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The moduli space of polynomial maps and their fixed-point multipliers ☆

Toshi Sugiyama

Nada High School, Uozaki-kita-machi 8-5-1, Higashinada, Kobe 658-0082, Japan

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ABSTRACT

We consider the family MP_d of affine conjugacy classes of polynomial maps of one complex variable with degree $d \geq 2$, and study the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$ which maps each $f \in \mathrm{MP}_d$ to the set of fixed-point multipliers of f. We show that the local fiber structure of the map Φ_d around $\bar{\lambda} \in \widetilde{\Lambda}_d$ is completely determined by certain two sets $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ which are subsets of the power set of $\{1, 2, \ldots, d\}$. Moreover for any $\bar{\lambda} \in \widetilde{\Lambda}_d$, we give an algorithm for counting the number of elements of each fiber $\Phi_d^{-1}(\bar{\lambda})$ only by using $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$. It can be carried out in finitely many steps, and often by hand.

1. Introduction

Let MP_d be the family of affine conjugacy classes of polynomial maps of one complex variable with degree $d \geq 2$, and $\mathbb{C}^d/\mathfrak{S}_d$ the set of unordered collections of d complex

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E-mail address: sugiyama.toshi@gmail.com.

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numbers. Then the aim of this paper is to give a *complete description* of the fiber structure of the map

$$\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$$

which maps each $f \in \mathrm{MP}_d$ to the set of fixed-point multipliers of f, except where $f \in \mathrm{MP}_d$ has multiple fixed points.

Since multipliers of fixed points have played a central role in the study of the complex dynamics, it is natural to ask to what extent fixed-point multipliers of f determine the original map f. For polynomial maps, since the set of fixed-point multipliers is invariant under the action of affine transformations, the question is to count the number of affine conjugacy classes of polynomial maps when the set of its fixed-point multipliers are given. It is formulated in the following form: how many elements there are on each fiber of the above map $\Phi_d : \mathrm{MP}_d \to \mathbb{C}^d/\mathfrak{S}_d$. Here, since the set of fixed-point multipliers always satisfies a certain relation by the fixed point theorem (see Proposition 1.1), the image of Φ_d is contained in a certain hyperplane $\widetilde{\Lambda}_d$ in $\mathbb{C}^d/\mathfrak{S}_d$. Hence the main object of our study is the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$.

For $d = 2$, it is easily verified that Φ_2 is bijective. In the case $d = 3$, Milnor [11] showed that Φ_3 is also bijective, which was the starting point of his study of the complex dynamics of cubic polynomials. For $d \geq 4$, Fujimura and Nishizawa have long studied the map Φ_d in their series of papers such as [16], [3] and [4]. Especially their achievement is summarized in Fujimura's paper [4], which includes the following:

- Φ_d is not surjective for $d \geq 4$. Moreover for $d = 4$ or 5, she found all $\bar{\lambda} \in \widetilde{\Lambda}_d$ whose inverse image of Φ_d is empty.
- Generic fiber of Φ_d consists of $(d-2)!$ points. Moreover if $\Phi_d^{-1}(\bar{\lambda})$ is finite, then $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) \leq (d-2)!$ always holds.
- For $d = 4$, she found $\#\left(\Phi_4^{-1}(\bar{\lambda})\right)$ for all $\bar{\lambda} \in \widetilde{\Lambda}_4$.

Here, we denote the cardinality of a set X by $\#(X)$. Similar results for rational maps are given by Milnor in [13, p. 152, Problem 12-d] and [12].

Based on the results above, this paper provides an algorithm for counting the number of elements of each fiber $\Phi_d^{-1}(\bar{\lambda})$ for *all* $\bar{\lambda} = \{\lambda_1, \ldots, \lambda_d\} \in \widetilde{\Lambda}_d$ and for *all* $d \geq 4$ except when $\lambda_i = 1$ for some i. In practice, for each $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda_d \subset \mathbb{C}^d$ with $\lambda_i \neq 1$, certain two subsets $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ of the power set of $\{1, 2, \ldots, d\}$ are defined, and the number of elements of a fiber $\Phi_d^{-1}(\bar{\lambda})$ is completely determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$. Moreover we give an algorithm for counting the number $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ only by using $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ (see Main Theorems I, III, Definition 1.7 and Section 2). The algorithm can be carried out in finitely many steps, and only by hand. Moreover in Main Theorem II we show that the local fiber structure of Φ_d around $\bar{\lambda}$ is also determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$.

We shall provide some more concerning results.

Several kinds of compactifications of MP_d have been constructed independently by Silverman [17], by DeMarco and McMullen [2] and by Fujimura and Taniguchi [5]. Silverman's is based on the GIT compactification of the moduli space of rational maps, while the compactifications of DeMarco and MuMullen and of Fujimura and Taniguchi are both based on the consideration of the multipliers of polynomial maps. Especially, Fujimura and Taniguchi's compactification is strongly related to the definition of the set $\mathcal{I}(\lambda)$ in this paper (see Remarks 1.3 and 1.6).

Regarding the moduli space of rational maps, let us recall an important result of McMullen [10]. He investigated the map $\overline{\Psi}_d$ which maps each Möbius conjugacy class of rational maps of $\widehat{\mathbb{C}}$ of degree d to the set of the multipliers of its periodic points of all periods, and showed that the map $\overline{\Psi}_d$ is finite-to-one with few exceptions. To state the result explicitly, we denote by MR_d the family of Möbius conjugacy classes of rational maps of degree d on the Riemann sphere $\widehat{\mathbb{C}}$, and define the map $\Psi_d^{(n)} : \mathrm{MR}_d \to \mathbb{C}^{d^n+1}/\mathfrak{S}_{d^n+1}$ which maps each $f \in \mathrm{MR}_d$ to the set of multipliers of n-periodic points of f. Under the above notation, he considered the map

$$\overline{\Psi}_d^N := \prod_{n=1}^{N} \Psi_d^{(n)} : \mathrm{MR}_d \to \prod_{n=1}^{N} \left(\mathbb{C}^{d^n+1}/\mathfrak{S}_{d^n+1}\right).$$

It is not hard to see that $\overline{\Psi}_2^1$ is an embedding, and in fact maps MR_2 isomorphically onto a hyperplane in $\mathbb{C}^3/\mathfrak{S}_3$ (see [12]). However by looking at (flexible or rigid) Lattès examples, we can no longer expect $\overline{\Psi}_d^N$ to be an embedding for many d even when N is sufficiently large (see [14] for Lattès examples). He showed that for sufficiently large N, the map $\overline{\Psi}_d^N$ is finite-to-one except when d is a square, in which case it is also finite-to-one away from the Lattès locus. Here, the Lattès locus consists of one or two points whose inverse images are one parameter families. Furthermore, rigid Lattès examples imply that for any positive integer h there exist infinitely many degrees d such that for any N, the map $\overline{\Psi}_d^N$ is at least h-to-one (see [10] for more detail). However, it does not appear to be known if $\overline{\Psi}_3^N$ is injective. Hutz and Tepper [9] showed that $\overline{\Psi}_3^2$ is 12-to-one map.

There are several other papers such as [6] and [7], that discuss the use of multipliers of periodic points to parameterize the moduli space of polynomial or rational maps.

In another direction, Bousch [1], Morton [15] and Silverman [17] have studied the algebraic properties of the hypersurfaces consisting of periodic points of polynomial or rational maps in the product space of $\widehat{\mathbb{C}}$ and some parameter space.

We have three main theorems in this paper. The rest of Introduction is devoted to state Main Theorems I, II and III. To state them explicitly, we fix our notation first.

For $d \geq 2$, we put

$$\mathrm{Poly}_d := \left\{ f \in \mathbb{C}[z] \;\middle|\; \deg f = d \right\} \quad \text{and} \quad \mathrm{Aut}(\mathbb{C}) := \left\{ \gamma(z) = az + b \;\middle|\; a, b \in \mathbb{C},\ a \neq 0 \right\}. \tag{1.1}$$

Since $\gamma \in \mathrm{Aut}(\mathbb{C})$ naturally acts on $f \in \mathrm{Poly}_d$ by $\gamma \cdot f := \gamma \circ f \circ \gamma^{-1}$, we can define its quotient $\mathrm{MP}_d := \mathrm{Poly}_d/\mathrm{Aut}(\mathbb{C})$, which we usually call the moduli space of polynomial maps of degree d. We put $\mathrm{Fix}(f) := \{z \in \mathbb{C} \mid f(z) = z\}$ for $f \in \mathrm{Poly}_d$, where $\mathrm{Fix}(f)$ is considered counted with multiplicity. Hence we always have $\#(\mathrm{Fix}(f)) = d$. Since the set of fixed-point multipliers $(f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$ is invariant under the action of $\mathrm{Aut}(\mathbb{C})$, we can naturally define the map $\Phi_d : \mathrm{MP}_d \to \mathbb{C}^d/\mathfrak{S}_d$ by $\Phi_d(f) := (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$. Here, $\mathfrak{S}_d$ denotes the d-th symmetric group which acts on $\mathbb{C}^d$ by the permutation of coordinates.

Note that a fixed point $\zeta \in \mathrm{Fix}(f)$ is multiple if and only if $f'(\zeta) = 1$.

Proposition 1.1 *(Fixed point theorem). Let d be a natural number with $d \geq 2$ and suppose that a polynomial map $f \in \mathrm{Poly}_d$ has no multiple fixed point. Then we have $\sum_{\zeta\in\mathrm{Fix}(f)} \frac{1}{1-f'(\zeta)} = 0$.*

Proposition 1.1 is shown by the integration $\frac{1}{2\pi\sqrt{-1}} \oint_{|z|=R} \frac{dz}{z-f(z)}$ for sufficiently large R. We put $\Lambda_d := \left\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \sum_{i=1}^{d} \prod_{j\neq i} (1-\lambda_j) = 0\right\}$, $\widetilde{\Lambda}_d := \Lambda_d/\mathfrak{S}_d$ and $pr : \Lambda_d \to \widetilde{\Lambda}_d$. Then the image of the map Φ_d is contained in $\widetilde{\Lambda}_d$ by Proposition 1.1 and by the fact that $(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ always belongs to Λ_d if at least two of λ_i are equal to 1. In the following, we consider the map

$$\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$$

defined by $f \mapsto (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$. In the main theorems of this paper, we restrict our attention to the map Φ_d on the domain where polynomial maps have no multiple fixed points, i.e., on the domains $V_d := \left\{(\lambda_1, \ldots, \lambda_d) \in \Lambda_d \;\middle|\; \lambda_i \neq 1 \text{ for any } 1 \leq i \leq d\right\}$ and $\widetilde{V}_d := V_d/\mathfrak{S}_d$, which are Zariski open subsets of Λ_d and $\widetilde{\Lambda}_d$ respectively. Throughout this paper, we always denote by $\bar{\lambda}$ the equivalence class of $\lambda \in \Lambda_d$ in $\widetilde{\Lambda}_d$, i.e., $\bar{\lambda} = pr(\lambda)$, and never denote the complex conjugate of λ.

It is not hard to see that in the case $d = 2$ or 3, the map Φ_d is bijective. However we can no longer expect Φ_d to be bijective if $d \geq 4$; yet we can expect Φ_d to be generically finite by the remark below:

Remark 1.2. We have $\mathrm{MP}_d \cong \mathbb{C}^{d-1}/(\mathbb{Z}/(d-1)\mathbb{Z})$ and $\widetilde{\Lambda}_d \cong \mathbb{C}^{d-1}$. Especially we have $\dim_{\mathbb{C}} \mathrm{MP}_d = \dim_{\mathbb{C}} \widetilde{\Lambda}_d = d-1$.

We now state the first main theorem in this paper.

Main Theorem I. *Let d be a natural number with $d \geq 4$ and suppose $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$. Then the following statements hold:*

(1) *We always have $0 \leq \#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) \leq (d-2)!$.*
(2) *The cardinality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ is a function of the two sets*

$$\mathcal{I}(\lambda) := \left\{ I \subsetneq \{1,2,\ldots,d\} \;\middle|\; I \neq \emptyset, \quad \sum_{i \in I} \frac{1}{1-\lambda_i} = 0 \right\} \textit{ and}$$

$$\mathcal{K}(\lambda) := \left\{ K \subseteq \{1,2,\ldots,d\} \;\middle|\; K \neq \emptyset. \ \textit{ If } i,j \in K, \textit{ then } \lambda_i = \lambda_j \right\}.$$

Moreover $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ *is computed in finitely many steps only by using* $\mathcal{I}(\lambda)$ *and* $\mathcal{K}(\lambda)$.

(3) *If* $\mathcal{I}(\lambda) \subseteq \mathcal{I}(\lambda')$ *and* $\mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda')$ *hold for* $\lambda, \lambda' \in V_d$, *then* $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) \geq \#\left(\Phi_d^{-1}\left(\bar{\lambda}'\right)\right)$ *holds.*

(4) *The equality* $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = (d-2)!$ *holds if and only if the set* $\mathcal{I}(\lambda)$ *is empty and the complex numbers* $\lambda_1, \ldots, \lambda_d$ *are mutually distinct.*

(5) *If there exist non-zero integers* $c_1, \ldots, c_d$ *which satisfy the conditions* $c_1(1-\lambda_1) = \cdots = c_d(1-\lambda_d)$ *and* $\sum_{i=1}^{d} |c_i| \leq 2(d-2)$, *then the set* $\Phi_d^{-1}\left(\bar{\lambda}\right)$ *is empty.*

(6) *In the case* $d \leq 7$, *the converse of the assertion (5) holds.*

(7) *In every degree* d, *the Chebyshev polynomial provides an example of an element of* $\Phi_d^{-1}\left(\bar{\lambda}\right)$ *if* $\lambda \in V_d$ *satisfies the condition* $c_1(1-\lambda_1) = \cdots = c_d(1-\lambda_d)$ *for some non-zero integers* c_i *with* $\sum_{i=1}^{d} c_i = 0$, $\sum_{i=1}^{d} |c_i| = 2(d-1)$ *and* $|c_i| \leq 2$ *for* $1 \leq i \leq d$.

