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AdS geometry from CFT on a general conformally flat manifold

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Abstract

We construct an anti-de-Sitter (AdS) geometry from a conformal field theory (CFT) defined on a general conformally flat manifold via a flow equation associated with the curved manifold, which we refer to as the primary flow equation. We explicitly show that the induced metric associated with the primary flow equation becomes AdS whose boundary is the curved manifold. Interestingly, it turns out that such an AdS metric with conformally flat boundary is obtained from the usual Poincare AdS by a simple bulk finite diffeomorphism. We also demonstrate that the emergence of such an AdS space is guaranteed only by the conformal symmetry at boundary, which converts to the AdS isometry after quantum averaging, as in the case of the flat boundary. As a side remark we show that a geometry with one warped direction becomes an Einstein manifold if and only if so is its boundary at the warped direction, and briefly discuss a possibility of a little extension beyond AdS/CFT correspondence by using a genuine Einstein geometry.

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1. Introduction

The holographic principle [1,2] provides a new perspective to investigate quantum gravity on space–time with a fixed boundary. Generally in order for holography to hold in two systems,

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there has to exist a mechanism to invalidate extra infinite degrees of freedom which the bulk system usually possesses. In the AdS/CFT correspondence [3], which is a testable realization of holography [4,5], diffeomorphism invariance clearly plays a key role to kill such extra degrees of freedom. Thus it is important to reveal how diffeomorphism invariance is encoded in the dual field theory living on a boundary.

The AdS/CFT correspondence, due to its holographic property, can be explored by extending the dual geometry gradually from a fixed boundary and constructing the dual gravitational theory from CFT [6,7] (see also [8–10]). One natural interpretation of the emergent AdS radial direction from CFT is a conventional renormalization group (RG) scale [3]. This interpretation was realized for a relevant RG flow from UV CFT to IR one by constructing the corresponding dual gravity solution [11–13] (see also [14] and references therein), though a direct analysis by finding out the cut-off of Wilsonian renormalization corresponding to the sharp cut-off in the AdS radial direction is difficult to achieve due to the appearance of non-locality in the bulk [15]. There were also different approaches to see a correspondence between a certain renormalization scale and the emergent AdS radial direction by using the entanglement entropy [16–18], the stochastic quantization [19], the bilocal field in vector models [20], and the flow equation [21–23]. Recently how the Einstein equation is encoded in the boundary side was investigated [24–26]. See also [27,28].

So far these analyses were almost all restricted to the asymptotic Poincaré or global AdS space and paid attention to only conformal structure on the boundary [29,30]. However, since the bulk theory enjoys diffeomorphism invariance, there is no reason for these analyses to be restricted on a particular AdS background. It should be possible for these analyses to be generalized to those on a different AdS with a more general curved boundary which admits CFT to live.

This paper aims at making progress in this direction by using the flow equation approach [21–23]. A flow equation was introduced to specify how to smear operators so as to resolve a UV singularity in the coincidence limit [31], which turned out to help numerical simulation in lattice QCD [32–35]. Recently it was proposed that a one higher dimensional geometry is emergent associated with a flow equation [21–23] and it turned out that the direction of the free flow time precisely matches the AdS radial direction for a generic conformal field theory on the flat background [36]. The goal of this paper is to generalize this result to an arbitrary conformally flat manifold in accord with the AdS/CFT correspondence. For this end we construct a flow equation for a scalar primary operator on the conformally flat background preserving the conformally symmetric structure, which we refer to as the *primary flow equation*. We find that the induced metric associated with the primary flow equation for a generic CFT describes an AdS space whose boundary is the conformally flat manifold. We also show that such an AdS metric connects to the usual Poincaré AdS metric by a bulk finite diffeomorphism. This new result may be regarded as a consequence of the fact that the bulk theory has diffeomorphism invariance.

The rest of this paper is organized as follows. In Sec. 2 we fix the setup and collect the standard technique of a conformal map to construct a CFT on a conformally flat manifold. In Sec. 3 we determine the primary flow equation. In Sec. 4 we compute the induced metric for this flow in a CFT on the conformally flat background, which turns out to describe the AdS space whose boundary is the conformally flat manifold. In Sec. 5 we demonstrate that the emergence of the AdS space is assured by the conformal symmetry as in the case of the flat boundary shown in [36]. Sec. 6 is devoted to summary and discussion. In Appendix A we prove that the induced metric obtained in Sec. 4 describes an AdS space by explicit computation. In Appendix B we show that a geometry with one warped direction becomes an Einstein manifold if and only if its boundary at the warped direction is also an Einstein one.

2. Conformal map of conformal field theory

In this section we fix our setup in this paper and collect the standard technique to study a CFT on a conformally flat manifold by using a conformal map.

Let us consider a real d -dimensional conformally flat manifold M_d . From the definition, there exists a conformal map from a local patch \mathbb{R}^d to a neighborhood around each point in M_d such that¹

$$\Omega_x = \Omega(x), \quad \Omega_x \in M_d, \quad x \in \mathbb{R}^d. \tag{2.1}$$

The distance in a local patch is measured by $ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu$ with $\mu, \nu = 1, \dots, d$, while that in the space M_d is

$$(ds^2)_{M_d} = g_{\mu\nu}(\Omega_x) d\Omega_x^\mu d\Omega_x^\nu = g^{\frac{1}{d}}(x) \delta_{\mu\nu} dx^\mu dx^\nu, \tag{2.2}$$

where $g^{\frac{1}{d}}(x)$ is a conformal factor associated with the curved manifold.

