizing the boundary metric as

$$\hat{\gamma}_{tt} = \frac{\epsilon^2}{L'^2} \tilde{\gamma}_{tt},$$

the boundary stress tensor is defined as

$$\langle \hat{T}_{tt} \rangle \equiv \frac{-2}{\sqrt{-\tilde{\gamma}_{tt}}} \frac{\delta S}{\delta \tilde{\gamma}^{tt}} = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{L'} \frac{-2}{\sqrt{-\tilde{\gamma}_{tt}}} \frac{\delta S}{\delta \tilde{\gamma}^{tt}}. \tag{4.21}$$

After all, $\langle \hat{T}_{tt} \rangle$ is evaluated as

$$\langle \hat{T}_{tt} \rangle = -\frac{\log(1 - 4\eta^2\mu)}{32\pi G\eta^2} + \frac{\log(1 + \frac{4}{3}GN\eta)}{32\pi G\eta^2} + \frac{N\sqrt{\mu}}{6\pi}. \tag{4.22}$$

This expression of $\langle \hat{T}_{tt} \rangle$ should be identified with thermodynamic energy E like

$$E = -\frac{\log(1 - 4\pi^2 T_H^2 \eta^2)}{32\pi G\eta^2} + \frac{\log(1 + \frac{4}{3}GN\eta)}{32\pi G\eta^2} + \frac{N}{6} T_H, \tag{4.23}$$

where we have used the expression of the Hawking temperature (4.12).

Then, by solving the thermodynamic relation,

$$dE = \frac{dS}{T_H}, \tag{4.24}$$

the entropy is obtained as

$$S = \frac{\text{arctanh}(2\pi T_H\eta)}{8G\eta} + \frac{N}{6} \log(T_H) + S_{T_H=0}. \tag{4.25}$$

Here $S_{T_H=0}$ has appeared as an integration constant that measures the entropy at zero temperature. Thus the resulting entropy is consistent with the Bekenstein–Hawking entropy (4.14), up to the temperature-independent constant.

5. Conclusion and discussion

In this paper we have revisited a deformed Jackiw–Teitelboim model with a hyperbolic dilaton potential, which was constructed in the preceding work [31,32]. By employing a Weyl transformation to this deformed model, we have discussed a Liouville type potential with a cosmological constant term. Then we have found regular solutions coupled to a conformal matter by using $SL(2)$ transformations.

For a black hole solution, its thermodynamic behavior has been investigated. In particular, the Bekenstein–Hawking entropy can be reproduced by evaluating the boundary stress tensor with a local counter-term, which is essentially provided by a Liouville type potential as in the undeformed case [26]. Notably, this counter-term is concise, in comparison to the previous ones utilized in [31,32].

It would be significant to argue the implication of the Weyl transformation utilized in our analysis in the context of Yang–Baxter deformation. An important observation is that the deformation is nothing but the overall factor in front of the AdS_2 metric. Hence the undeformed AdS_2 may be realized by performing an appropriate Weyl transformation. After all, the vestiges of the deformation is encoded into the dilaton and other matter fields. Of course, in higher dimensions, it would not be possible to realize the undeformed metric completely. But it should be interesting to concentrate on the AdS_2 subregion of higher-dimensional Yang–Baxter deformed backgrounds. In particular, Yang–Baxter deformed backgrounds suffer from a naked singularity. A famous example is the η -deformed background [40] with a classical r -matrix of the Drinfeld–Jimbo type and an η -deformed $\text{AdS}_2 \times S^2$ is discussed in [44,45]. As an exercise, it would be nice to apply our argument to this η -deformed $\text{AdS}_2 \times S^2$ case. We will report some results on this issue in the near future [46].