The algorithm of the computation in Main Theorem I(2) is given later in Definition 1.7 and Main Theorem III.

Remark 1.3. There is some overlap between Main Theorem I above and the results by Fujimura.

- She showed in [4] that if $\Phi_d^{-1}\left(\bar{\lambda}\right)$ is finite for $\bar{\lambda} \in \widetilde{\Lambda}_d$, then $0 \leq \#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) \leq (d-2)!$ holds. We removed the assumption that $\Phi_d^{-1}\left(\bar{\lambda}\right)$ is finite in the case $\bar{\lambda} \in \widetilde{V}_d$.
- She showed in [4, Theorem 6] that if $\mathcal{I}(\lambda)$ is empty, then $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = (d-2)!$ holds counted with multiplicity. Main Theorem I(4) is a strengthening of this result.
- She also gave a sufficient condition for $\Phi_d^{-1}(\bar{\lambda})$ to be empty in [4, Theorem 12]. For $d \leq 5$, her condition is equivalent to that in Main Theorem I(5). However for $d \geq 6$, her condition is stricter than ours. In the case $d = 6$, Example 1 in Section 2 in this paper is the unique example which satisfies our condition (5) but not Fujimura's condition in her Theorem 12.
- In the case $d \leq 5$, she also showed Main Theorem I(6) in [4, Theorem 5].
- Fujimura and Taniguchi's compactification [5] gives us a geometric insight of the fiber structure of Φ_d. Especially it provides an intuitional explanation of the reasons why $\mathcal{I}(\lambda)$ naturally arises in the computation of $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$. See also Remark 1.6.

Remark 1.4. The importance of this paper is that we can *completely* count the number of elements of each fiber $\Phi_d^{-1}(\bar{\lambda})$ for *all* $\lambda \in V_d$ *without exception* as we will see in Main Theorem III and Section 2. The main technical tools that we use for the proof of main theorems are a certain extension of Bezout's theorem on projective space $\mathbb{P}^n$ (see Proposition 5.3) and the relation between intersection multiplicity and the degree of finite branched covering (see Propositions 7.3, 7.5, 8.4, 8.7 and 8.10), which are common in the area of complex algebraic geometry.

Remark 1.5. The assertion (7) shows that the estimate $\sum_{i=1}^{d}|c_i| \le 2(d-2)$ in the assertion (5) is sharp, because $\sum_{i=1}^{d}|c_i|$ must be even. However this does not assure the converse of (5).

Conjecture 1.

(1) *The converse of the assertion (5) also holds in the case $d \geq 8$.*
(2) *If $\mathcal{I}(\lambda) \subsetneq \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) \subseteq \mathcal{K}(\lambda')$ hold for $\lambda, \lambda' \in V_d$, then $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) > \#\left(\Phi_d^{-1}\left(\bar{\lambda}'\right)\right)$ holds.*

The above conjecture is completely reduced to the problems on combinatorics by Main Theorem III.

The local fiber structure of the map Φ_d is also determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ as in the following:

Main Theorem II.

(1) *For any $\lambda, \lambda' \in V_d$ with $\mathcal{I}(\lambda) = \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda) = \mathcal{K}(\lambda')$, there exist open neighborhoods $\widetilde{U} \ni \bar{\lambda}$, $\widetilde{U}' \ni \bar{\lambda}'$ in $\widetilde{V}_d$ and biholomorphic maps $\mathfrak{L} : \Phi_d^{-1}\big(\widetilde{U}\big) \to \Phi_d^{-1}\big(\widetilde{U}'\big)$, $\widetilde{L} : \widetilde{U} \to \widetilde{U}'$ and $L : U \to U'$ with $L(\lambda) = \lambda'$ such that the following conditions (1a) and (1b) are satisfied, where U, U' are the connected components of $pr^{-1}\big(\widetilde{U}\big)$, $pr^{-1}\big(\widetilde{U}'\big)$ in V_d containing λ, λ' respectively.*
   (a) *The equalities $\Phi_d \circ \mathfrak{L} = \widetilde{L} \circ \Phi_d|_{\Phi_d^{-1}(\widetilde{U})}$ and $pr \circ L = \widetilde{L} \circ pr|_U$ hold.*
   (b) *For any $\lambda'' \in U$, the equalities $\mathcal{I}(\lambda'') = \mathcal{I}(L(\lambda''))$ and $\mathcal{K}(\lambda'') = \mathcal{K}(L(\lambda''))$ hold.*
(2) *For any $(\mathcal{I}, \mathcal{K}) \in \big\{(\mathcal{I}(\lambda), \mathcal{K}(\lambda)) \bigm| \lambda \in V_d\big\}$, the following properties (2a), (2b) and (2c) hold for the sets*

$$\widetilde{V}(\mathcal{I}, \mathcal{K}) := \left\{\bar{\lambda} \in \widetilde{V}_d \;\middle|\; \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I} \text{ and } \mathcal{K}(\lambda) = \mathcal{K}\right\},$$

$$\widetilde{V}(\mathcal{I}, *) := \left\{\bar{\lambda} \in \widetilde{V}_d \;\middle|\; \lambda \in V_d,\ \mathcal{I}(\lambda) = \mathcal{I}\right\},$$

$$\widetilde{V}(*, \mathcal{K}) := \left\{\bar{\lambda} \in \widetilde{V}_d \;\middle|\; \lambda \in V_d,\ \mathcal{K}(\lambda) = \mathcal{K}\right\}:$$

   (a) *the map $\Phi_d|_{\Phi_d^{-1}(\widetilde{V}(\mathcal{I},*))} : \Phi_d^{-1}\big(\widetilde{V}(\mathcal{I}, *)\big) \to \widetilde{V}(\mathcal{I}, *)$ is proper.*
   (b) *The map $\Phi_d|_{\Phi_d^{-1}(\widetilde{V}(*,\mathcal{K}))} : \Phi_d^{-1}\big(\widetilde{V}(*, \mathcal{K})\big) \to \widetilde{V}(*, \mathcal{K})$ is locally homeomorphic.*
   (c) *For each connected component X of $\Phi_d^{-1}\big(\widetilde{V}(\mathcal{I}, \mathcal{K})\big)$, the map $\Phi_d|_X : X \to \widetilde{V}(\mathcal{I}, \mathcal{K})$ is an unbranched covering.*

Remark 1.6. The above assertion (2a) implies that $\mathcal{I}(\lambda)$ dominates the information on the number of 'holes' on each fiber of the map Φ_d. Fujimura and Taniguchi [5] showed that the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$ is extended to the map $\widehat{\Psi}_d : \widehat{M_d} \to \mathbb{P}^{d-1}$, where $\widehat{M_d}$ is their compactification of MP_d. In our context, the condition $\mathcal{I}(\lambda) \neq \emptyset$ holds for $\lambda \in V_d$ if and only if $\widehat{\Psi}_d^{-1}(\bar{\lambda}) \cap \left(\widehat{M_d} \setminus \mathrm{MP}_d\right) \neq \emptyset$.

On the other hand, the above assertion (2b) implies that the condition $\mathcal{K}(\lambda) \supsetneq \{\{1\}, \ldots, \{d\}\}$ holds for $\lambda \in V_d$ if $\bar{\lambda}$ lies on the branch locus of the map Φ_d.

To state Main Theorem III explicitly, we need some more notations, which are defined in Definition 1.7 and are often used later in the proof of the main theorems. After reading Sections 4, 5, 6 and 9, the readers will find that the process in Main Theorem III is natural.

Definition 1.7. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of V_d. Then

- we put

$$\mathfrak{I}(\lambda) := \left\{ \{I_1, \ldots, I_l\} \;\middle|\; \begin{array}{c} I_1 \amalg \cdots \amalg I_l = \{1, \ldots, d\}, \quad l \geq 2, \\ I_u \in \mathcal{I}(\lambda) \text{ for each } 1 \leq u \leq l \end{array} \right\},$$

  where $I_1 \amalg \cdots \amalg I_l$ denotes the disjoint union of $I_1, \ldots, I_l$. The partial order $\prec$ in $\mathfrak{I}(\lambda)$ is defined by the refinement of partitions, namely, for $\mathbb{I}, \mathbb{I}' \in \mathfrak{I}(\lambda)$, the relation $\mathbb{I} \prec \mathbb{I}'$ holds if and only if $\mathbb{I}'$ is a refinement of $\mathbb{I}$ as partitions of $\{1, \ldots, d\}$. Note that $\mathfrak{I}(\lambda)$ gives the equivalent information as $\mathcal{I}(\lambda)$. (For more detail, see Remark 6.4 and Section 2.)
- We denote by $K_1, \ldots, K_q$ the collection of maximal elements of $\mathcal{K}(\lambda)$ with respect to the inclusion relations, i.e.,

$$\{K_1, \ldots, K_q\} = \left\{ K \in \mathcal{K}(\lambda) \;\middle|\; i \in K,\ j \in \{1, \ldots, d\} \setminus K \Longrightarrow \lambda_i \neq \lambda_j \right\}.$$

  Note that the equality $K_1 \amalg \cdots \amalg K_q = \{1, \ldots, d\}$ always holds by definition. We put $\kappa_w := \#(K_w)$ for $1 \leq w \leq q$ and denote by g_w the greatest common divisor of $\kappa_1, \ldots, \kappa_{(w-1)}, (\kappa_w) - 1, \kappa_{(w+1)}, \ldots, \kappa_q$ for each $1 \leq w \leq q$.
- We define the function m by $m(z) := \frac{1}{1-z}$ for $z \in \mathbb{C} \setminus \{1\}$.
- We may assume $\lambda \in V_d$ to be in the form

$$\lambda = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\kappa_1}, \ldots, \underbrace{\lambda_q, \ldots, \lambda_q}_{\kappa_q}),$$

  where $\lambda_1, \ldots, \lambda_q$ are mutually distinct. For each $1 \leq w \leq q$ and for each divisor t of g_w with $t \geq 2$, we put $d[t] := \frac{d-1}{t} + 1$ and denote by $\lambda[t]$ the element of $V_{d[t]}$ such that

$$\lambda[t] := (\underbrace{m^{-1}(tm(\lambda_1)), \ldots, m^{-1}(tm(\lambda_1))}_{\frac{\kappa_1}{t}}, \ldots,$$

$$\underbrace{m^{-1}(tm(\lambda_w)), \ldots, m^{-1}(tm(\lambda_w))}_{\frac{(\kappa_w)-1}{t}}, \ldots, \underbrace{m^{-1}(tm(\lambda_q)), \ldots, m^{-1}(tm(\lambda_q))}_{\frac{\kappa_q}{t}}, \lambda_w).$$

Note that w is determined by t and that $\mathcal{I}(\lambda[t])$ is determined by $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ and t.

Main Theorem III. *For $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, the cardinality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right)$ is computed in the following steps.*

- *For each $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we define the number $e_{\mathbb{I}}(\lambda)$ inductively by the equality*

$$e_{\mathbb{I}}(\lambda) := \left(\prod_{u=1}^{l} (\#(I_u) - 1)!\right) - \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I},\ \mathbb{I}' \neq \mathbb{I}}} \left(e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k\right)\right), \tag{1.2}$$

  where we put $\chi_u(\mathbb{I}') := \#\left(\left\{I' \in \mathbb{I}' \,\middle|\, I' \subseteq I_u\right\}\right)$ for $\mathbb{I}' \succ \mathbb{I}$. Note that in the case $\chi_u(\mathbb{I}') = 1$, we assume that $\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k = \prod_{k=\#(I_u)}^{\#(I_u)-1} k = 1$.
- *We put*

$$s_d(\lambda) := (d-2)! - \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-2} k\right). \tag{1.3}$$

  Note that in the case $\#(\mathbb{I}) = 2$, we assume that $\prod_{k=d-\#(\mathbb{I})+1}^{d-2} k = \prod_{k=d-1}^{d-2} k = 1$.
- *Moreover we define the numbers $c_t(\lambda)$ for $t \in \bigcup_{1 \le w \le q} \left\{t \,\middle|\, t|g_w\right\}$ by the equalities*

$$\sum_{t|b,\ b|g_w} \frac{t}{b}\, c_b(\lambda) = \frac{s_{d[t]}(\lambda[t])}{\left(\frac{\kappa_1}{t}\right)! \cdots \left(\frac{\kappa_{(w-1)}}{t}\right)! \left(\frac{(\kappa_w)-1}{t}\right)! \left(\frac{\kappa_{(w+1)}}{t}\right)! \cdots \left(\frac{\kappa_q}{t}\right)!} \tag{1.4}$$

  for $(w,t) \in \left\{(w,t) \,\middle|\, 1 \le w \le q,\ t|g_w,\ t \ge 2\right\}$, and

$$c_1(\lambda) + \sum_{w=1}^{q} \left(\sum_{t|g_w,\ t \ge 2} \frac{1}{t}\, c_t(\lambda)\right) = \frac{s_d(\lambda)}{\kappa_1! \cdots \kappa_q!}, \tag{1.5}$$

  where $t|b$ denotes that t divides b for positive integers t and b.
- *Then the numbers $e_{\mathbb{I}}(\lambda)$, $s_d(\lambda)$ and $c_t(\lambda)$ are non-negative integers. Moreover we have*

$$\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = \sum_t c_t(\lambda) = c_1(\lambda) + \sum_{w=1}^{q}\left(\sum_{t|g_w,\ t\geq 2} c_t(\lambda)\right). \tag{1.6}$$

Remark 1.8. Note that all the numbers defined in Main Theorem III are determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$. Especially the number $s_d(\lambda)$ is determined only by $\mathcal{I}(\lambda)$. If we count the number $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ with multiplicity, then we always have $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = s_d(\lambda)$. However in our context, we do not consider $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ counted with multiplicity, and therefore need some more computation. The number $s_d(\lambda)$ is the cardinality of the set $S_d(\lambda)$ which will be defined in Definition 4.2.