Take the d -dimensional sphere $M_d = \mathbb{S}^d$ with the radius L as an example. In this case, a conformal map which covers the neighborhood around the north pole is given by a stereographic projection from the north pole $(L, 0)$ to the d -dimensional plane $(0, \mathbb{R}^d)$:

$$\Omega_x^0 = \frac{x^2 - 4L^2}{x^2 + 4L^2} L, \quad \Omega_x^\mu = \frac{4L^2}{x^2 + 4L^2} x^\mu, \tag{2.3}$$

where $x^2 = \delta_{\mu\nu} x^\mu x^\nu$ and the sphere is embedded into \mathbb{R}^{d+1} . The conformal factor is computed from the embedding metric as

$$g^{\frac{1}{d}}(x) = \left(\frac{4L^2}{x^2 + 4L^2} \right)^2. \tag{2.4}$$

We are interested in a CFT on M_d which contains a primary operator $O(\Omega_x)$ with a general conformal dimension Δ . To construct such a CFT on M_d , we prepare a CFT on a local patch \mathbb{R}^d containing a primary operator $O(x)$ with the conformal dimension Δ , whose conformal transformation is given by

$$\begin{aligned} \delta^{\text{conf}} O(x) &= (-\delta x^\mu \partial_{x^\mu} - \frac{\Delta}{d} \frac{\partial \delta x^\mu}{\partial x^\mu}) O(x), \\ \delta x^\mu &= a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b_\nu x^\nu), \end{aligned} \tag{2.5}$$

where $a^\mu, \omega^{\mu\nu}, \lambda$ and b^μ are infinitesimal parameters of the translation, the rotation, the dilatation and the special conformal transformation, respectively. The scalar primary operator inserted at Ω_x is related to the one at x by the pull-back of the conformal map:

$$\Omega * O(\Omega_x) := U_\Omega^{-1} O(\Omega_x) U_\Omega = g^{-\frac{\Delta}{2d}}(x) O(x), \tag{2.6}$$

where U_Ω is a unitary transformation which maps a state on \mathbb{R}^d to the corresponding one on M_d .

Since the vacuum states on two spaces are related as $|0\rangle_{M_d} = U_\Omega |0\rangle_{\mathbb{R}^d}$, correlation functions on M_d are related to those on \mathbb{R}^d as

¹ For simplicity, we consider a Riemannian manifold with the Euclidean signature in this paper. A generalization to a different signature is straightforward by changing signs suitably. Thus, strictly speaking, the word ‘‘AdS’’ in the main text should read ‘‘Euclidean AdS’’.

$$\begin{aligned} & \langle O(\Omega_{x_1})O(\Omega_{x_2})\cdots O(\Omega_{x_n}) \rangle_{M_d} \\ &= g^{-\frac{\Delta}{2d}}(x_1)g^{-\frac{\Delta}{2d}}(x_2)\cdots g^{-\frac{\Delta}{2d}}(x_n)\langle O(x_1)O(x_2)\cdots O(x_n) \rangle_{\mathbb{R}^d} \end{aligned}$$

where $\langle O \rangle_{X_d} = X_d \langle 0|O|0 \rangle_{X_d}$ with $X_d = M_d, \mathbb{R}^d$. For example, the two point function of the scalar primary operator on M_d is evaluated as

$$\langle O(\Omega_x)O(\Omega_y) \rangle_{M_d} = g^{-\frac{\Delta}{2d}}(x)g^{-\frac{\Delta}{2d}}(y)\frac{C}{(x-y)^{2\Delta}} = \frac{C}{|\Omega_x - \Omega_y|^{2\Delta}}, \tag{2.7}$$

where $|\Omega_x - \Omega_y|^2 := g^{\frac{1}{2d}}(x)g^{\frac{1}{2d}}(y)(x-y)^2$ and the two point function on \mathbb{R}^d is normalized as

$$\langle O(x)O(y) \rangle_{\mathbb{R}^d} = \frac{C}{(x-y)^{2\Delta}}, \quad C = \frac{\Gamma(\Delta)}{4^{d/2-\Delta}\pi^{d/2}\Gamma(d/2-\Delta)}. \tag{2.8}$$

A conformal transformation of $O(\Omega_x)$ is computed as

$$\delta^{\text{conf}}O(\Omega_x) = (-\delta\Omega_x^\mu\partial_{\Omega_x^\mu} - \frac{\Delta}{d}\frac{\partial\delta x^\mu}{\partial x^\mu} - \delta x^\mu\partial_\mu \log g^{\frac{\Delta}{2d}})O(\Omega_x) \tag{2.9}$$

where $\delta\Omega_x^\mu = \frac{\partial\Omega_x^\mu}{\partial x^\nu}\delta x^\nu$. Since correlation functions on \mathbb{R}^d are invariant under any conformal transformation, so are those on M_d :

$$\langle \delta^{\text{conf}}\{O(\Omega_{x_1})O(\Omega_{x_2})\cdots O(\Omega_{x_n})\} \rangle_{M_d} = 0. \tag{2.10}$$

3. Primary flow equation

In this section we construct a certain free flow equation for a primary scalar operator on a conformally flat background. For this purpose we begin with the one on a local patch \mathbb{R}^d [36].

$$\frac{\partial O(x; t)}{\partial t} = \partial^2 O(x; t), \quad O(x; 0) = O(x), \tag{3.1}$$

where $\partial^2 = \delta^{\mu\nu}\partial_\mu\partial_\nu$. A question is how we extend this equation to the one on a conformally flat manifold M_d respecting the structure of conformal property in the previous section.

To answer this, we request a flow equation of a scalar primary operator on M_d to satisfy the following properties.