Acknowledgements

We are very grateful to Masafumi Fukuma, Hikaru Kawai, Hideki Kyono and Jun-ichi Sakamoto for valuable comments and useful discussions. We appreciate Daniel Grumiller for letting us know significant references. The work of K.Y. was supported by the Supporting Program for Interaction-based Initiative Team Studies (SPIRITS) from Kyoto University, a JSPS Grant-in-Aid for Scientific Research (B) No. 18H01214 and a JSPS Grant-in-Aid for Scientific Research (C) No. 15K05051. This work is also supported in part by the JSPS Japan–Russia Research Cooperative Program.

Appendix A

A.1. Comparison of two black hole solutions

In Sec. 4, we have considered a black hole solution in the proper frame. This solution is slightly different from the one presented in Sec. 2.2 (or equivalently found in [31,32]). In the following, we show the difference explicitly by comparing the two solutions after mapping the previous solution in Sec. 2.2 to the proper frame.

A.2. The deformed black hole solution in Sec. 2.2

First of all, let us revisit the black hole solution [(2.23) and (2.24)]. In the following, we will focus upon the case without matters for simplicity (i.e., the case with $N = 0$).

In Sec. 2.2, we employed the following linear fractional transformation:

$$\begin{aligned} X_1^+(x^+) &= \frac{(1 - \eta\beta)X^+(x^+) - 2\eta\alpha}{-2\eta\gamma X^+(x^+) + (1 + \eta\beta)}, & X_1^-(x^-) &= X^-(x^-), \\ X_2^+(x^+) &= \frac{(1 + \eta\beta)X^+(x^+) + 2\eta\alpha}{2\eta\gamma X^+(x^+) + (1 - \eta\beta)}, & X_2^-(x^-) &= X^-(x^-). \end{aligned} \quad (\text{A.1})$$

Then, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are given by

$$\begin{aligned}
 e^{2\tilde{\omega}_1} &= \frac{4(1 - \eta^2(\beta^2 + 4\alpha\gamma)) \partial_+ X^+ \partial_- X^-}{(X^+ - X^- - \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+ X^-))^2}, \\
 e^{2\tilde{\omega}_2} &= \frac{4(1 - \eta^2(\beta^2 + 4\alpha\gamma)) \partial_+ X^+ \partial_- X^-}{(X^+ - X^- + \eta(2\alpha + \beta(X^+ + X^-) - 2\gamma X^+ X^-))^2}.
 \end{aligned}
 \tag{A.2}$$

In particular, by taking the following parameters

$$\alpha = \frac{1}{2}, \quad \beta = 0, \quad \gamma = \frac{\mu}{2}
 \tag{A.3}$$

and the black hole coordinates

$$X^\pm = \frac{1}{\sqrt{\mu}} \tanh[\sqrt{\mu}(T \pm Z)],
 \tag{A.4}$$

the deformed black hole solution [(2.23) and (2.24)] can be reproduced.

A.3. Mapping to the proper frame

The next task is to map the solution [(2.23) and (2.24)] to the proper frame.

In order to realize the rigid AdS₂ metric in the proper frame, we need to perform the following coordinate transformation:

$$X^+(x^+) = \frac{(1 - \eta\beta)\tilde{X}^+ + 2\eta\alpha}{2\eta\gamma\tilde{X}^+ + (1 - \eta\beta)}, \quad X^-(x^+) = \tilde{X}^-.
 \tag{A.5}$$

Then the holomorphic functions are transformed as

$$X_1^+ = \tilde{X}^+, \quad X_2^+ = \frac{((1 + \eta\beta)^2 + 4\eta^2\alpha\gamma)\tilde{X}^+ + 4\eta\alpha}{4\eta\gamma\tilde{X}^+ + ((1 - \eta\beta)^2 + 4\eta^2\alpha\gamma)}.
 \tag{A.6}$$