Remark 1.9. Under the isomorphism $\mathrm{MP}_d \cong \mathbb{C}^{d-1}/\left(\mathbb{Z}/(d-1)\mathbb{Z}\right)$ in Remark 1.2, the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ on $\mathbb{C}^{d-1}$ is not free, and MP_d has the set of singular points $\mathrm{Sing}(\mathrm{MP}_d)$ for $d \geq 4$. If $\bar{\lambda} \in \widetilde{V}_d$ lies away from the locus $\Phi_d(\mathrm{Sing}(\mathrm{MP}_d))$, then the set $\left\{(w,t) \,\middle|\, 1 \leq w \leq q,\ t|g_w,\ t \geq 2\right\}$ in the third step in Main Theorem III is empty, and therefore we have $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = c_1(\lambda) = s_d(\lambda)/(\kappa_1! \cdots \kappa_q!)$.

Problem. *Give a combinatorial proof of the fact that for any $\lambda \in V_d$ and for any t, the number $c_t(\lambda)$ defined above is a non-negative integer. Note that the proof given in this paper is not combinatorial.*

For parameters $\lambda \in \Lambda_d \setminus V_d$, we have the following:

Remark 1.10. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda_d \setminus V_d$ with $\#\left\{i \,\middle|\, \lambda_i = 1\right\} \geq 4$, some connected components of the inverse image $\Phi_d^{-1}\left(\bar{\lambda}\right)$ may have dimension greater than or equal to 1. However, if we put

$$\mathrm{MP}''_d := \left\{f \in \mathrm{MP}_d \,\middle|\, f \text{ has at most one multiple fixed point}\right\},$$

then the map $\Phi_d|_{\mathrm{MP}''_d} : \mathrm{MP}''_d \to \widetilde{\Lambda}_d$ is finite. Moreover similar results to the main theorems hold for $\Phi_d|_{\mathrm{MP}''_d}$ and for any $\lambda \in \Lambda_d \setminus V_d$, whose proofs are also similar to those of the main theorems.

We shall also comment about $f \in \mathrm{MP}_d$ having more than two multiple fixed points. For any $\zeta \in \mathrm{Fix}(f)$, the holomorphic index of f at ζ is defined to be the complex number $\iota(f,\zeta) := \frac{1}{2\pi\sqrt{-1}} \oint_{|z-\zeta|=\epsilon} \frac{dz}{z-f(z)}$, where ϵ is a sufficiently small positive real number. The index $\iota(f,\zeta)$ is invariant under biholomorphic transformations, and is equal to $\frac{1}{1-f'(\zeta)}$ if ζ is not multiple. We denote by $m(f,\zeta)$ the fixed-point multiplicity of f at $\zeta \in \mathrm{Fix}(f)$. Then we always have $\sum_{\zeta\in\mathrm{Fix}(f)} m(f,\zeta) = \deg f$ and $\sum_{\zeta\in\mathrm{Fix}(f)} \iota(f,\zeta) = 0$. Moreover we have $\iota(f,\zeta) \neq 0$ whenever $m(f,\zeta) = 1$. Note that $\mathrm{Fix}(f)$ is not considered counted with multiplicity only here and in the following conjecture.

Conjecture 2. *We consider the map $\widetilde{\Phi}_d$, instead of Φ_d, which assigns $\widetilde{\Phi}_d(f) = \left([\iota(f,\zeta), m(f,\zeta)]\right)_{\zeta\in\mathrm{Fix}(f)}$ to each $f \in \mathrm{MP}_d$, so that the target space of $\widetilde{\Phi}_d$ is defined to be the* *family of unordered collections of pairs* $[m_i, d_i]$ *with* $m_i \in \mathbb{C},\ d_i \in \mathbb{Z},\ d_i \geq 1,\ \sum_i d_i = d$ *and* $\sum_i m_i = 0$. *Then it is conjectured that the map* $\widetilde{\Phi}_d$ *is finite and that similar results to the main theorems hold for* $\widetilde{\Phi}_d$ *and for any parameter value without exception.*

We have ten sections in this paper. In Section 2, we give some examples which illustrate the calculation of $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ in Main Theorem III. In Section 3, we give the detailed program of the remaining sections. Sections from 4 to 10 are devoted to the proofs of Main Theorems I, II and III.

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2. Some examples

In this section, we give three examples which illustrate the calculation of $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ in Main Theorem III.

Example 1. We consider an element $\lambda = (\lambda_1, \ldots, \lambda_6) \in V_6$ satisfying the equality

$$\frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_6} = 1 : 1 : 2 : -1 : -1 : -2.$$

In this case we have $\#\left(\Phi_6^{-1}\left(\bar{\lambda}\right)\right) = 0$ by the assertion (5) in Main Theorem I; however in this example we shall find it again by following the steps in Main Theorem III.

By definition, we have $\mathfrak{I}(\lambda) = \left\{\mathbb{I}_\omega \;\middle|\; 1 \leq \omega \leq 8\right\}$, where

$$\begin{aligned}
&\mathbb{I}_1 = \left\{\{1,4\},\{2,5\},\{3,6\}\right\},\ \mathbb{I}_2 = \left\{\{1,5\},\{2,4\},\{3,6\}\right\},\\
&\mathbb{I}_3 = \left\{\{1,2,4,5\},\{3,6\}\right\},\ \mathbb{I}_4 = \left\{\{1,4\},\{2,3,5,6\}\right\},\ \mathbb{I}_5 = \left\{\{2,5\},\{1,3,4,6\}\right\},\\
&\mathbb{I}_6 = \left\{\{1,5\},\{2,3,4,6\}\right\},\ \mathbb{I}_7 = \left\{\{2,4\},\{1,3,5,6\}\right\} \text{ and } \mathbb{I}_8 = \left\{\{1,2,6\},\{3,4,5\}\right\}.
\end{aligned}$$

We have $\mathbb{I}_3 \prec \mathbb{I}_1, \mathbb{I}_4 \prec \mathbb{I}_1, \mathbb{I}_5 \prec \mathbb{I}_1, \mathbb{I}_3 \prec \mathbb{I}_2, \mathbb{I}_6 \prec \mathbb{I}_2$ and $\mathbb{I}_7 \prec \mathbb{I}_2$; hence the maximal elements of $\mathfrak{I}(\lambda)$ are $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_8$.

By the equality (1.2), we have $e_{\mathbb{I}_1}(\lambda) = e_{\mathbb{I}_2}(\lambda) = (2-1)! \cdot (2-1)! \cdot (2-1)! = 1$ and $e_{\mathbb{I}_8}(\lambda) = (3-1)! \cdot (3-1)! = 4$. Moreover we have $e_{\mathbb{I}_3}(\lambda) = (4-1)! \cdot (2-1)! - (e_{\mathbb{I}_1}(\lambda) \cdot 3 + e_{\mathbb{I}_2}(\lambda) \cdot 3) = 6 - (3+3) = 0$, $e_{\mathbb{I}_4}(\lambda) = e_{\mathbb{I}_5}(\lambda) = (2-1)! \cdot (4-1)! - e_{\mathbb{I}_1}(\lambda) \cdot 3 = 6 - 3 = 3$ and $e_{\mathbb{I}_6}(\lambda) = e_{\mathbb{I}_7}(\lambda) = (2-1)! \cdot (4-1)! - e_{\mathbb{I}_2}(\lambda) \cdot 3 = 6 - 3 = 3$. Hence by the equality (1.3), we have $s_6(\lambda) = (6-2)! - \left(\sum_{\omega=1}^{2} e_{\mathbb{I}_\omega}(\lambda) \cdot 4 + \sum_{\omega=3}^{8} e_{\mathbb{I}_\omega}(\lambda)\right) = 24 - (4+4+0+3+3+3+3+4) = 0$, which implies $\#\left(\Phi_6^{-1}\left(\bar{\lambda}\right)\right) = c_1(\lambda) = 0$.

Example 2. In this example we consider $\lambda = (\lambda_1, \dots, \lambda_{31}) \in V_{31}$ with

$$\frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_{31}} = \underbrace{6 : \cdots : 6}_{25} : \underbrace{-25 : \cdots : -25}_{6}.$$

In this case we have $\mathfrak{I}(\lambda) = \emptyset$ and $s_{31}(\lambda) = 29!$ by the equality (1.3).

On the other hand, by Definition 1.7, we have $q = 2$, $K_1 = \{1, \dots, 25\}$, $K_2 = \{26, \dots, 31\}$, $\kappa_1 = 25$, $\kappa_2 = 6$, $g_1 = \gcd(\kappa_1 - 1, \kappa_2) = 6$, $g_2 = 5$, $\bigcup_{1 \le w \le 2} \{t \mid t | g_w\} = \{1, 2, 3, 6, 5\}$, $d[2] = \frac{31-1}{2} + 1 = 16$, $d[3] = 11$, $d[6] = 6$ and $d[5] = 7$. Moreover we have $\lambda[2] = (\lambda[2]_1, \dots, \lambda[2]_{16}) \in V_{16}$ with

$$\frac{1}{1-\lambda[2]_1} : \cdots : \frac{1}{1-\lambda[2]_{16}} = \underbrace{12 : \cdots : 12}_{\frac{\kappa_1-1}{2}=12} : \underbrace{-50 : -50 : -50}_{\frac{\kappa_2}{2}=3} : 6.$$

Similarly we have

$\lambda[3] = (\lambda[3]_1, \dots, \lambda[3]_{11}) \in V_{11}$ with $\frac{1}{1-\lambda[3]_1} : \cdots : \frac{1}{1-\lambda[3]_{11}} = \underbrace{18 : \cdots : 18}_{8} : -75 : -75 : 6$,

$\lambda[6] = (\lambda[6]_1, \dots, \lambda[6]_6) \in V_6$ with $\frac{1}{1-\lambda[6]_1} : \cdots : \frac{1}{1-\lambda[6]_6} = \underbrace{36 : \cdots : 36}_{4} : -150 : 6$ and

$\lambda[5] = (\lambda[5]_1, \dots, \lambda[5]_7) \in V_7$ with $\frac{1}{1-\lambda[5]_1} : \cdots : \frac{1}{1-\lambda[5]_7} = \underbrace{30 : \cdots : 30}_{\frac{\kappa_1}{5}=5} : \underbrace{-125}_{\frac{\kappa_2-1}{5}=1} : -25.$

Since $\mathfrak{I}(\lambda) = \emptyset$, we have $\mathfrak{I}(\lambda[t]) = \emptyset$ for $t = 2, 3, 6, 5$, which implies $s_{16}(\lambda[2]) = 14!$, $s_{11}(\lambda[3]) = 9!$, $s_6(\lambda[6]) = 4!$ and $s_7(\lambda[5]) = 5!$ by the equality (1.3). By the equality (1.4) for $(w, t) = (1, 6)$, $(1, 3)$, $(1, 2)$ and $(2, 5)$, we have

$$\begin{aligned}
\frac{6}{6} c_6(\lambda) &= \frac{s_6(\lambda[6])}{\left(\frac{\kappa_1-1}{6}\right)! \cdot \left(\frac{\kappa_2}{6}\right)!} = \frac{4!}{4! \cdot 1!} = 1, \\
\frac{3}{3} c_3(\lambda) + \frac{3}{6} c_6(\lambda) &= \frac{s_{11}(\lambda[3])}{\left(\frac{\kappa_1-1}{3}\right)! \cdot \left(\frac{\kappa_2}{3}\right)!} = \frac{9!}{8! \cdot 2!} = \frac{9}{2}, \\
\frac{2}{2} c_2(\lambda) + \frac{2}{6} c_6(\lambda) &= \frac{s_{16}(\lambda[2])}{\left(\frac{\kappa_1-1}{2}\right)! \cdot \left(\frac{\kappa_2}{2}\right)!} = \frac{14!}{12! \cdot 3!} = \frac{91}{3}, \\
\frac{5}{5} c_5(\lambda) &= \frac{s_7(\lambda[5])}{\left(\frac{\kappa_1}{5}\right)! \cdot \left(\frac{\kappa_2-1}{5}\right)!} = \frac{5!}{5! \cdot 1!} = 1
\end{aligned}$$

respectively, which implies $c_6(\lambda) = 1$, $c_3(\lambda) = 4$, $c_2(\lambda) = 30$ and $c_5(\lambda) = 1$. Moreover by the equality (1.5), we have

$$c_1(\lambda) + \frac{1}{2} c_2(\lambda) + \frac{1}{3} c_3(\lambda) + \frac{1}{6} c_6(\lambda) + \frac{1}{5} c_5(\lambda) = \frac{s_{31}(\lambda)}{\kappa_1! \cdot \kappa_2!} = \frac{29!}{25! \cdot 6!} = \frac{7917}{10},$$

which implies $c_1(\lambda) = 775$. Hence by (1.6), we have

$$\#\left(\Phi_{31}^{-1}\left(\bar{\lambda}\right)\right) = c_1(\lambda) + c_2(\lambda) + c_3(\lambda) + c_6(\lambda) + c_5(\lambda) = 775 + 30 + 4 + 1 + 1 = 811.$$