- (i) There exists a flow time \tilde{t} associated with M_d corresponding to the flow time t on \mathbb{R}^d such that the flowed operator inserted at Ω_x is related to the flowed one at x by the pullback as (2.6):

$$\Omega * O(\Omega_x; \tilde{t}) = U_\Omega^{-1}O(\Omega_x; \tilde{t})U_\Omega = g^{-\frac{\Delta}{2d}}(x)O(x; t). \tag{3.2}$$

- (ii) The flow equation is invariant under the scale transformation.

We refer to a flow equation satisfying these conditions as the primary flow equation, which is determined as follows. The condition (i) fixes a differential equation consistent with (3.1) as

$$\frac{\partial}{\partial \tilde{t}}O(\Omega_x; \tilde{t}) = g^{-\frac{\Delta}{2d}}(x)\partial^2 g^{\frac{\Delta}{2d}}(x)O(\Omega_x; \tilde{t}), \quad O(\Omega_x; 0) = O(\Omega_x). \tag{3.3}$$

Then \tilde{t} is determined by the condition (ii), which is met if the scaling dimension of \tilde{t} becomes two. Since \tilde{t} is associated with the manifold M_d , namely \tilde{t} is related to t through the conformal factor, \tilde{t} is fixed as

$$\tilde{t} = g^{\frac{1}{d}}(x)t \tag{3.4}$$

up to an overall constant. Introducing a copy of the local patch with coordinates \tilde{x}^μ which is independent of \tilde{t} , we obtain the primary flow equation as

$$\frac{\partial}{\partial \tilde{t}} O(\Omega_{\tilde{x}}; \tilde{t}) = D \cdot O(\Omega_{\tilde{x}}; \tilde{t}), \quad O(\Omega_{\tilde{x}}; 0) = O(\Omega_{\tilde{x}}), \tag{3.5}$$

with $D = g^{-\frac{1}{d}}(\tilde{x})g^{-\frac{\Delta}{2d}}(\tilde{x})\tilde{\partial}^2 \cdot g^{\frac{\Delta}{2d}}(\tilde{x})$. Remark that (x^μ, t) and $(\tilde{x}^\mu, \tilde{t})$ are two sets of independent variables, which are related by $x^\mu = \tilde{x}^\mu$ and (3.4). In what follows, however, we often abuse \tilde{x}^μ and x^μ just to avoid notational clutter.

It may be instructive to mention that, in the case that $O(x)$ has the canonical dimension as $\Delta = (d - 2)/2$, the operator D has a conformally covariant expression such that

$$D = g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{d-2}{4(d-1)}R^{M_d}, \tag{3.6}$$

where $g^{\mu\nu}$, ∇_μ and R^{M_d} are the metric, the covariant derivative and the scalar curvature on M_d , respectively.

Since the two point function of the flowed primary field on a local chart is known as [36]

$$\langle O(x; t)O(y; s) \rangle_{\mathbb{R}^d} = \frac{1}{(t+s)^\Delta} F\left(\frac{(x-y)^2}{t+s}\right), \tag{3.7}$$

where F is a smooth function depending on each CFT, that on M_d is determined by using eq. (3.2) as

$$\langle O(\Omega_x; \tilde{t})O(\Omega_y; \tilde{s}) \rangle_{M_d} = \frac{1}{\tilde{\eta}_+^\Delta} F\left(\frac{|\Omega_x - \Omega_y|^2}{\tilde{\eta}_+}\right) \tag{3.8}$$

where

$$\tilde{\eta}_+ := g^{\frac{1}{2d}}(x)g^{\frac{1}{2d}}(y)(t+s) = \frac{g^{\frac{1}{2d}}(y)}{g^{\frac{1}{2d}}(x)}\tilde{t} + \frac{g^{\frac{1}{2d}}(x)}{g^{\frac{1}{2d}}(y)}\tilde{s}. \tag{3.9}$$

Then the two point function of the normalized flow field

$$\sigma(\Omega_x; \tilde{t}) := \frac{1}{\sqrt{\langle O(\Omega_x; \tilde{t})O(\Omega_x; \tilde{t}) \rangle_{M_d}}} O(\Omega_x; \tilde{t}) = \sqrt{\frac{(2\tilde{t})^\Delta}{F(0)}} O(\Omega_x; \tilde{t}) \tag{3.10}$$

is given by

$$\langle \sigma(\Omega_x; \tilde{t})\sigma(\Omega_y; \tilde{s}) \rangle_{M_d} = \left(\frac{2\sqrt{\tilde{t}\tilde{s}}}{\tilde{\eta}_+}\right)^\Delta \bar{F}\left(\frac{|\Omega_x - \Omega_y|^2}{\tilde{\eta}_+}\right), \tag{3.11}$$

where $\bar{F}(x) = F(x)/F(0)$.

Note that the normalized flow field on M_d satisfies eq. (3.2) with the vanishing conformal dimension:

$$U_\Omega^{-1} \sigma(\Omega_x; \tilde{t}) U_\Omega = \Omega * \sigma(\Omega_x; \tilde{t}) = \sigma(x; t). \tag{3.12}$$