Again, by choosing the parameters (A.3) and the black hole coordinates (A.4), the black hole solution in the proper frame can be obtained as

$$\begin{aligned}
 e^{2\tilde{\omega}_1} &= \frac{4\mu L^2}{\sinh^2(2\sqrt{\mu}Z)}, \\
 e^{2\tilde{\omega}_2} &= \left(\frac{1 - \eta^2\mu}{1 + \eta^2\mu}\right)^2 \frac{4\mu L^2}{\left(\sinh(2\sqrt{\mu}Z) + \frac{2\eta\sqrt{\mu}}{1 + \eta^2\mu} \cosh(2\sqrt{\mu}Z)\right)^2}.
 \end{aligned}
 \tag{A.7}$$

The dilaton is given by

$$\Phi^2 = \frac{1}{2\eta} \log \left| 1 + \frac{2\eta\sqrt{\mu}}{1 + \eta^2\mu} \coth(2\sqrt{\mu}Z) \right| + \frac{1}{2\eta} \log \left(\frac{1 + \eta^2\mu}{1 - \eta^2\mu} \right).
 \tag{A.8}$$

These expressions are clearly different from the solution with (4.6) and (4.7) for the $N = 0$ case.

References

- [1] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, *Int. J. Theor. Phys.* 38 (1999) 1113; *Adv. Theor. Math. Phys.* 2 (1998) 231, arXiv:hep-th/9711200.
- [2] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett. B* 428 (1998) 105, arXiv:hep-th/9802109.
- [3] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998) 253, arXiv:hep-th/9802150.
- [4] G. 't Hooft, Dimensional reduction in quantum gravity, *Salamfest 1993:0284–296*, arXiv:gr-qc/9310026.
- [5] L. Susskind, The World as a hologram, *J. Math. Phys.* 36 (1995) 6377, arXiv:hep-th/9409089.
- [6] A. Kitaev, A simple model of quantum holography, <http://online.kitp.ucsb.edu/online/entangled15/kitaev/>, <http://online.kitp.ucsb.edu/online/entangled15/kitaev2/>, talks at KITP, April 7, 2015 and May 27, 2015.
- [7] S. Sachdev, J.-w. Ye, Gapless spin fluid ground state in a random, quantum Heisenberg magnet, *Phys. Rev. Lett.* 70 (1993) 3339, arXiv:cond-mat/9212030.
- [8] J. Polchinski, V. Rosenhaus, The spectrum in the Sachdev–Ye–Kitaev model, *J. High Energy Phys.* 1604 (2016) 001, arXiv:1601.06768 [hep-th].
- [9] J. Maldacena, D. Stanford, Remarks on the Sachdev–Ye–Kitaev model, *Phys. Rev. D* 94 (10) (2016) 106002, arXiv:1604.07818 [hep-th].
- [10] S.H. Shenker, D. Stanford, Black holes and the butterfly effect, *J. High Energy Phys.* 1403 (2014) 067, arXiv:1306.0622 [hep-th].
- [11] S.H. Shenker, D. Stanford, Stringy effects in scrambling, *J. High Energy Phys.* 1505 (2015) 132, arXiv:1412.6087 [hep-th].
- [12] D.J. Gross, V. Rosenhaus, A line of CFTs: from generalized free fields to SYK, *J. High Energy Phys.* 1707 (2017) 086, arXiv:1706.07015 [hep-th].