Example 3. Here we consider a little complicated example, which is $\lambda = (\lambda_1, \dots, \lambda_9) \in V_9$ with $\frac{1}{1-\lambda_1} : \cdots : \frac{1}{1-\lambda_9} = 2 : 2 : 2 : 2 : -1 : -1 : -2 : -2 : -2$. In this case, by Definition 1.7, we have $q = 3$, $\kappa_1 = 4$, $\kappa_2 = 2$, $\kappa_3 = 3$, $g_1 = g_2 = 1$ and $g_3 = 2$. Hence we must find $s_9(\lambda)$ and $s_5(\lambda[2])$, and after that by the equalities (1.4) and (1.5) we have

$$\frac{2}{2}c_2(\lambda) = \frac{s_5(\lambda[2])}{(4/2)! \cdot (2/2)! \cdot ((3-1)/2)!} = \frac{s_5(\lambda[2])}{2} \quad \text{and} \quad c_1(\lambda) + \frac{1}{2}c_2(\lambda) = \frac{s_9(\lambda)}{4! \cdot 2! \cdot 3!}. \tag{2.1}$$

We shall find $s_5(\lambda[2])$ first. Since $\lambda[2] = (\lambda[2]_1, \dots, \lambda[2]_5) \in V_5$ with $\frac{1}{1-\lambda[2]_1} : \cdots : \frac{1}{1-\lambda[2]_5} = 4 : 4 : -2 : -4 : -2$, we have $\mathfrak{I}(\lambda[2]) = \{\mathbb{I}'_1, \mathbb{I}'_2\}$, where $\mathbb{I}'_1 = \big\{\{1,4\},\{2,3,5\}\big\}$ and $\mathbb{I}'_2 = \big\{\{2,4\},\{1,3,5\}\big\}$. Hence we have $e_{\mathbb{I}'_1}(\lambda[2]) = e_{\mathbb{I}'_2}(\lambda[2]) = (2-1)! \cdot (3-1)! = 2$ and $s_5(\lambda[2]) = (5-2)! - \big(e_{\mathbb{I}'_1}(\lambda[2]) + e_{\mathbb{I}'_2}(\lambda[2])\big) = 6 - (2+2) = 2$, which implies $c_2(\lambda) = \frac{2}{2} = 1$ by the equality (2.1).

On the other hand, the computation of $s_9(\lambda)$ is much more complicated than that of $s_5(\lambda[2])$. First of all, $\mathfrak{I}(\lambda)$ consists of 130 elements, and we shall express them by

$$\begin{aligned}\mathfrak{I}(\lambda) = \left\{\mathbb{I}_{(1,\omega)} \;\middle|\; 1 \le \omega \le 24\right\} &\cup \left\{\mathbb{I}_{(2,\omega)} \;\middle|\; 1 \le \omega \le 36\right\} \cup \left\{\mathbb{I}_{(3,\omega)} \;\middle|\; 1 \le \omega \le 36\right\} \\ &\cup \left\{\mathbb{I}_{(4,\omega)} \;\middle|\; 1 \le \omega \le 12\right\} \cup \left\{\mathbb{I}_{(5,\omega)} \;\middle|\; 1 \le \omega \le 18\right\} \cup \left\{\mathbb{I}_{(6,\omega)} \;\middle|\; 1 \le \omega \le 4\right\}.\end{aligned}$$

Here $\mathbb{I}_{(1,\omega)}$ for $1 \le \omega \le 24$ are of the form $\big\{\{\sigma(1),5,6\},\{\sigma(2),7\},\{\sigma(3),8\},\{\sigma(4),9\}\big\}$ for $\sigma \in \mathfrak{S}_4 = \mathrm{Aut}(\{1,2,3,4\})$. Similarly $\mathbb{I}_{(2,\omega)}$, $\mathbb{I}_{(3,\omega)}$, $\mathbb{I}_{(4,\omega)}$, $\mathbb{I}_{(5,\omega)}$ and $\mathbb{I}_{(6,\omega)}$ are of the form

$$\begin{gathered}\big\{\{\sigma(1),\sigma(2),5,6,\tau(7)\},\{\sigma(3),\tau(8)\},\{\sigma(4),\tau(9)\}\big\},\\ \big\{\{\sigma(1),5,6\},\{\sigma(2),\tau(7)\},\{\sigma(3),\sigma(4),\tau(8),\tau(9)\}\big\},\\ \big\{\{\sigma(1),\sigma(2),\sigma(3),5,6,\tau(7),\tau(8)\},\{\sigma(4),\tau(9)\}\big\},\\ \big\{\{\sigma(1),\sigma(2),5,6,\tau(7)\},\{\sigma(3),\sigma(4),\tau(8),\tau(9)\}\big\}\\ \text{and } \big\{\{\sigma(1),5,6\},\{\sigma(2),\sigma(3),\sigma(4),7,8,9\}\big\}\end{gathered}$$

respectively for $\sigma \in \mathfrak{S}_4 = \mathrm{Aut}(\{1,2,3,4\})$ and $\tau \in \mathfrak{S}_3 = \mathrm{Aut}(\{7,8,9\})$. By (1.2) we have $e_{\mathbb{I}_{(1,\omega)}}(\lambda) = 2! \cdot 1! \cdot 1! \cdot 1! = 2$. For each $1 \le \omega \le 36$, we have $\#\left(\left\{\omega' \mid \mathbb{I}_{(2,\omega)} \prec \mathbb{I}_{(1,\omega')}\right\}\right) = 2$; hence by (1.2) we have $e_{\mathbb{I}_{(2,\omega)}}(\lambda) = 4! \cdot 1! \cdot 1! - 2 \cdot 4 \times 2 = 8$. Similarly for each $1 \le \omega \le 36$, we have $\#\left(\left\{\omega' \mid \mathbb{I}_{(3,\omega)} \prec \mathbb{I}_{(1,\omega')}\right\}\right) = 2$, which implies $e_{\mathbb{I}_{(3,\omega)}}(\lambda) = 2! \cdot 1! \cdot 3! - 2 \cdot 3 \times 2 = 0$. Since $\#\left(\left\{\omega' \mid \mathbb{I}_{(4,\omega)} \prec \mathbb{I}_{(1,\omega')}\right\}\right) = 6$, $\#\left(\left\{\omega' \mid \mathbb{I}_{(4,\omega)} \prec \mathbb{I}_{(2,\omega')}\right\}\right) = 6$ and $\#\left(\left\{\omega' \mid \mathbb{I}_{(4,\omega)} \prec \mathbb{I}_{(3,\omega')}\right\}\right) = 3$, we have $e_{\mathbb{I}_{(4,\omega)}}(\lambda) = 6! \cdot 1! - (2 \cdot (5 \cdot 6) \times 6 + 8 \cdot 6 \times 6 + 0 \cdot 6 \times 3) = 72$. Similarly we have $e_{\mathbb{I}_{(5,\omega)}}(\lambda) = 4! \cdot 3! - (2 \cdot 4 \cdot 3 \times 4 + 8 \cdot 3 \times 2 + 0 \cdot 4 \times 2) = 0$

and $e_{\mathbb{I}_{(6,\omega)}}(\lambda) = 2! \cdot 5! - (2 \cdot (4 \cdot 5) \times 6 + 0 \cdot 5 \times 9) = 0$. Therefore by (1.3) we have $s_9(\lambda) = 7! - (2 \cdot (6 \cdot 7) \times 24 + 8 \cdot 7 \times 36 + 72 \times 12) = 144$.

To summarize, we have $c_2(\lambda) = 1$ and $c_1(\lambda) + \frac{1}{2}c_2(\lambda) = \frac{144}{4! \cdot 2! \cdot 3!}$ by (2.1), which implies $c_1(\lambda) = 0$ and $\#\left(\Phi_9^{-1}\left(\bar{\lambda}\right)\right) = c_1(\lambda) + c_2(\lambda) = 0 + 1 = 1$. Here, the unique element of $\Phi_9^{-1}\left(\bar{\lambda}\right)$ is represented by $f_9(x)$ which is the one defined in the proof of Proposition 4.10.

3. Detailed program of the proof

In this section, we describe the detailed program of the proof of the main theorems. Sections from 4 to 10 are devoted to the proofs of Main Theorems I, II and III. The proofs are self-contained except for the basic knowledge of the intersection theory on the projective space $\mathbb{P}^n$ (see Section 4 of Chapter 0 and Section 3 of Chapter 1 in [8]) and the theory on finite branched coverings. The most important tool for the proof, which is stated in Proposition 5.3, is an extension of Bezout's theorem on $\mathbb{P}^n$ especially in the case that some components of the common zeros of n homogeneous polynomials are not points or are components which are proper subsets of other components. The most difficult and most crucial part in the proof of the main theorems is the proof of Theorem B. Theorem B is stated in Section 6, and its proof is described in Section 8. Main Theorem II is naturally proved in the process of proving Main Theorems I and III. The assertions (5) and (7) in Main Theorem I are proved in Section 4, and the assertions (1) and (4) in Main Theorem I are proved in Section 6. On the other hand, the proofs of the rest are completed in Section 10.

In Section 4 we rewrite the set $\Phi_d^{-1}\left(\bar{\lambda}\right)$ as follows: for each $\lambda \in V_d$, we define the subsets $T_d(\lambda)$, $S_d(\lambda)$ and $B_d(\lambda)$ of $\mathbb{P}^{d-2}$, where $T_d(\lambda)$ is the set of the common zeros of some $(d-2)$ homogeneous polynomials $\varphi_1, \ldots, \varphi_{d-2}$ on $\mathbb{P}^{d-2}$, and $T_d(\lambda) = S_d(\lambda) \amalg B_d(\lambda)$. We define the subgroup $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ of $\mathfrak{S}_d$ acting on $S_d(\lambda)$, and show the existence of the bijection $\overline{\pi(\lambda)} : S_d(\lambda)/\mathfrak{S}\left(\mathcal{K}(\lambda)\right) \cong \Phi_d^{-1}\left(\bar{\lambda}\right)$ in Proposition 4.3. By Proposition 4.3, we can divide the proof of Main Theorems I and III into two steps: the first one is to determine the cardinality $\#\left(S_d(\lambda)\right)$; the second one is to analyze the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$.

In Section 5 we review the intersection theory on $\mathbb{P}^n$ and give an extension of Bezout's theorem on $\mathbb{P}^n$ in Proposition 5.3, which will be utilized crucially for determining the cardinality $\#\left(S_d(\lambda)\right)$ afterward. In Definitions 5.1 and 5.2, we define the family $\mathcal{C}(\varphi_1, \ldots, \varphi_m)$ of irreducible varieties for homogeneous polynomials $\varphi_1, \ldots, \varphi_m$ on $\mathbb{P}^n$ and the number $\mathrm{mult}_C(\varphi_1, \ldots, \varphi_m)$ for each $C \in \mathcal{C}(\varphi_1, \ldots, \varphi_m)$ with $\operatorname{codim} C = m$. Here, $\mathcal{C}(\varphi_1, \ldots, \varphi_m)$ stands for the family of the "components" of the common zeros of $\varphi_1, \ldots, \varphi_m$ in $\mathbb{P}^n$. In practice, it contains all the irreducible components of the common zeros of $\varphi_1, \ldots, \varphi_m$, and may also contain some irreducible varieties which are proper subsets of some irreducible components of the common zeros of $\varphi_1, \ldots, \varphi_m$. On the other hand, the number $\mathrm{mult}_C(\varphi_1, \ldots, \varphi_m)$ stands for the "intersection multiplicity" of $\varphi_1, \ldots, \varphi_m$ along C; if C is an irreducible component, then it is the usual intersection multiplicity of $\varphi_1, \ldots, \varphi_m$ along C. Proposition 5.3 gives the relation among these numbers, which is also reduced to the usual Bezout's theorem if $\mathcal{C}(\varphi_1, \ldots, \varphi_n)$ consists only of points.

In Sections 6, 7 and 8 we determine the cardinality $\#\left(S_d(\lambda)\right)$, based on Section 5. More precisely, in Section 6, we give the explicit expression of the set $B_d(\lambda)$ in Lemma 6.5, and determine the number $\mathrm{mult}_C(\varphi_1, \ldots, \varphi_m)$ for each $C \in \mathcal{C}(\varphi_1, \ldots, \varphi_{d-2})$ with $\operatorname{codim} C = m$ and $C \subseteq B_d(\lambda)$ in Theorems A and B. Some of the elements of $\mathcal{C}(\varphi_1, \ldots, \varphi_{d-2})$ may be proper subsets of other elements, which makes their computation much complicated. Proposition 5.3, Theorems A and B give the exact expression of the cardinality $\#\left(S_d(\lambda)\right)$. Sections 7 and 8 are devoted to the proofs of Theorems A and B respectively.

In most cases, the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ is free. However in some cases, it is rather complicated. In Section 9 we analyze the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ in detail, and give the exact relation between the cardinalities of $S_d(\lambda)$ and $\Phi_d^{-1}\left(\bar{\lambda}\right)$ in Theorem E. To summarize, in Section 10 we complete the proof of the main theorems.

4. Another expression of the set $\boldsymbol{\Phi_d^{-1}\left(\bar{\lambda}\right)}$

In this section we start proving the main theorems. In the rest of this paper, we always assume that d is a natural number with $d \geq 4$.