4. Induced metric of CFT on a conformally flat manifold

4.1. Induced metric

The induced metric is defined by

$$\tilde{g}_{MN}(z) := R^2 \left\langle \frac{\partial \sigma(\Omega_x; \tilde{t})}{\partial z^M} \frac{\partial \sigma(\Omega_x; \tilde{t})}{\partial z^N} \right\rangle_{M_d}, \tag{4.1}$$

where $z^M = (x^\mu, \sqrt{2d\tilde{t}})$, $w^M = (y^\mu, \sqrt{2d\tilde{s}})$ and R is an arbitrary length parameter. Then the induced line element is given by

$$ds^2 = \tilde{G}_{MN}(Z) dZ^M dZ^N = \tilde{g}_{MN}(z) dz^M dz^N, \tag{4.2}$$

where $Z^M = (\Omega_x^\mu, \tilde{\tau})$ with $\tilde{\tau} = \sqrt{2d\tilde{t}}$. Explicitly the induced metric is computed as²

$$\begin{aligned} \tilde{g}_{\tilde{\tau}\tilde{\tau}}(z) &= R^2 \frac{\Delta}{\tilde{\tau}^2}, \\ \tilde{g}_{\tilde{\tau}\mu}(z) = g_{\mu\tilde{\tau}}(z) &= -R^2 \frac{\Delta}{\tilde{\tau}} \frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\}, \\ \tilde{g}_{\mu\nu}(z) &= R^2 \Delta \left[\frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\} \frac{\partial}{\partial x^\nu} \log\{g^{\frac{1}{2d}}(x)\} + \frac{\delta_{\mu\nu} g^{\frac{1}{d}}(x)}{\tilde{\tau}^2} \right], \end{aligned} \tag{4.4}$$

where we use $2d\tilde{F}'(0) = -\Delta$. Remark that there appear nontrivial off-diagonal elements.

An explicit calculation in Appendix A leads to

$$G_{MN}(z) = -\Lambda \tilde{g}_{MN}(z), \quad \Lambda = -\frac{d(d-1)}{2R^2\Delta}, \tag{4.5}$$

where $G_{MN}(z)$ is the Einstein tensor. As a result the induced metric turns out to be the $d + 1$ dimensional (Euclidean) AdS space ($\Lambda < 0$) at $d > 1$, whose radius is given by $R_{\text{AdS}} = R\sqrt{\Delta}$. In addition, since

$$\lim_{\tilde{\tau} \rightarrow 0} ds^2 = \frac{R^2\Delta}{\tilde{\tau}^2} \left(d\tilde{\tau}^2 + g^{\frac{1}{d}}(x) \delta_{\mu\nu} dx^\mu dx^\nu \right) + O(\tilde{\tau}^{-1}), \tag{4.6}$$

the metric $\tilde{g}_{MN}(z)$ indeed describes the (local) AdS space in $d + 1$ dimensions with the d dimensional curved space M_d as its boundary.³

² In the previous example of the d dimensional sphere, $M_d = \mathbb{S}^d$, these are computed by using (2.4) as

$$\begin{aligned} \tilde{g}_{\tilde{\tau}\mu}(z) = g_{\mu\tilde{\tau}}(z) &= R^2 \frac{\Delta}{\tilde{\tau}} \frac{2x_\mu}{x^2 + 4L^2}, \\ \tilde{g}_{\mu\nu}(z) &= R^2 \Delta \left(\frac{4L^2}{r^2 + 4L^2} \right)^2 \left[\frac{4x_\mu x_\nu}{(4L^2)^2} + \frac{\delta_{\mu\nu}}{\tilde{\tau}^2} \right]. \end{aligned} \tag{4.3}$$

³ The total AdS space whose boundary is M_d is obtained by gluing a set of local AdS spaces whose boundaries are open covering of M_d , as is usual with the standard construction of a manifold.

4.2. Diffeomorphism and AdS metrics

In this subsection, we show that the metric $\tilde{g}_{MN}(z)$ is obtained from the usual Poincare AdS metric by a finite diffeomorphism. The AdS metric in the Poincare patch is given by

$$ds_{\text{PAdS}}^2 = \frac{R^2 \Delta}{\tau^2} \left(d\tau^2 + \delta_{\mu\nu} dx^\mu dx^\nu \right), \tag{4.7}$$

which was obtained as the result of the induced metric of a CFT on the flat space \mathbb{R}^d [36]. Under the $d + 1$ dimensional finite diffeomorphism that $(x^\mu, \tau) \rightarrow (\tilde{x}^\mu, \tilde{\tau})$ with $\tilde{x}^\mu = x^\mu$ and $\tilde{\tau} = \tau g^{\frac{1}{2d}}(x)$, we have

$$d\tau = g^{\frac{-1}{2d}}(x) \left[d\tilde{\tau} - \tilde{\tau} \frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\} dx^\mu \right], \tag{4.8}$$

which leads to

$$ds_{\text{PAdS}}^2 \rightarrow \frac{R^2 \Delta}{\tilde{\tau}^2} \left[d\tilde{\tau}^2 - 2\tilde{\tau} \frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\} dx^\mu d\tilde{\tau} + \left(\tilde{\tau}^2 \frac{\partial}{\partial x^\mu} \log\{g^{\frac{1}{2d}}(x)\} \frac{\partial}{\partial x^\nu} \log\{g^{\frac{1}{2d}}(x)\} + g^{\frac{1}{d}}(x) \delta_{\mu\nu} \right) dx^\mu dx^\nu \right]. \tag{4.9}$$

This gives the metric identical to the one in eqs. (4.4). Therefore the induced metric in the previous subsection must describe the AdS space, since it connects to the Poincare AdS by a finite diffeomorphism, although the resulting metric has a different boundary, M_d . This is a consequence of the fact that the choice of the AdS solution in the Poincare patch breaks the diffeomorphism invariance at boundary.

5. Bulk symmetry from boundary symmetry

In this section we prove that the emergence of the AdS space is assured only by the conformal symmetry at boundary without explicit calculation. This was shown in the case of the flat boundary in [36]. Here we argue that this is the case also for a conformally flat boundary.