- [13] D.J. Gross, V. Rosenhaus, The bulk dual of SYK: cubic couplings, *J. High Energy Phys.* 1705 (2017) 092, arXiv:1702.08016 [hep-th].
- [14] D.J. Gross, V. Rosenhaus, All point correlation functions in SYK, *J. High Energy Phys.* 1712 (2017) 148, arXiv:1710.08113 [hep-th].
- [15] D.J. Gross, V. Rosenhaus, A generalization of Sachdev–Ye–Kitaev, *J. High Energy Phys.* 1702 (2017) 093, arXiv:1610.01569 [hep-th].
- [16] W. Fu, D. Gaiotto, J. Maldacena, S. Sachdev, Supersymmetric SYK models, *Phys. Rev. D* 95 (2) (2017) 026009, arXiv:1610.08917 [hep-th].
- [17] S.R. Das, A. Jevicki, K. Suzuki, Three dimensional view of the SYK/AdS duality, *J. High Energy Phys.* 1709 (2017) 017, arXiv:1704.07208 [hep-th];
S.R. Das, A. Ghosh, A. Jevicki, K. Suzuki, Three dimensional view of arbitrary q SYK models, arXiv:1711.09839 [hep-th];
Space-time in the SYK model, arXiv:1712.02725 [hep-th].
- [18] A.M. Garcia-Garcia, J.J.M. Verbaarschot, Analytical spectral density of the Sachdev–Ye–Kitaev model at finite N , *Phys. Rev. D* 96 (6) (2017) 066012, arXiv:1701.06593 [hep-th].
- [19] E. Witten, An SYK-like model without disorder, arXiv:1610.09758 [hep-th].
- [20] T. Nishinaka, S. Terashima, A note on Sachdev–Ye–Kitaev like model without random coupling, *Nucl. Phys. B* 926 (2018) 321, arXiv:1611.10290 [hep-th].
- [21] I.R. Klebanov, G. Tarnopolsky, Uncolored random tensors, melon diagrams, and the SYK models, *Phys. Rev. D* 95 (4) (2017) 046004, arXiv:1611.08915 [hep-th];
On large N limit of symmetric traceless tensor models, *J. High Energy Phys.* 1710 (2017) 037, arXiv:1706.00839 [hep-th];
S. Giombi, I.R. Klebanov, G. Tarnopolsky, Bosonic tensor models at large N and small ϵ , *Phys. Rev. D* 96 (10) (2017) 106014, arXiv:1707.03866 [hep-th];
K. Bulychева, I.R. Klebanov, A. Milekhin, G. Tarnopolsky, Spectra of operators in large N tensor models, arXiv:1707.09347 [hep-th].
- [22] H. Itoyama, A. Mironov, A. Morozov, Rainbow tensor model with enhanced symmetry and extreme melonic dominance, *Phys. Lett. B* 771 (2017) 180, arXiv:1703.04983 [hep-th];
Ward identities and combinatorics of rainbow tensor models, *J. High Energy Phys.* 1706 (2017) 115, arXiv:1704.08648 [hep-th];
Cut and join operator ring in Aristotelian tensor model, arXiv:1710.10027 [hep-th].
- [23] R. Jackiw, Lower dimensional gravity, *Nucl. Phys. B* 252 (1985) 343–356.
- [24] C. Teitelboim, Gravitation and Hamiltonian structure in two space-time dimensions, *Phys. Lett. B* 126 (1983) 41–45.