An arbitrary polynomial map $f(z) \in \mathbb{C}[z]$ of degree d can be expressed in the form $f(z) = z+\rho(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_d)$, where $\zeta_1, \zeta_2, \ldots, \zeta_d$ and ρ are complex numbers with $\rho \neq 0$. In this expression we have $\mathrm{Fix}(f) = \{\zeta_1, \zeta_2, \ldots, \zeta_d\}$ and $f'(\zeta_i) = 1+\rho\prod_{j\neq i}(\zeta_i-\zeta_j)$ for $1 \leq i \leq d$. Hence to show Main Theorems I and III, we only need to count the number of the solutions of the equations $1 + \rho\prod_{j\neq i}(\zeta_i - \zeta_j) = \lambda_i$ for $1 \leq i \leq d$ modulo affine conjugacy. However we do not take this method. The following is the key for the proof of the main theorems.

Key Lemma. *Let f be a polynomial map of degree d expressed in the form*

$$f(z) = z + \rho(z - \zeta_1)(z - \zeta_2)\cdots(z - \zeta_d),$$

where $\zeta_1, \ldots, \zeta_d$ and ρ are complex numbers with $\rho \neq 0$. Then for $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, the equalities $f'(\zeta_i) = \lambda_i$ hold for $1 \leq i \leq d$ if and only if the equalities

$$\sum_{i=1}^{d} \frac{1}{1-\lambda_i}\zeta_i^k = \begin{cases} 0 & (1 \leq k \leq d-2) \\ -\frac{1}{\rho} & (k = d-1) \end{cases} \tag{4.1}$$

hold and $\zeta_1, \ldots, \zeta_d$ are mutually distinct.

Remark 4.1. Similar result to Key Lemma is already given by Fujimura in [4, Lemma 9], while her proof is different from the following.

Proof. The integration $\frac{1}{2\pi\sqrt{-1}}\oint_{|z|=R}\frac{z^k}{z-f(z)}\,dz$ for large real number R implies the equalities

$$\sum_{i=1}^{d} \frac{1}{1-f'(\zeta_i)} \zeta_i^k = \begin{cases} 0 & (0 \le k \le d-2) \\ -\frac{1}{\rho} & (k = d-1) \end{cases} \tag{4.2}$$

if $\zeta_1, \ldots, \zeta_d$ are mutually distinct. Since $\lambda_i \neq 1$ for $1 \le i \le d$, the equalities $f'(\zeta_i) = \lambda_i$ for $1 \le i \le d$ imply the mutual distinctness of $\zeta_1, \ldots, \zeta_d$ and the equalities (4.2), which verifies the necessary condition of the lemma.

Suppose oppositely the equalities (4.1) and the mutual distinctness of $\zeta_1, \ldots, \zeta_d$. Note that the equalities (4.1) are equivalent to

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \zeta_1 & \zeta_2 & \cdots & \zeta_d \\ \zeta_1^2 & \zeta_2^2 & \cdots & \zeta_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_1^{d-1} & \zeta_2^{d-1} & \cdots & \zeta_d^{d-1} \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda_1} \\ \frac{1}{1-\lambda_2} \\ \vdots \\ \frac{1}{1-\lambda_d} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{\rho} \end{pmatrix}. \tag{4.3}$$

The mutual distinctness of $\zeta_1, \ldots, \zeta_d$ implies (4.2), which are equivalent to the equality obtained from (4.3) by replacing λ_i by $f'(\zeta_i)$ for $1 \le i \le d$. Therefore the invertibility of the square matrix in the left hand side of the equality (4.3) implies $\frac{1}{1-f'(\zeta_i)} = \frac{1}{1-\lambda_i}$ for $1 \le i \le d$, which completes the proof of Key Lemma. □

By Key Lemma, we associate the set $\Phi_d^{-1}\left(\bar{\lambda}\right)$ with some other one whose cardinality is expected to be easier to count. Recall that $\mathbb{P}^{d-2}$ denotes the complex projective space of dimension $d-2$.

Definition 4.2. For any $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, we put

$$\begin{aligned} T_d(\lambda) &:= \left\{ (\zeta_1 : \cdots : \zeta_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \sum_{i=1}^{d-1} \frac{1}{1-\lambda_i} \zeta_i^k = 0 \;\text{ for } 1 \le k \le d-2 \right\}, \\ S_d(\lambda) &:= \left\{ (\zeta_1 : \cdots : \zeta_{d-1}) \in T_d(\lambda) \;\middle|\; \zeta_1, \ldots, \zeta_{d-1} \text{ and } 0 \text{ are mutually distinct} \right\}, \\ B_d(\lambda) &:= T_d(\lambda) \setminus S_d(\lambda) \quad \text{and} \\ \mathfrak{S}\left(\mathcal{K}(\lambda)\right) &:= \left\{ \sigma \in \mathfrak{S}_d \;\middle|\; \lambda_{\sigma(i)} = \lambda_i \text{ holds for any } i \right\}. \end{aligned}$$

Note that $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ is a subgroup of $\mathfrak{S}_d$ determined by $\mathcal{K}(\lambda)$ and is isomorphic to the group $\mathfrak{S}_{\kappa_1} \times \cdots \times \mathfrak{S}_{\kappa_q}$, where $\kappa_1, \ldots, \kappa_q$ and $K_1, \ldots, K_q$ are those defined in Definition 1.7.

Proposition 4.3. *For $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, we can define the surjection $\pi(\lambda) : S_d(\lambda) \to \Phi_d^{-1}\left(\bar{\lambda}\right)$ by*

$$(\zeta_1 : \cdots : \zeta_{d-1}) \mapsto f(z) = z + \rho z(z-\zeta_1)\cdots(z-\zeta_{d-1}),$$

where $-\frac{1}{\rho} = \sum_{i=1}^{d-1} \frac{1}{1-\lambda_i} \zeta_i^{d-1}$. The group $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ acts on $S_d(\lambda)$ by the permutation of the coordinates $\zeta_1, \ldots, \zeta_{d-1}$ and 0, namely, it is defined by

$$\sigma \cdot (\zeta_1 : \cdots : \zeta_{d-1}) := (\zeta_{\sigma^{-1}(1)} - \zeta_{\sigma^{-1}(d)} : \cdots : \zeta_{\sigma^{-1}(d-1)} - \zeta_{\sigma^{-1}(d)}),$$

where we are assuming $\zeta_d = 0$. Finally the map $\pi(\lambda) : S_d(\lambda) \to \Phi_d^{-1}\left(\bar{\lambda}\right)$ induces the bijection

$$\overline{\pi(\lambda)} : S_d(\lambda)/\mathfrak{S}\left(\mathcal{K}(\lambda)\right) \overset{\cong}{\to} \Phi_d^{-1}\left(\bar{\lambda}\right).$$

To prove Proposition 4.3, we consider the auxiliary definitions, lemma and proposition.

Definition 4.4. We put

$$Q_d(\lambda) := \left\{ (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \;\middle|\; \begin{array}{c} \sum_{i=1}^d \frac{1}{1-\lambda_i} \zeta_i^k = 0 \quad \text{for } 1 \le k \le d-2 \\ \zeta_1, \ldots, \zeta_d \text{ are mutually distinct} \end{array} \right\},$$

and denote by G the projection map $G : \mathrm{Poly}_d \to \mathrm{MP}_d = \mathrm{Poly}_d/\mathrm{Aut}(\mathbb{C})$, where $\mathrm{Aut}(\mathbb{C})$ and its action on Poly_d are those defined in (1.1).

The groups $\mathrm{Aut}(\mathbb{C}), \mathfrak{S}_d$ and its subgroup $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ naturally act on $\mathbb{C}^d$, and the actions of $\mathrm{Aut}(\mathbb{C})$ and $\mathfrak{S}_d$ on $\mathbb{C}^d$ commute.

Lemma 4.5. *Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be an element of V_d. Then*

(1) *we can define the map $\varpi(\lambda) : Q_d(\lambda) \to G^{-1} \circ \Phi_d^{-1}\left(\bar{\lambda}\right)$ by*

$$(\zeta_1, \ldots, \zeta_d) \mapsto f(z) := z + \rho(z - \zeta_1) \cdots (z - \zeta_d),$$

where $-\frac{1}{\rho} = \sum_{i=1}^d \frac{1}{1-\lambda_i} \zeta_i^{d-1}$.

(2) *The map $\varpi(\lambda)$ is surjective.*

(3) *The set $Q_d(\lambda)$ is invariant under the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathbb{C}^d$.*

(4) *The actions of $\mathrm{Aut}(\mathbb{C})$ on $Q_d(\lambda)$ and on $G^{-1} \circ \Phi_d^{-1}\left(\bar{\lambda}\right)$ commute with the map $\varpi(\lambda)$, i.e., the equality $\varpi(\lambda)(\gamma \cdot \zeta) = \gamma \circ \varpi(\lambda)(\zeta) \circ \gamma^{-1}$ holds for any $\zeta \in Q_d(\lambda)$ and $\gamma \in \mathrm{Aut}(\mathbb{C})$.*

(5) *The set $Q_d(\lambda)$ is invariant under the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $\mathbb{C}^d$.*

(6) *For $\zeta, \zeta' \in Q_d(\lambda)$, the equality $\varpi(\lambda)(\zeta) = \varpi(\lambda)(\zeta')$ holds if and only if the equality $\zeta' = \sigma \cdot \zeta$ holds for some $\sigma \in \mathfrak{S}\left(\mathcal{K}(\lambda)\right)$.*

Proof. Most of the assertions are obvious by Key Lemma. We only check the existence of the complex number ρ in the assertion (1) and the necessary condition of the assertion (6).

If we cannot determine $\rho \in \mathbb{C}^*$, then we have $\sum_{i=1}^d \frac{1}{1-\lambda_i} \zeta_i^{d-1} = 0$, which implies $\frac{1}{1-\lambda_i} = 0$ for $1 \le i \le d$ by the equality (4.3). Hence the contradiction assures the existence of ρ.

Let $\zeta = (\zeta_1, \ldots, \zeta_d), \zeta' = (\zeta'_1, \ldots, \zeta'_d)$ be elements of $Q_d(\lambda)$ with $\varpi(\lambda)(\zeta) = \varpi(\lambda)(\zeta') =: f$. Then by the definition of $\varpi(\lambda)$, there exists a permutation $\sigma \in \mathfrak{S}_d$

with $\zeta' = \sigma \cdot \zeta$. On the other hand, by Key Lemma, we have $f'(\zeta_i) = f'(\zeta'_i) = \lambda_i$ for $1 \le i \le d$. Since $\zeta'_i = \zeta_{\sigma^{-1}(i)}$ for $1 \le i \le d$, we have $\lambda_i = \lambda_{\sigma(i)}$ for $1 \le i \le d$, which implies $\sigma \in \mathfrak{S}(\mathcal{K}(\lambda))$. Thus the necessary condition of (6) is verified. □

Definition 4.6. We put $\widetilde{Q}_d(\lambda) := Q_d(\lambda)/\mathrm{Aut}(\mathbb{C})$.

Proposition 4.7. *For $\lambda = (\lambda_1, \dots, \lambda_d) \in V_d$, the map $\varpi(\lambda)$ in Lemma 4.5 induces the surjection $\widetilde{\varpi}(\lambda) : \widetilde{Q}_d(\lambda) \to \Phi_d^{-1}\left(\bar{\lambda}\right)$. The group $\mathfrak{S}(\mathcal{K}(\lambda))$ acts on $\widetilde{Q}_d(\lambda)$, which induces the bijection*

$$\overline{\varpi(\lambda)} : \widetilde{Q}_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \to \Phi_d^{-1}\left(\bar{\lambda}\right).$$

Moreover $\widetilde{Q}_d(\lambda)$ is canonically identified with $S_d(\lambda)$ by the bijection $\iota(\lambda) : S_d(\lambda) \to \widetilde{Q}_d(\lambda)$ which maps $(\zeta_1 : \cdots : \zeta_{d-1}) \in S_d(\lambda)$ to the equivalence class of $(\zeta_1, \dots, \zeta_{d-1}, 0)$ in $\widetilde{Q}_d(\lambda)$. Under this identification, $\widetilde{\varpi}(\lambda) \circ \iota(\lambda) = \pi(\lambda)$ holds, and the actions of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$ and on $\widetilde{Q}_d(\lambda)$ commute with the map $\iota(\lambda)$.

Proof of Propositions 4.7 and 4.3. Proposition 4.7 is a direct consequence of Lemma 4.5, whereas Proposition 4.3 is just a corollary of Proposition 4.7. □

We make use of the bijection $\iota(\lambda) : S_d(\lambda) \cong \widetilde{Q}_d(\lambda)$ in the proof; in the process of determining the cardinality $\#(S_d(\lambda))$ we consider only $S_d(\lambda)$, while we utilize $\widetilde{Q}_d(\lambda)$ in the process of analyzing the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$.

Proposition 4.8. *The assertion (5) in Main Theorem I holds.*

Remark 4.9. As already mentioned in Remark 1.3, Fujimura [4, Theorem 12] proved a weaker statement of Proposition 4.8, while her proof is very similar to the following.