Following [36] we decompose the infinitesimal conformal transformation for the normalized field σ on a local patch derived from (2.5) as

$$\delta^{\text{conf}} \sigma(x; t) = \delta^{\text{diff}} \sigma(x; t) + \delta^{\text{extra}} \sigma(x; t), \tag{5.1}$$

where

$$\begin{aligned} \delta^{\text{diff}} \sigma(x; t) &= -(\bar{\delta}t \partial_t + \bar{\delta}x^\mu \partial_\mu) \sigma(x; t), \\ \delta^{\text{extra}} \sigma(x; t) &= 4R^2 t^2 b^\mu \partial_\mu \left(\partial_t + \frac{\Delta + 2}{2t} \right) \sigma(x; t) \end{aligned} \tag{5.2}$$

with $\bar{\delta}x^\mu = \delta x^\mu + 2dR^2 t b^\mu$, $\bar{\delta}t = (2\lambda - 4(b_\mu x^\mu))t$. Then, using eq. (3.12), we derive the conformal transformation to the normalized flow field on M_d as

$$\delta^{\text{conf}} \sigma(\Omega_x; \tilde{t}) = \delta^{\text{diff}} \sigma(\Omega_x; \tilde{t}) + \delta^{\text{extra}} \sigma(\Omega_x; \tilde{t}), \tag{5.3}$$

where

$$\delta^{\text{diff}}\sigma(\Omega_x; \tilde{t}) = (-\bar{\delta}\tilde{t}\partial_{\tilde{t}} - \bar{\delta}\Omega_x^\mu\partial_\mu)\sigma(\Omega_x; \tilde{t}), \tag{5.4}$$

$$\delta^{\text{extra}}\sigma(\Omega_x; \tilde{t}) = 4g^{-\frac{1}{d}}(x)\tilde{t}^2b^\mu(\partial_\mu\log\{g^{\frac{1}{d}}(x)\} + \partial_\mu)\left(\partial_{\tilde{t}} + \frac{\Delta + 2}{2\tilde{t}}\right)\sigma(\Omega_x; \tilde{t}), \tag{5.5}$$

with $\bar{\delta}\tilde{t} = \bar{\delta}t g^{\frac{1}{d}}(x) + \bar{\delta}x^\nu \tilde{t} \partial_\nu \log\{g^{\frac{1}{d}}(x)\}$, $\bar{\delta}\Omega_x^\mu = \bar{\delta}x^\nu \frac{\partial\Omega_x^\mu}{\partial x^\nu}$.

Let us show that the induced metric (4.1) is invariant under the transformation δ^{diff} :

$$\begin{aligned} \delta^{\text{diff}}\tilde{g}_{MN}(z) &= \lim_{w \rightarrow z} \frac{\partial}{\partial z^M} \frac{\partial}{\partial w^N} (\delta^{\text{conf}} - \delta^{\text{extra}})\langle\sigma(\Omega_x; \tilde{t})\sigma(\Omega_y; \tilde{s})\rangle_{M_d} \\ &= -\lim_{w \rightarrow z} \frac{\partial}{\partial z^M} \frac{\partial}{\partial w^N} \delta^{\text{extra}}\langle\sigma(\Omega_x; \tilde{t})\sigma(\Omega_y; \tilde{s})\rangle_{M_d}, \end{aligned} \tag{5.6}$$

where we used the conformal symmetry of correlation functions (2.10):

$$\delta^{\text{conf}}\langle\sigma(\Omega_x; \tilde{t})\sigma(\Omega_y; \tilde{s})\rangle_{M_d} = 0.$$

Since

$$\begin{aligned} \delta^{\text{extra}}\langle\sigma(\Omega_x; \tilde{t})\sigma(\Omega_y; \tilde{s})\rangle_{M_d} \\ = -8 \frac{\sqrt{4\tilde{t}\tilde{s}}^\Delta}{(\tilde{\eta}_+)^{\Delta}} \left(\frac{\tilde{t}}{g^{\frac{1}{d}}(x)} - \frac{\tilde{s}}{g^{\frac{1}{d}}(y)} \right) b_\mu(x-y)^\mu (x-y)^2 F''\left(\frac{|\Omega_x - \Omega_y|^2}{\tilde{\eta}_+}\right), \end{aligned}$$

which vanishes in the $w \rightarrow z$ limit, so does eq. (5.6). Explicitly

$$\delta^{\text{diff}}\tilde{g}_{MN}(z) = -\bar{\delta}z^K\partial_K\tilde{g}_{MN}(z) - \partial_M\bar{\delta}z^K\tilde{g}_{KN}(z) - \partial_N\bar{\delta}z^K\tilde{g}_{MK}(z) = 0, \tag{5.7}$$

with $\bar{\delta}x^\mu = \delta x^\mu + g^{-\frac{1}{d}}(x)\tilde{t}^2b^\mu$, $\bar{\delta}\tilde{t} = \tilde{t}\left[\lambda - 2(b \cdot x) + \bar{\delta}x^\mu\partial_\mu\log\{g^{\frac{1}{d}}(x)\}\right]$. This means that the induced metric is invariant under the infinitesimal AdS isometry transformations expressed in the coordinates (x^μ, \tilde{t}) . As a result, the induced metric has to be the AdS one since it is a maximally symmetric space. This completes the proof of our claim.

6. Discussion

In this paper, we have extended our previous investigation [36] on the proposal [21–23] that the bulk geometry is constructed from a boundary CFT on the flat background to the case with an arbitrary boundary CFT on curved spaces within a conformally flat class by using a canonical flow equation called the primary one. We have shown that the resulting induced metric becomes AdS whose boundary is the curved manifold. We have found that such AdS metrics can be constructed from the usual Poincare AdS metric by a finite diffeomorphism such that the AdS radial coordinate is scaled by the conformal factor with the other directions fixed. We have also shown that the conformal symmetry at the boundary generates the AdS isometry for the vacuum expectation value of the metric operator, so that the bulk geometry must be AdS with a given boundary, as in the case of the flat boundary.