- [25] D. Grumiller, W. Kummer, D.V. Vassilevich, Dilaton gravity in two-dimensions, *Phys. Rep.* 369 (2002) 327, arXiv: hep-th/0204253.
- [26] A. Almheiri, J. Polchinski, Models of AdS₂ backreaction and holography, *J. High Energy Phys.* 1511 (2015) 014, arXiv:1402.6334 [hep-th].
- [27] K. Jensen, Chaos in AdS₂ holography, *Phys. Rev. Lett.* 117 (11) (2016) 111601, arXiv:1605.06098 [hep-th].
- [28] J. Maldacena, D. Stanford, Z. Yang, Conformal symmetry and its breaking in two dimensional nearly Anti-de-Sitter space, *PTEP* 2016 (12) (2016) 12C104, arXiv:1606.01857 [hep-th].
- [29] J. Engelsöy, T.G. Mertens, H. Verlinde, An investigation of AdS₂ backreaction and holography, *J. High Energy Phys.* 1607 (2016) 139, arXiv:1606.03438 [hep-th].
- [30] D. Stanford, E. Witten, Fermionic localization of the Schwarzian theory, arXiv:1703.04612 [hep-th].
- [31] H. Kyono, S. Okumura, K. Yoshida, Deformations of the Almheiri–Polchinski model, *J. High Energy Phys.* 1703 (2017) 173, arXiv:1701.06340 [hep-th].
- [32] H. Kyono, S. Okumura, K. Yoshida, Comments on 2D dilaton gravity system with a hyperbolic dilaton potential, *Nucl. Phys. B* 923 (2017) 126, arXiv:1704.07410 [hep-th].
- [33] C. Klimcik, Yang–Baxter sigma models and dS/AdS T duality, *J. High Energy Phys.* 0212 (2002) 051, arXiv: hep-th/0210095;
On integrability of the Yang–Baxter sigma-model, *J. Math. Phys.* 50 (2009) 043508, arXiv:0802.3518 [hep-th].
- [34] F. Delduc, M. Magro, B. Vicedo, On classical q -deformations of integrable sigma-models, *J. High Energy Phys.* 1311 (2013) 192, arXiv:1308.3581 [hep-th];
An integrable deformation of the AdS₅ × S⁵ superstring action, *Phys. Rev. Lett.* 112 (2014) 051601, arXiv:1309.5850 [hep-th];
Derivation of the action and symmetries of the q -deformed AdS₅ × S⁵ superstring, *J. High Energy Phys.* 1410 (2014) 132, arXiv:1406.6286 [hep-th].
- [35] T. Matsumoto, K. Yoshida, Yang–Baxter sigma models based on the CYBE, *Nucl. Phys. B* 893 (2015) 287, arXiv: 1501.03665 [hep-th];
I. Kawaguchi, T. Matsumoto, K. Yoshida, Jordanian deformations of the AdS₅ × S⁵ superstring, *J. High Energy Phys.* 1404 (2014) 153, arXiv:1401.4855 [hep-th].
- [36] V.P. Frolov, A. Zelnikov, On the Liouville 2D dilaton gravity models with sinh-Gordon matter, arXiv:1712.07700 [hep-th].
- [37] N. Ikeda, K.I. Izawa, General form of dilaton gravity and nonlinear gauge theory, *Prog. Theor. Phys.* 90 (1993) 237, arXiv:hep-th/9304012.
- [38] D. Grumiller, R. Jackiw, Liouville gravity from Einstein gravity, in: S. Gosh, G. Kar (Eds.), *Recent Developments in Theoretical Physics*, World Scientific, Singapore, 2010, p. 331, arXiv:0712.3775 [gr-qc].
- [39] T. Kameyama, K. Yoshida, A new coordinate system for q -deformed AdS₅ × S⁵ and classical string solutions, *J. Phys. A* 48 (7) (2015) 075401, arXiv:1408.2189 [hep-th].
- [40] G. Arutyunov, R. Borsato, S. Frolov, S-matrix for strings on η -deformed AdS₅ × S⁵, *J. High Energy Phys.* 1404 (2014) 002, arXiv:1312.3542 [hep-th].
- [41] M. Cvetič, I. Papadimitriou, AdS₂ holographic dictionary, *J. High Energy Phys.* 1612 (2016) 008, arXiv:1608.07018 [hep-th].
- [42] D. Grumiller, R. McNees, Thermodynamics of black holes in two (and higher) dimensions, *J. High Energy Phys.* 0704 (2007) 074, arXiv:hep-th/0703230 [hep-th].
- [43] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel, D. Vassilevich, Menagerie of AdS₂ boundary conditions, *J. High Energy Phys.* 1710 (2017) 203, arXiv:1708.08471 [hep-th].
- [44] B. Hoare, R. Roiban, A.A. Tseytlin, On deformations of AdS_{*n*} × S^{*n*} supercosets, *J. High Energy Phys.* 1406 (2014) 002, arXiv:1403.5517 [hep-th].
- [45] O. Lunin, R. Roiban, A.A. Tseytlin, Supergravity backgrounds for deformations of AdS_{*n*} × S^{*n*} supercoset string models, *Nucl. Phys. B* 891 (2015) 106, arXiv:1411.1066 [hep-th].
- [46] S. Okumura, J. Sakamoto, K. Yoshida, in preparation.