Proof. Since the map $G \circ \varpi(\lambda) : Q_d(\lambda) \to \Phi_d^{-1}\left(\bar{\lambda}\right)$ is surjective, it suffices to show that the set $Q_d(\lambda)$ is empty. Note first that the conditions $\sum_{i=1}^{d}(1-\lambda_i)^{-1} = 0$ and $c_1(1-\lambda_1) = \cdots = c_d(1-\lambda_d)$ imply $\sum_{i=1}^{d} c_i = 0$. We may assume that the integers $c_1, \dots, c_j$ are positive and that the rests are negative. Then the condition $\sum_{i=1}^{d} |c_i| \le 2(d-2)$ is equivalent to $\sum_{i=1}^{j} c_i = \sum_{i=j+1}^{d} -c_i \le d-2$, and the defining equations $\sum_{i=1}^{d} \frac{1}{1-\lambda_i}\zeta_i^k = 0$ for $1 \le k \le d-2$ are equivalent to the equations

$$\underbrace{\zeta_1^k + \cdots + \zeta_1^k}_{c_1} + \cdots + \underbrace{\zeta_j^k + \cdots + \zeta_j^k}_{c_j} = \underbrace{\zeta_{j+1}^k + \cdots + \zeta_{j+1}^k}_{-c_{j+1}} + \cdots + \underbrace{\zeta_d^k + \cdots + \zeta_d^k}_{-c_d}$$

for $1 \le k \le d-2$. Hence the k-th fundamental symmetric expressions of

$$\underbrace{\zeta_1, \dots, \zeta_1}_{c_1}, \dots, \underbrace{\zeta_j, \dots, \zeta_j}_{c_j} \quad \text{and} \quad \underbrace{\zeta_{j+1}, \dots, \zeta_{j+1}}_{-c_{j+1}}, \dots, \underbrace{\zeta_d, \dots, \zeta_d}_{-c_d} \tag{4.4}$$

coincide for $1 \leq k \leq d-2$. Therefore the condition $\sum_{i=1}^{j} c_i = \sum_{i=j+1}^{d} -c_i \leq d-2$ assures that the left half of (4.4) is some permutation of the right half of (4.4), which contradicts the mutual distinctness of $\zeta_1, \dots, \zeta_d$. Thus the set $Q_d(\lambda)$ is empty. $\square$

Proposition 4.10. *The assertion (7) in Main Theorem I holds.*

Proof. To prove the proposition, we may assume that $c_1, \dots, c_d$ is a permutation of $1, -1, \underbrace{2, \dots, 2}_{\frac{d}{2}-1}, \underbrace{-2, \dots, -2}_{\frac{d}{2}-1}$ or $1, 1, \underbrace{2, \dots, 2}_{\frac{d-3}{2}}, \underbrace{-2, \dots, -2}_{\frac{d-1}{2}}$ according as d is even or odd.

Let $U_{d-2}(z)$ be Chebyshev polynomial of the second kind of degree $d-2$. By definition, $U_{d-2}(z)$ is a polynomial of degree $d-2$ satisfying the equality $U_{d-2}(\cos\theta) = \sin(d-1)\theta/\sin\theta$. Put $f_d(z) = z + \rho(z^2-1)U_{d-2}(z)$ for $\rho \in \mathbb{C} \setminus \{0\}$. Then we have $\mathrm{Fix}(f_d) = \{\cos(k\pi/(d-1)) \mid k = 0, 1, \dots, d-1\}$. Moreover by a direct calculation we have $f_d'(1) = 1 + \rho \cdot 2(d-1)$, $f_d'(-1) = 1 + \rho \cdot 2(-1)^{d-1}(d-1)$ and $f_d'\left(\cos\frac{k\pi}{d-1}\right) = 1 + \rho \cdot (-1)^k(d-1)$ for $1 \leq k \leq d-2$. Hence for any $\lambda \in V_d$ with $c_1(1-\lambda_1) = \cdots = c_d(1-\lambda_d)$, we have $\Phi_d(f_d) = \bar{\lambda}$ for suitable ρ. $\square$

Remark 4.11. In practice, for any d, a similar computation to Example 3 in Section 2 assures the equality $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) = 1$ for $\lambda \in V_d$ given in the proof of Proposition 4.10. Hence the unique element of $\Phi_d^{-1}(\bar{\lambda})$ is represented by the above $f_d(z)$ for any d.

5. Review of the intersection theory on $\mathbb{P}^n$

This section summarizes the facts about the intersection theory on $\mathbb{P}^n$, and states extended Bezout's theorem in Proposition 5.3. For detailed explanation of the basic knowledge of this section, see Section 4 of Chapter 0 and Section 3 of Chapter 1 in [8].

Let C be an algebraic variety of dimension k in $\mathbb{P}^n$. Then generic $(n-k)$-plane $\mathbb{P}^{n-k} \subset \mathbb{P}^n$ intersects C transversely; we may thus define the degree of C to be the number of intersection points of C with a generic linear subspace $\mathbb{P}^{n-k}$, which does not depend on the choice of $\mathbb{P}^{n-k}$. For example, for any homogeneous polynomial $\varphi(\zeta)$ of degree d on $\mathbb{P}^n$, the degree of the zeros of φ is always d.

Secondly we remember the definition of the intersection multiplicity $\mathrm{mult}_{C_\mu}(C, C')$ of varieties C and C' in $\mathbb{P}^n$ along an irreducible component C_μ of $C \cap C'$ with $\dim C_\mu = \dim C + \dim C' - n$. If C_μ is a point, then the intersection multiplicity is defined as follows: in a local coordinate having the origin as C_μ, C meets $C' + \epsilon$ transversely around the origin for generic small $\epsilon \in \mathbb{C}^n$, where $C' + \epsilon$ denotes the translation of C' by ϵ with respect to the given local coordinate; we may thus define the intersection multiplicity $\mathrm{mult}_{C_\mu}(C, C')$ to be the number of intersection points of C and $C' + \epsilon$ around the origin for sufficiently small generic ϵ, which does not depend on the choice of ϵ nor a local coordinate. In the general case with $\dim C_\mu = \dim C + \dim C' - n$, the intersection multiplicity $\mathrm{mult}_{C_\mu}(C, C')$ is defined to be the number $\mathrm{mult}_p(C \cap H, C' \cap H)$ on H, where p is a generic smooth point of C_μ and H is a submanifold in a neighborhood of p intersecting C_μ transversely at p and with complementary dimension of C_μ.

Next we state the relation among the intersection multiplicities defined above. Let C, C' be algebraic varieties in $\mathbb{P}^n$ with $\dim C = k$ and $\dim C' = k'$, and $C_1, \dots, C_r$ the irreducible components of $C \cap C'$. Suppose that the equality $\dim C_\mu = \dim C + \dim C' - n$ holds for any μ. Then the topological intersection of C and C' is given by $(C \cdot C') = \sum_{\mu=1}^{r} \mathrm{mult}_{C_\mu}(C, C') \cdot C_\mu$, which implies the equality

$$\deg C \cdot \deg C' = \sum_{\mu=1}^{r} \mathrm{mult}_{C_\mu}(C, C') \cdot \deg C_\mu. \tag{5.1}$$

On the basis of those mentioned above, we state Definitions 5.1, 5.2 and Proposition 5.3.

Definition 5.1. We define the family $\mathcal{C}(\varphi_1, \dots, \varphi_m)$ for homogeneous polynomials $\varphi_1, \dots, \varphi_m$ on $\mathbb{P}^n$ inductively as follows: if $m = 1$, then $\mathcal{C}(\varphi_1)$ is the family of the irreducible components of the zeros of φ_1 in $\mathbb{P}^n$; in the case $m \geq 2$, putting

$$\mathcal{C}' := \left\{ C' \in \mathcal{C}(\varphi_1, \dots, \varphi_{m-1}) \;\middle|\; C' \subseteq \{\varphi_m = 0\} \right\} \quad \text{and} \quad \mathcal{C}'' := \mathcal{C}(\varphi_1, \dots, \varphi_{m-1}) \setminus \mathcal{C}',$$

we define the family $\mathcal{C}(\varphi_1, \dots, \varphi_m)$ by

$$\mathcal{C}(\varphi_1, \dots, \varphi_m) := \mathcal{C}' \cup \bigcup_{C'' \in \mathcal{C}''} \left\{ C \;\middle|\; C \text{ is an irreducible component of } C'' \cap \{\varphi_m = 0\} \right\}.$$

By definition, a variety C in $\mathbb{P}^n$ is an irreducible component of the common zeros of $\varphi_1, \dots, \varphi_m$ if and only if C is a maximal element of $\mathcal{C}(\varphi_1, \dots, \varphi_m)$ with respect to the inclusion relations. Making use of the family $\mathcal{C}(\varphi_1, \dots, \varphi_m)$, we are able to consider "components" of the common zeros which are proper subsets of some irreducible components of the common zeros.

Definition 5.2. We shall define the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ for homogeneous polynomials $\varphi_1, \dots, \varphi_m$ on $\mathbb{P}^n$ and an irreducible variety C in $\mathbb{P}^n$ with $\operatorname{codim} C = m$. If $C \notin \mathcal{C}(\varphi_1, \dots, \varphi_m)$, then we put $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m) = 0$; if $C \in \mathcal{C}(\varphi_1, \dots, \varphi_m)$, we define $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ by induction of m in the following manner: if $m = 1$, then the number $\mathrm{mult}_C(\varphi_1)$ is the usual order of zeros of φ_1 along C; in the case $m \geq 2$, the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ is defined by the equality

$$\mathrm{mult}_C(\varphi_1, \dots, \varphi_m) = \sum_{C' \in \mathcal{C}_C} \mathrm{mult}_{C'}(\varphi_1, \dots, \varphi_{m-1}) \cdot \mathrm{mult}_C(C', \varphi_m), \tag{5.2}$$

where $\mathcal{C}_C = \left\{ C' \in \mathcal{C}(\varphi_1, \dots, \varphi_{m-1}) \;\middle|\; \operatorname{codim} C' = m - 1,\ C \subseteq C',\ C' \nsubseteq \{\varphi_m = 0\} \right\}$. Here, for a homogeneous polynomial φ, an irreducible variety C' with $C' \nsubseteq \{\varphi = 0\}$ and an irreducible component C of $C' \cap \{\varphi = 0\}$, the number $\mathrm{mult}_C(C', \varphi)$ is defined by

$$\mathrm{mult}_C(C', \varphi) := \sum_{C'' \in \mathcal{C}(\varphi),\ C \subseteq C''} \mathrm{mult}_C(C', C'') \cdot \mathrm{mult}_{C''}(\varphi).$$

Note that the notation $\mathrm{mult}_C(C', \varphi)$ is also used in the following sections.

At any rate, Definition 5.2 assigns a positive integer $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ to each $C \in \mathcal{C}(\varphi_1, \dots, \varphi_m)$ with $\operatorname{codim} C = m$. By definition, if C is an irreducible component of the common zeros of $\varphi_1, \dots, \varphi_m$ with $\operatorname{codim} C = m$, then the number $\mathrm{mult}_C(\varphi_1, \dots, \varphi_m)$ defined above is the usual intersection multiplicity of $\varphi_1, \dots, \varphi_m$ along C. We state the relation among the numbers defined above in Proposition 5.3.

Proposition 5.3. *Let $\varphi_1, \dots, \varphi_n$ be homogeneous polynomials on $\mathbb{P}^n$, put $\operatorname{codim} C =: l_C$ for each $C \in \mathcal{C}(\varphi_1, \dots, \varphi_n)$, and suppose that the inclusion relation*

$$\left\{C \in \mathcal{C}(\varphi_1, \dots, \varphi_k) \;\middle|\; \operatorname{codim} C < k\right\} \subseteq \mathcal{C}(\varphi_1, \dots, \varphi_n) \tag{5.3}$$

holds for every $1 \le k \le n$. Then we have $\left\{C \in \mathcal{C}(\varphi_1, \dots, \varphi_n) \;\middle|\; l_C = k\right\} \subseteq \mathcal{C}(\varphi_1, \dots, \varphi_k)$ for every $1 \le k \le n$. Moreover we have the equality

$$\prod_{k=1}^{n} \deg \varphi_k = \sum_{C \in \mathcal{C}(\varphi_1, \dots, \varphi_n)} \left(\deg C \cdot \mathrm{mult}_C(\varphi_1, \dots, \varphi_{l_C}) \cdot \prod_{k=l_C+1}^{n} \deg \varphi_k \right). \tag{5.4}$$

Here, in the case $l_C = n$, we assume that $\prod_{k=l_C+1}^{n} \deg \varphi_k = \prod_{k=n+1}^{n} \deg \varphi_k = 1$.

Proof. We put $\mathcal{C}_k := \left\{C \in \mathcal{C}(\varphi_1, \dots, \varphi_k) \;\middle|\; \operatorname{codim} C = k,\ C \subseteq \{\varphi_{k+1} = 0\}\right\}$ for each $1 \le k \le n-1$. Then by Definition 5.1 and the assumption (5.3), we have $\mathcal{C}_1 \amalg \cdots \amalg \mathcal{C}_k \subseteq \mathcal{C}(\varphi_1, \dots, \varphi_k)$ and $\left\{C \in \mathcal{C}(\varphi_1, \dots, \varphi_k) \;\middle|\; \operatorname{codim} C = k\right\} = \mathcal{C}(\varphi_1, \dots, \varphi_k) \setminus (\mathcal{C}_1 \amalg \cdots \amalg \mathcal{C}_{k-1})$ for every $1 \le k \le n-1$, which implies the former assertion of the proposition.