In Appendix B, we showed that a geometry with one warped direction becomes an Einstein manifold if and only if so is its boundary. This geometry would be theoretically interesting because the usual AdS/CFT correspondence can be extended to a more general gravity/gauge one by using this geometry.⁴ An issue in this generalization is that it may be nontrivial whether a

⁴ We would like to thank Prof. Kinya Oda for his valuable comment on this possibility.

CFT can be realized on a genuine Einstein manifold or not. If a genuine Einstein manifold does not admit for any CFT to live, one needs to know what kind of quantum field theory is realized on the boundary Einstein manifold and corresponds to gravity theory on a bulk Einstein one. Further investigation of “Einstein/QFT” correspondence is awaited.

In the previous paper [36], we had shown that the induced metric associated with the free flow equation from the CFT on the flat boundary describes the usual Poincare AdS space as (4.7). Let us recast this in the following way. Since the flowed field σ is dimensionless thank to the (NLSM) normalization and free from the UV divergence at non-zero flow time, the Poincare symmetry as well as an absence of dimensionful constants in a CFT on the flat boundary demand the metric to be of a form such that $ds_{d+1}^2 = R^2\{A(x^2/\tau^2)d\tau^2 + B(x^2/\tau^2)\delta_{\mu\nu}dx^\mu dx^\nu\}/\tau^2$. We had shown that the symmetry argument constrains the functions A and B to be an equal constant while an explicit calculation determines its value as Δ , so that the induced metric results in the Poincare AdS space. This kind of analysis, however, may not be available if the boundary space is curved. First of all, it is not guaranteed that there exists the AdS metric with a curved boundary in the Poincare patch. Secondly, extra dimensionful parameters such as the radius of the sphere existing in a curved space make the ansatz of the metric much more complicated. Regardless of these difficulties, the method proposed in Refs. [21–23] still works well for a CFT on a conformally flat boundary such as a d dimensional sphere. The induced metric for such a CFT was explicitly given by (4.9), which was shown to describe the AdS space with the boundary metric $g^{\frac{1}{d}}(x)\delta_{\mu\nu}$. To the best of our knowledge, construction of an AdS metric with a general conformally flat boundary was not known, or, at least our result (4.9) gives a new expression for such a non-trivial AdS space with the manifest Poincare symmetry.⁵

Although it is interesting to generalize our analysis to the case with a more general curved boundary including de-Sitter space or a gravity solution corresponding to a RG flow on a curved manifold, one of the most important and urgent issues in the proposal [21–23] is to clarify how this formalism encodes the bulk dynamics beyond the geometry. For this purpose, the equation of motion for (the fluctuation of) the metric must be determined by calculating, for instance, the 2-point correlation function of the metric operator $g_{MN}(z)$. Works along this direction are ongoing.

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⁵ A bulk diffeomorphism corresponding to a Weyl transformation at boundary for a general boundary metric is known as the PBH transformation [37,38], which is constructed order by order so as to preserve the Fefferman–Graham gauge. Thus the PBH transformation is different from the one given in this paper, which encodes a local conformal map on a conformally flat manifold at boundary. We leave the problem to clarify the relation between these two to a future work. We would like to thank Prof. Sumit Das for his valuable comment on this after the submission of the first version.

Appendix A. Explicit calculation of the Einstein tensor

In this appendix we explicitly compute the Einstein tensor for the induced metric (4.4) to result in the (Euclidean) AdS space. The metric and its inverse are given by

$$\begin{aligned} \tilde{g}_{\tilde{\tau}\tilde{\tau}}(z) &= \frac{R_{\text{AdS}}^2}{\tilde{\tau}^2}, & \tilde{g}_{\tilde{\tau}\mu}(z) &= \tilde{g}_{\mu\tilde{\tau}}(z) = -\frac{R_{\text{AdS}}^2}{\tilde{\tau}} F_\mu(x), \\ \tilde{g}_{\mu\nu}(z) &= R_{\text{AdS}}^2 \left[F_\mu(x) F_\nu(x) + \frac{\delta_{\mu\nu} \Omega^2(x)}{\tilde{\tau}^2} \right], & \tilde{g}^{\tilde{\tau}\tilde{\tau}}(z) &= \frac{\tilde{\tau}^2}{R_{\text{AdS}}^2} \left[1 + \tau^2 F^2(x) \right], \\ \tilde{g}^{\tilde{\tau}\mu}(z) &= \tilde{g}^{\mu\tilde{\tau}}(z) = \frac{\tilde{\tau} \tau^2}{R_{\text{AdS}}^2} F^\mu, & \tilde{g}^{\mu\nu}(z) &= \frac{\tau^2}{R_{\text{AdS}}^2} \delta^{\mu\nu}, \end{aligned} \tag{A.1}$$

where $R_{\text{AdS}}^2 = R^2 \Delta$, $\Omega(x) = g^{\frac{1}{2d}}(x)$,

$$F_\mu(x) = \frac{\partial}{\partial x^\mu} \log \Omega(x), \quad F^\mu(x) = \delta^{\mu\nu} F_\nu(x), \quad F^2(x) = F^\mu(x) F_\mu(x). \tag{A.2}$$