To prove the latter, it suffices to show the equality

$$\prod_{l=1}^{k} \deg \varphi_l = \sum_{C \in \mathcal{C}(\varphi_1, \dots, \varphi_k)} \left(\deg C \cdot \mathrm{mult}_C(\varphi_1, \dots, \varphi_{l_C}) \cdot \prod_{l=l_C+1}^{k} \deg \varphi_l \right) \tag*{$(5.5)_k$}$$

by induction of k, because $(5.5)_n$ is the same as (5.4). The equality $(5.5)_1$ is in the form $\deg \varphi_1 = \sum_{C \in \mathcal{C}(\varphi_1)} \deg C \cdot \mathrm{mult}_C(\varphi_1)$, which obviously holds. Multiplying both sides of the equality $(5.5)_k$ by $\deg \varphi_{k+1}$, we have

$$\begin{aligned} \prod_{l=1}^{k+1} \deg \varphi_l &= \deg \varphi_{k+1} \cdot \sum_{C \in \mathcal{C}(\varphi_1, \dots, \varphi_k)} \left(\deg C \cdot \mathrm{mult}_C(\varphi_1, \dots, \varphi_{l_C}) \cdot \prod_{l=l_C+1}^{k} \deg \varphi_l \right) \\ &= \sum_{C \in \mathcal{C}_1 \amalg \cdots \amalg \mathcal{C}_k} \left(\deg C \cdot \mathrm{mult}_C(\varphi_1, \dots, \varphi_{l_C}) \cdot \prod_{l=l_C+1}^{k+1} \deg \varphi_l \right) \end{aligned}$$

$$+ \sum_{C\in\mathcal{C}(\varphi_1,\ldots,\varphi_k)\setminus(\mathcal{C}_1\amalg\cdots\amalg\mathcal{C}_k)} \Big(\deg\varphi_{k+1}\cdot\deg C\cdot \mathrm{mult}_C(\varphi_1,\ldots,\varphi_k)\Big).$$

We put $\mathcal{C}'_k := \mathcal{C}(\varphi_1,\ldots,\varphi_k)\setminus(\mathcal{C}_1\amalg\cdots\amalg\mathcal{C}_k)$ for the brevity of notation. Then for every $C\in\mathcal{C}'_k$, we have $\deg\varphi_{k+1}\cdot\deg C = \sum_{\mu=1}^{r}\mathrm{mult}_{C_\mu}(C,\varphi_{k+1})\cdot\deg C_\mu$ by (5.1) and by the definition of $\mathrm{mult}_{C_\mu}(C,\varphi_{k+1})$, where $C_1,\ldots,C_r$ are the irreducible components of $C\cap\{\varphi_{k+1}=0\}$. Therefore, putting $\mathrm{mult}_{C'}(C,\varphi_{k+1})=0$ for C' different from $C_1,\ldots,C_r$, we have

$$\begin{aligned}
&\sum_{C\in\mathcal{C}'_k}\Big(\deg\varphi_{k+1}\cdot\deg C\cdot\mathrm{mult}_C(\varphi_1,\ldots,\varphi_k)\Big)\\
&=\sum_{C\in\mathcal{C}'_k}\left(\left(\sum_{C'\in\mathcal{C}(\varphi_1,\ldots,\varphi_{k+1})\setminus(\mathcal{C}_1\amalg\cdots\amalg\mathcal{C}_k)}\mathrm{mult}_{C'}(C,\varphi_{k+1})\cdot\deg C'\right)\cdot\mathrm{mult}_C(\varphi_1,\ldots,\varphi_k)\right)\\
&=\sum_{C'\in\mathcal{C}(\varphi_1,\ldots,\varphi_{k+1})\setminus(\mathcal{C}_1\amalg\cdots\amalg\mathcal{C}_k)}\left(\deg C'\cdot\left(\sum_{C\in\mathcal{C}'_k}\mathrm{mult}_C(\varphi_1,\ldots,\varphi_k)\cdot\mathrm{mult}_{C'}(C,\varphi_{k+1})\right)\right)\\
&=\sum_{C'\in\mathcal{C}(\varphi_1,\ldots,\varphi_{k+1})\setminus(\mathcal{C}_1\amalg\cdots\amalg\mathcal{C}_k)}\Big(\deg C'\cdot\mathrm{mult}_{C'}(\varphi_1,\ldots,\varphi_{k+1})\Big) \qquad (5.6)
\end{aligned}$$

by Definition 5.2. To summarize, we have $(5.5)_{k+1}$. □

Proposition 5.3 is reduced to the usual Bezout's theorem if $\mathcal{C}(\varphi_1,\ldots,\varphi_n)$ consists only of points. Proposition 5.3 is utilized crucially for determining the cardinality $\#\left(S_d(\lambda)\right)$ in Section 6.

Remark 5.4. The family $\mathcal{C}(\varphi_1,\ldots,\varphi_m)$ and the number $\mathrm{mult}_C(\varphi_1,\ldots,\varphi_m)$ may vary when the order of $\varphi_1,\ldots,\varphi_m$ changes. Hence Definitions 5.1 and 5.2 may appear to be a little strange in some sense; however this works very well for the computation of the cardinality $\#\left(S_d(\lambda)\right)$. In the following, we give an example in which the family $\mathcal{C}(\varphi_1,\varphi_2)$ and the number $\mathrm{mult}_{\mathrm{P}_2}(\varphi_1,\varphi_2)$ differ from $\mathcal{C}(\varphi_2,\varphi_1)$ and $\mathrm{mult}_{\mathrm{P}_2}(\varphi_2,\varphi_1)$ respectively. Consider $\varphi_1 = y(y-x)$ and $\varphi_2 = y\left(yz^2+x^3-2x^2z\right)$ on $\mathbb{P}^2=\{(x:y:z)\}$. We put $\mathrm{P}_1=\{(1:1:1)\}$, $\mathrm{P}_2=\{(0:0:1)\}$, $\mathrm{P}_3=\{(2:0:1)\}$, $C_0=\{y=0\}$, $C_1=\{x=y\}$ and $C_2=\{yz^2+x^3-2x^2z=0\}$. Then we have $\mathcal{C}(\varphi_1,\varphi_2)=\{C_0,\mathrm{P}_1,\mathrm{P}_2\}$ and $\mathcal{C}(\varphi_2,\varphi_1)=\{C_0,\mathrm{P}_1,\mathrm{P}_2,\mathrm{P}_3\}$. Moreover we have $\mathrm{mult}_{\mathrm{P}_2}(\varphi_1,\varphi_2)=\mathrm{mult}_{C_1}(\varphi_1)\cdot\mathrm{mult}_{\mathrm{P}_2}(C_1,\varphi_2)=1\cdot 2=2$ and $\mathrm{mult}_{\mathrm{P}_2}(\varphi_2,\varphi_1)=\mathrm{mult}_{C_2}(\varphi_2)\cdot\mathrm{mult}_{\mathrm{P}_2}(C_2,\varphi_1)=1\cdot 3=3$. However Proposition 5.3 holds as we will see

$$\begin{aligned}
&\deg C_0\cdot\mathrm{mult}_{C_0}(\varphi_1)\cdot\deg\varphi_2+\deg\mathrm{P}_1\cdot\mathrm{mult}_{\mathrm{P}_1}(\varphi_1,\varphi_2)+\deg\mathrm{P}_2\cdot\mathrm{mult}_{\mathrm{P}_2}(\varphi_1,\varphi_2)\\
&\qquad\qquad\qquad\qquad\qquad = 1\cdot1\cdot4+1\cdot2+1\cdot2=8=\deg\varphi_1\cdot\deg\varphi_2,\\
&\deg C_0\cdot\mathrm{mult}_{C_0}(\varphi_2)\cdot\deg\varphi_1+\deg\mathrm{P}_1\cdot\mathrm{mult}_{\mathrm{P}_1}(\varphi_2,\varphi_1)+\deg\mathrm{P}_2\cdot\mathrm{mult}_{\mathrm{P}_2}(\varphi_2,\varphi_1)\\
&\qquad +\deg\mathrm{P}_3\cdot\mathrm{mult}_{\mathrm{P}_3}(\varphi_2,\varphi_1)=1\cdot1\cdot2+1\cdot2+1\cdot3+1\cdot1=8=\deg\varphi_2\cdot\deg\varphi_1.
\end{aligned}$$

6. Outline of determining the cardinality $\#(S_d(\lambda))$

In this section we give an outline of determining the cardinality of the set $S_d(\lambda)$ defined in Definition 4.2 for each $\lambda \in V_d$. The assertions (1) and (4) in Main Theorem I are also proved in this section.

For the brevity of notation we put

$$m_i := \frac{1}{1-\lambda_i} \quad \text{and} \quad \varphi_k(\zeta) := \sum_{i=1}^{d-1} m_i \zeta_i^k$$

for each i and k, and we always assume that $\zeta_d = 0$. Therefore $T_d(\lambda)$ is the set of the common zeros of $\varphi_1, \ldots, \varphi_{d-2}$ in $\mathbb{P}^{d-2}$, and $S_d(\lambda)$ consists of an element $\zeta = (\zeta_1 : \cdots : \zeta_{d-1}) \in T_d(\lambda)$ with mutually distinct $\zeta_1, \ldots, \zeta_{d-1}$ and ζ_d. Moreover we may also consider that $\varphi_k(\zeta) = \sum_{i=1}^{d} m_i \zeta_i^k$.

Lemma 6.1. *Let λ be an element of V_d. Then $S_d(\lambda)$ is discrete in $\mathbb{P}^{d-2}$. Moreover we always have* $\operatorname{mult}_{\zeta_0}(\varphi_1, \ldots, \varphi_{d-2}) = 1$ *for any $\zeta_0 \in S_d(\lambda)$.*

Proof. We consider the row vectors

$$\frac{\partial \varphi_k}{\partial \zeta} = \left(\frac{\partial \varphi_k}{\partial \zeta_1}, \ldots, \frac{\partial \varphi_k}{\partial \zeta_{d-1}}\right) = \left(k m_1 \zeta_1^{k-1}, \ldots, k m_{d-1} \zeta_{d-1}^{k-1}\right)$$

at $\zeta = \zeta_0 \in S_d(\lambda)$ for $1 \le k \le d-1$. Since $\zeta_1, \ldots, \zeta_{d-1}$ are mutually distinct at $\zeta = \zeta_0$ and since $m_i \neq 0$ for any i, we have

$$\det {}^t\!\left({}^t\!\left(\frac{\partial \varphi_1}{\partial \zeta}\right), \ldots, {}^t\!\left(\frac{\partial \varphi_{d-1}}{\partial \zeta}\right)\right) = (d-1)! \cdot \prod_{i=1}^{d-1} m_i \cdot \det \begin{pmatrix} 1 & \cdots & 1 \\ \zeta_1 & \cdots & \zeta_{d-1} \\ \vdots & \ddots & \vdots \\ \zeta_1^{d-2} & \cdots & \zeta_{d-1}^{d-2} \end{pmatrix} \neq 0.$$

Therefore the row vectors $\frac{\partial \varphi_1}{\partial \zeta}, \ldots, \frac{\partial \varphi_{d-2}}{\partial \zeta}$ are linearly independent at $\zeta = \zeta_0$, which proves the lemma. $\square$

Proposition 6.2. *The assertion (1) in Main Theorem I holds*

Proof. Since the map $\pi(\lambda) : S_d(\lambda) \to \Phi_d^{-1}(\bar{\lambda})$ is surjective, it suffices to show the inequality $\#(S_d(\lambda)) \le (d-2)!$ for $\lambda \in V_d$. The following argument is similar to the proof of Proposition 5.3.

Note first that if C, C' are irreducible varieties in $\mathbb{P}^{d-2}$ with $\operatorname{codim} C' = 1$ and $C \not\subseteq C'$, then all the irreducible components of $C \cap C'$ have codimension $\operatorname{codim} C + 1$. Hence a component $C \in \mathcal{C}(\varphi_1, \ldots, \varphi_k)$ with $\operatorname{codim} C < k$ does not "generate" any elements of $S_d(\lambda)$ since all the components of $S_d(\lambda)$ have "codim $= d-2$" by the discreteness of $S_d(\lambda)$. Therefore putting $\mathcal{C}_1 = \mathcal{C}(\varphi_1)$ and

$$\mathcal{C}'_k := \{C \in \mathcal{C}_k \mid C \nsubseteq \{\varphi_{k+1} = 0\}\},$$
$$\mathcal{C}_{k+1} := \bigcup_{C' \in \mathcal{C}'_k} \{C \mid C \text{ is an irreducible component of } C' \cap \{\varphi_{k+1} = 0\}\}$$

inductively, we have $\operatorname{codim} C = k$ for every $C \in \mathcal{C}_k$ and also have $\{\{\zeta\} \mid \zeta \in S_d(\lambda)\} \subseteq \mathcal{C}_{d-2}$. Here, note that $\mathcal{C}_k$ and $\mathcal{C}'_k$ above are different from those defined in the proof of Proposition 5.3. Applying the equalities (5.1) and (5.2) repeatedly, we have

$$\begin{aligned}
\deg \varphi_1 &= \sum_{C_1 \in \mathcal{C}_1} \operatorname{mult}_{C_1}(\varphi_1) \cdot \deg C_1 \\
\deg \varphi_2 \cdot \sum_{C_1 \in \mathcal{C}'_1} \operatorname{mult}_{C_1}(\varphi_1) \cdot \deg C_1 &= \sum_{C_2 \in \mathcal{C}_2} \operatorname{mult}_{C_2}(\varphi_1, \varphi_2) \cdot \deg C_2 \\
\deg \varphi_3 \cdot \sum_{C_2 \in \mathcal{C}'_2} \operatorname{mult}_{C_2}(\varphi_1, \varphi_2) \cdot \deg C_2 &= \sum_{C_3 \in \mathcal{C}_3} \operatorname{mult}_{C_3}(\varphi_1, \varphi_2, \varphi_3) \cdot \deg C_3 \\
&\vdots \\
\deg \varphi_{d-2} \cdot \sum_{C_{d-3} \in \mathcal{C}'_{d-3}} \operatorname{mult}_{C_{d-3}}(\varphi_1, \ldots, \varphi_{d-3}) \cdot \deg C_{d-3} & \\
= \sum_{C_{d-2} \in \mathcal{C}_{d-2}} \operatorname{mult}_{C_{d-2}}(\varphi_1, \ldots, \varphi_{d-2}) \cdot \deg C_{d-2} &
\end{aligned}$$

by a similar calculation to (5.6). Hence we have

$$\prod_{k=1}^{d-2} \deg \varphi_k \geq \sum_{C_{d-2} \in \mathcal{C}_{d-2}} \operatorname{mult}_{C_{d-2}}(\varphi_1, \ldots, \varphi_{d-2}) \cdot \deg C_{d-2} \geq \#(\mathcal{C}_{d-2}) \geq \#(S_d(\lambda)).$$

Since $\deg \varphi_k = k$, we have $\#(S_d(\lambda)) \leq \prod_{k=1}^{d-2} \deg \varphi_k = \prod_{k=1}^{d-2} k = (d-2)!$, which completes the proof of the proposition $\square$

Proposition 5.3 and Lemma 6.1 imply that in order to determine the cardinality $\#(S_d(\lambda))$, we only need to find the degree $\deg C$ and the number $\operatorname{mult}_C(\varphi_1, \ldots, \varphi_{d-l})$ for each $2 \leq l \leq d-1$ and $C \in \mathcal{C}(\varphi_1, \ldots, \varphi_{d-2})$ with $\dim C = l-2$ included in $B_d(\lambda)$. To state the explicit expression of the set $B_d(\lambda)$, we shall make a definition of $E_d(\mathbb{I})$ for each $\mathbb{I} \in \mathfrak{I}(\lambda)$. Recall the definition of $\mathfrak{I}(\lambda)$ for $\lambda \in V_d$ defined in Definition 1.7.