We have

$$\begin{aligned} \partial_{\tilde{\tau}} \tilde{g}_{\tilde{\tau}\tilde{\tau}} &= -2 \frac{R_{\text{AdS}}^2}{\tilde{\tau}^3}, & \partial_\mu \tilde{g}_{\tilde{\tau}\tilde{\tau}} &= 0, & \partial_{\tilde{\tau}} \tilde{g}_{\tilde{\tau}\mu} &= \frac{R_{\text{AdS}}^2}{\tilde{\tau}^2} F_\mu, & \partial_\nu \tilde{g}_{\tilde{\tau}\mu} &= -\frac{R_{\text{AdS}}^2}{\tilde{\tau}} \partial_\nu F_\mu, \\ \partial_{\tilde{\tau}} \tilde{g}_{\mu\nu} &= -2 \frac{R_{\text{AdS}}^2 \Omega^2}{\tilde{\tau}^3} \delta_{\mu\nu}, & \partial_\alpha \tilde{g}_{\mu\nu} &= R_{\text{AdS}}^2 \left[\partial_\alpha (F_\mu F_\nu) + 2 \frac{F_\alpha}{\tau^2} \delta_{\mu\nu} \right]. \end{aligned} \tag{A.3}$$

Christoffel symbol,

$$\Gamma_{LM}^K = \frac{1}{2} \tilde{g}^{KN} \left[\partial_M \tilde{g}_{NL} + \partial_L \tilde{g}_{NM} - \partial_N \tilde{g}_{LM} \right], \tag{A.4}$$

is evaluated as

$$\begin{aligned} \Gamma_{\tilde{\tau}\tilde{\tau}}^{\tilde{\tau}} &= -\frac{1}{\tilde{\tau}}, & \Gamma_{\tilde{\tau}\tilde{\tau}}^\mu &= 0, & \Gamma_{\tilde{\tau}\mu}^{\tilde{\tau}} &= -F_\mu, & \Gamma_{\mu\nu}^{\tilde{\tau}} &= \frac{\Omega^2}{\tilde{\tau}} \delta_{\mu\nu} + 3\tilde{\tau} F_\mu F_\nu - \tilde{\tau} F_{\mu\nu}, \\ \Gamma_{\tilde{\tau}\nu}^\mu &= -\frac{1}{\tilde{\tau}} \delta_\nu^\mu, & \Gamma_{\mu\nu}^\alpha &= F_\nu \delta_\mu^\alpha + F_\mu \delta_\nu^\alpha, \end{aligned} \tag{A.5}$$

where $F_{\mu\nu} = F_{\nu\mu} := \frac{\partial_\mu \partial_\nu \Omega}{\Omega}$. Then the Riemann tensor is defined as

$$R_{LKM}^N := \partial_K \Gamma_{LM}^N - \partial_M \Gamma_{LK}^N + \Gamma_{JK}^N \Gamma_{LM}^J - \Gamma_{JM}^N \Gamma_{LK}^J = -R_{LKM}^N, \tag{A.6}$$

whose non-zero components are given by

$$\begin{aligned} R_{\tilde{\tau}\tilde{\tau}\mu}^{\tilde{\tau}} &= \frac{F_\mu}{\tilde{\tau}}, & R_{\mu\tilde{\tau}\nu}^{\tilde{\tau}} &= -\left[\frac{\Omega}{\tilde{\tau}^2} + F_\mu F_\nu \right], & R_{\tilde{\tau}\tilde{\tau}\nu}^\mu &= \frac{\delta_{\mu\nu}}{\tilde{\tau}^2}, & R_{\nu\tilde{\tau}\alpha}^\mu &= -\frac{F_\nu \delta_\alpha^\mu}{\tilde{\tau}}, \\ R_{\tilde{\tau}\nu\alpha}^\mu &= \frac{F_\alpha \delta_\nu^\mu - F_\nu \delta_\alpha^\mu}{\tilde{\tau}}, & R_{\nu\alpha\beta}^\mu &= \frac{\Omega}{\tilde{\tau}^2} \left(\delta_\beta^\mu \delta_{\nu\alpha} - \delta_\alpha^\mu \delta_{\nu\beta} \right) + \delta_\beta^\mu F_\nu F_\alpha - \delta_\alpha^\mu F_\nu F_\beta. \end{aligned} \tag{A.7}$$

Thus the Riemann tensor can be written as

$$R_{LKM}^N = -\frac{1}{R_{\text{AdS}}^2} \left(\delta_K^N \tilde{g}_{LM} - \delta_M^N \tilde{g}_{LK} \right), \tag{A.8}$$

which shows that the \tilde{g}_{MN} indeed describes the AdS space. Therefore Ricci and scalar curvatures are simply given as

$$R_{LM} := R_{LNM}^N = -\frac{d}{R_{\text{AdS}}^2} \tilde{g}_{LM}, \quad R := \tilde{g}^{LM} R_{LM} = -\frac{d(d+1)}{R_{\text{AdS}}^2}. \quad (\text{A.9})$$

We finally obtain the Einstein tensor as

$$G_{MN}(z) := R_{LM} - \frac{1}{2} \tilde{g}_{LM} R = -\Lambda \tilde{g}_{MN}(z), \quad \Lambda = -\frac{d(d-1)}{2R_{\text{AdS}}^2}, \quad (\text{A.10})$$

where Λ is the cosmological constant.

Appendix B. Warped Einstein geometry

In this appendix we present another Einstein geometry whose boundary is not necessarily conformally flat. For this purpose we consider a generic warped geometry with one warped dimension, whose metric is written as

$$ds'^2 = f_1(\tau) d\tau^2 + f_2(\tau) ds_{\mathcal{M}}^2, \quad (\text{B.1})$$

where \mathcal{M} is a general d dimensional manifold whose metric $ds_{\mathcal{M}}^2 = g_{\mu\nu} dx^\mu dx^\nu$ independent of τ . We claim that this warped geometry becomes an Einstein manifold by a specific choice of $f_{1,2}(\tau)$ if and only if the base manifold \mathcal{M} is also an Einstein one. The proof of this statement is given as follows.

The Einstein equation with a cosmological term is

$$G'_{MN} + g'_{MN} \Lambda = 0, \quad (\text{B.2})$$

where $G'_{MN} = R'_{MN} - \frac{1}{2} g'_{MN} R'$ and

$$g'_{\tau\tau} = f_1, \quad g'_{\mu\nu} = f_2 g_{\mu\nu}, \quad g'_{\mu\tau} = 0. \quad (\text{B.3})$$

We emphasize that the cosmological term Λ here is not necessarily constant and could be a function of coordinates.

We first perform the Weyl transformation so that

$$ds^2 = e^{-2\phi} ds'^2 = g_{\tau\tau} d\tau^2 + ds_{\mathcal{M}}^2 = g_{MN} dx^M dx^N, \quad (\text{B.4})$$

where $\phi = \frac{1}{2} \log f_2(\tau)$, $g_{\tau\tau} = f_1(\tau)/f_2(\tau)$. The Ricci tensor becomes

$$\begin{aligned} R'_{NL} &= R_{NL} - g_{NL} \partial^2 \phi + (d-1)(-\nabla_N \partial_L \phi + \partial_N \phi \partial_L \phi - g_{NL} \partial_M \phi \partial^M \phi), \\ R' &= e^{-2\phi} (R - (2d-1) \partial^2 \phi - d(d-1) \partial_M \phi \partial^M \phi). \end{aligned} \quad (\text{B.5})$$

Since the function $g_{\tau\tau}$ can be absorbed in the τ coordinate by diffeomorphism, there is no non-trivial curvature in τ direction. Thus

$$R_{\mu\nu} = R_{\mu\nu}^{\mathcal{M}}, \quad R = R^{\mathcal{M}}, \quad R_{\mu\tau} = R_{\tau\tau} = 0. \quad (\text{B.6})$$

Therefore the Ricci tensor is computed as

$$\begin{aligned}
R'_{\mu\nu} &= R_{\mu\nu} - g_{\mu\nu}\partial^2\phi + (d-1)(-g_{\mu\nu}g^{\tau\tau}\partial_\tau\phi\partial_\tau\phi), \\
R'_{\tau\nu} &= R_{\tau\nu}, \\
R'_{\tau\tau} &= R_{\tau\tau} - g_{\tau\tau}\partial^2\phi + (d-1)(-\nabla_\tau\partial_\tau\phi), \\
R' &= e^{-2\phi}(R - (2d-1)\partial^2\phi - d(d-1)g^{\tau\tau}\partial_\tau\phi\partial_\tau\phi).
\end{aligned}
\tag{B.7}$$

Thus the Einstein tensor is given as

$$\begin{aligned}
G'_{\mu\nu} &= G_{\mu\nu} + (d-1)g_{\mu\nu}\partial^2\phi + \frac{1}{2}(d-1)(d-2)g_{\mu\nu}g^{\tau\tau}\partial_\tau\phi\partial_\tau\phi, \\
G'_{\tau\nu} &= R_{\tau\nu}, \\
G'_{\tau\tau} &= G_{\tau\tau} + \frac{1}{2}d(d-1)\partial_\tau\phi\partial_\tau\phi.
\end{aligned}
\tag{B.8}$$

By using (B.6), the Einstein equation becomes

$$\begin{aligned}
G^{\mathcal{M}}_{\mu\nu} + (d-1)g_{\mu\nu}\partial^2\phi + \frac{1}{2}(d-1)(d-2)g_{\mu\nu}g^{\tau\tau}\partial_\tau\phi\partial_\tau\phi + f_2g_{\mu\nu}\Lambda &= 0, \\
R_{\tau\nu} &= 0, \\
-\frac{1}{2}g_{\tau\tau}R^{\mathcal{M}} + \frac{1}{2}d(d-1)\partial_\tau\phi\partial_\tau\phi + f_1\Lambda &= 0.
\end{aligned}
\tag{B.9}$$

By using the 3rd equation (B.9) the 1st equation becomes

$$\text{Ric}^{\mathcal{M}}_{\mu\nu} - \frac{1}{d}g_{\mu\nu}R^{\mathcal{M}} + (d-1)g_{\mu\nu}\nabla^2\phi + \frac{2}{d}f_2g_{\mu\nu}\Lambda = 0.
\tag{B.10}$$

Contracting with $g^{\mu\nu}$ leads to

$$\Lambda = -e^{-2\phi}\frac{d(d-1)}{2}\nabla^2\phi,
\tag{B.11}$$

which implies that Λ is dependent only on the radial coordinate τ . Then (B.9) becomes

$$R^{\mathcal{M}} = d(d-1)\left(\frac{f_2}{f_1}\partial_\tau\phi\partial_\tau\phi - \nabla^2\phi\right),
\tag{B.12}$$

which suggests that the scalar curvature $R^{\mathcal{M}}$ has to be constant since the right-hand side depends only on τ . Plugging this back gives

$$\text{Ric}^{\mathcal{M}}_{\mu\nu} - \frac{1}{d}g_{\mu\nu}R^{\mathcal{M}} = 0,
\tag{B.13}$$

which, together with the constant $R^{\mathcal{M}}$, means that \mathcal{M} is an Einstein manifold. This completes the proof of our claim.

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