Definition 6.3. Let λ be an element of V_d. For each $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we define the subset $E_d(\mathbb{I})$ of $\mathbb{P}^{d-2}$ by

$$E_d(\mathbb{I}) := \left\{ (\zeta_1 : \cdots : \zeta_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \begin{array}{c} \text{If } i, j \in \{1, \ldots, d\} \text{ belong to the same } I_u \\ \text{for some } u, \text{ then } \zeta_i = \zeta_j \text{ holds.} \end{array} \right\}.$$

In the definition of $E_d(\mathbb{I})$, we are assuming $\zeta_d = 0$. By definition, the relation $\mathbb{I} \prec \mathbb{I}'$ holds for $\mathbb{I}, \mathbb{I}' \in \mathfrak{I}(\lambda)$ if and only if $E_d(\mathbb{I}) \subseteq E_d(\mathbb{I}')$ holds. Moreover if $\#(\mathbb{I}) = l$, then $E_d(\mathbb{I})$ is an $(l-2)$-dimensional complex plane in $\mathbb{P}^{d-2}$; hence the degree of $E_d(\mathbb{I})$ is always 1. To help the reader to understand the definition of $E_d(\mathbb{I})$, we give an example.

Example 4. Let us consider again $\lambda \in V_6$ with $m_1 : \cdots : m_6 = 1 : 1 : 2 : -1 : -1 : -2$ introduced in Example 1. The notation follows that in Example 1. In this case, we have

$$E_6(\mathbb{I}_1) = \left\{(\zeta_1 : \zeta_2 : 0 : \zeta_1 : \zeta_2) \in \mathbb{P}^4 \;\middle|\; (\zeta_1 : \zeta_2) \in \mathbb{P}^1\right\},$$

$$E_6(\mathbb{I}_2) = \left\{(\zeta_1 : \zeta_2 : 0 : \zeta_2 : \zeta_1) \in \mathbb{P}^4 \;\middle|\; (\zeta_1 : \zeta_2) \in \mathbb{P}^1\right\},$$

$$E_6(\mathbb{I}_3) = \{(1:1:0:1:1)\},\ E_6(\mathbb{I}_4) = \{(1:0:0:1:0)\},\ E_6(\mathbb{I}_5) = \{(0:1:0:0:1)\},$$

$$E_6(\mathbb{I}_6) = \{(1:0:0:0:1)\},$$

$$E_6(\mathbb{I}_7) = \{(0:1:0:1:0)\} \text{ and } E_6(\mathbb{I}_8) = \{(0:0:1:1:1)\}.$$

$E_6(\mathbb{I}_1)$ and $E_6(\mathbb{I}_2)$ are complex lines in $\mathbb{P}^4$, whereas $E_6(\mathbb{I}_\omega)$ are points for $3 \le \omega \le 8$. We have $E_6(\mathbb{I}_\omega) \subset E_6(\mathbb{I}_1)$ for $\omega = 3, 4$ and 5, and $E_6(\mathbb{I}_\omega) \subset E_6(\mathbb{I}_2)$ for $\omega = 3, 6$ and 7.

Remark 6.4. Since we always have the equality $\sum_{i=1}^{d} m_i = 0$, we have

$$\mathfrak{I}(\lambda) = \left\{\mathbb{I} \subseteq \mathcal{I}(\lambda) \;\middle|\; \coprod_{I\in\mathbb{I}} I = \{1, \ldots, d\}\right\} \quad \text{and} \quad \mathcal{I}(\lambda) = \bigcup_{\mathbb{I}\in\mathfrak{I}(\lambda)} \mathbb{I}.$$

Hence $\mathfrak{I}(\lambda)$ gives the equivalent information as $\mathcal{I}(\lambda)$.

Now we are in a position to give the explicit expression of the set $B_d(\lambda)$.

Lemma 6.5. *Let λ be an element of V_d. Then we have the equality*

$$B_d(\lambda) = \bigcup_{\mathbb{I}\in\mathfrak{I}(\lambda)} E_d(\mathbb{I}). \tag{6.1}$$

More strictly, $B_d(\lambda)$ is a union of $E_d(\mathbb{I})$ only for maximal elements $\mathbb{I}$ of $\mathfrak{I}(\lambda)$ as set. However as we will see later in Example 5, it is better to consider components $E_d(\mathbb{I})$ for $\mathbb{I}$ which are not necessarily maximal in $\mathfrak{I}(\lambda)$. Note that the equality (6.1) is only an equality as set.

Proof. For any point $\zeta_0 = (\zeta_1 : \cdots : \zeta_{d-1}) \in B_d(\lambda)$, we put

$$\mathbb{I}(\zeta_0) := \left\{ I \subsetneq \{1,2,\ldots,d\} \;\middle|\; \begin{array}{c} I \neq \emptyset. \quad \text{ If } i,j \in I, \text{ then } \zeta_i = \zeta_j. \\ \text{If } i \in I \text{ and } j \in \{1,2,\ldots,d\} \setminus I, \text{ then } \zeta_i \neq \zeta_j. \end{array} \right\},$$

$\#\left(\mathbb{I}(\zeta_0)\right) =: l$, $\mathbb{I}(\zeta_0) =: \{I_1,\ldots,I_l\}$ and $\alpha_u := \zeta_i$ for $i \in I_u$ for each $1 \le u \le l$. Then by definition, $\{1,2,\ldots,d\}$ is a disjoint union of $I_1,\ldots,I_l$, and $\alpha_1,\ldots,\alpha_l$ are mutually distinct, one of which is zero since $\zeta_d = 0$ and $d \in I_u$ for some $1 \le u \le l$. Moreover since $\zeta_0 \in B_d(\lambda)$, we have $2 \le l \le d-1$.

Under the notation above, the defining equations $\varphi_k(\zeta_0) = \sum_{u=1}^{l} \left(\sum_{i \in I_u} m_i\right) \alpha_u^k = 0$ for $1 \le k \le d-2$ are equivalent to the equality

$$\begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_l \\ \vdots & \ddots & \vdots \\ \alpha_1^{d-2} & \cdots & \alpha_l^{d-2} \end{pmatrix} \begin{pmatrix} \sum_{i \in I_1} m_i \\ \vdots \\ \sum_{i \in I_l} m_i \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

which implies $\sum_{i \in I_u} m_i = 0$ for $1 \le u \le l$ since $l \le d-1$. Therefore we have $\mathbb{I}(\zeta_0) \in \mathfrak{I}(\lambda)$ and $\zeta_0 \in E_d\left(\mathbb{I}(\zeta_0)\right)$ for any $\zeta_0 \in B_d(\lambda)$, which assures $B_d(\lambda) \subseteq \bigcup_{\mathbb{I} \in \mathfrak{I}(\lambda)} E_d(\mathbb{I})$. The opposite inclusion relation is clear, which completes the proof of the lemma. $\square$

Proposition 6.6. *The assertion (4) in Main Theorem I holds.*

Proof. By Proposition 4.3, the equality $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right) = (d-2)!$ holds if and only if $\#\left(S_d(\lambda)\right) = (d-2)!$ holds and that the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ is trivial. Here, if $\lambda_i = \lambda_j$ holds for some $i \neq j$, then the action of the permutation $(i,j) \in \mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ is not trivial since $d \ge 4$. Hence the action of $\mathfrak{S}\left(\mathcal{K}(\lambda)\right)$ on $S_d(\lambda)$ is trivial if and only if $\lambda_1,\ldots,\lambda_d$ are mutually distinct. Moreover by Lemma 6.5, $\mathcal{I}(\lambda)$ is empty if and only if $B_d(\lambda)$ is empty. Hence to complete the proof of the proposition, we only need to show that the condition $\#\left(S_d(\lambda)\right) = (d-2)!$ is equivalent to the condition that $B_d(\lambda)$ is empty.

In the following, we use notations defined in the proof of Proposition 6.2. Looking at the proof of Proposition 6.2 carefully, we can find that the condition $\#\left(S_d(\lambda)\right) = (d-2)!$ is equivalent to the conditions

$$\mathcal{C}'_k = \mathcal{C}_k \text{ for } 1 \le k \le d-3 \quad \text{and} \quad \mathcal{C}_{d-2} = S_d(\lambda), \tag{6.2}$$

since $\deg P = 1$ for a point P and $\operatorname{mult}_\zeta(\varphi_1,\ldots,\varphi_{d-2}) = 1$ for $\zeta \in S_d(\lambda)$. Here, we identify $\zeta \in S_d(\lambda)$ with $\{\zeta\}$ by abuse of notation.

If the conditions $\mathcal{C}'_k = \mathcal{C}_k$ hold for every $1 \le k \le d-3$, then the set of common zeros of $\varphi_1,\ldots,\varphi_{d-2}$, which we denote by $T_d(\lambda)$ in this paper, consists of discrete points; hence we have $T_d(\lambda) = B_d(\lambda) \amalg S_d(\lambda) = \mathcal{C}_{d-2}$. Therefore in this case, $\mathcal{C}_{d-2} = S_d(\lambda)$ holds if and only if $B_d(\lambda)$ is empty.

On the other hand, if $\mathcal{C}'_k \subsetneq \mathcal{C}_k$ for some $1 \le k \le d-3$, then for $C_k \in \mathcal{C}_k \setminus \mathcal{C}'_k$, all the irreducible components of $C_k \cap \{\varphi_{k+2} = \cdots = \varphi_{d-2} = 0\}$ have codimension greater than or equal to 1. Hence in this case $T_d(\lambda) = B_d(\lambda) \amalg S_d(\lambda)$ contains components greater than or equal to 1. Since $S_d(\lambda)$ consists of discrete points, $B_d(\lambda)$ is not empty.

To summarize, we have shown that the condition (6.2) is equivalent to the condition that $B_d(\lambda)$ is empty, which completes the proof of the proposition. $\square$

In the rest of this section we give an example and some theorems that exactly give the number $\mathrm{mult}_C(\varphi_1, \ldots, \varphi_{d-l})$ for each $C \in \mathcal{C}(\varphi_1, \ldots, \varphi_{d-2})$ with $\dim C = l - 2$. However their proofs, which are the most crucial and difficult part in the proof of the main theorems, will be given later in Sections 7 and 8.

Theorem A. *Let λ be an element of V_d, and $\mathbb{I} = \{I_1, \ldots, I_l\}$ a maximal element of $\mathfrak{I}(\lambda)$. Then $E_d(\mathbb{I})$ is an irreducible component of the common zeros of $\varphi_1, \ldots, \varphi_{d-l}$ with its intersection multiplicity*

$$\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l}) = \prod_{u=1}^{l} (\#(I_u) - 1)!.$$

Example 5. We consider again $\lambda \in V_6$ introduced in Examples 1 and 4. The notation follows that in Examples 1 and 4 again. In this case, we have $\Phi_6^{-1}\left(\bar{\lambda}\right) = \emptyset$ by the assertion (5) in Main Theorem I, which implies $S_6(\lambda) = \emptyset$. Hence in this example, we verify $S_6(\lambda) = \emptyset$ by the calculation of intersection multiplicities.

By Example 4 and Lemma 6.5, we have $B_6(\lambda) = E_6(\mathbb{I}_1) \cup E_6(\mathbb{I}_2) \cup E_6(\mathbb{I}_8)$ as set. Moreover by Theorem A, we have $\mathrm{mult}_{E_6(\mathbb{I}_\omega)}(\varphi_1, \varphi_2, \varphi_3) = ((2-1)!)^3 = 1$ for $\omega = 1, 2$, and $\mathrm{mult}_{E_6(\mathbb{I}_8)}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = ((3-1)!)^2 = 4$. Hence the common zeros of φ_1, φ_2 and φ_3 are composed of $E_6(\mathbb{I}_1), E_6(\mathbb{I}_2)$ and some curve C whose degree is $\deg C = 3! - (1+1) = 4$. Moreover since $\deg C \cdot \deg \varphi_4 = 4 \cdot 4 = 16$, we have

$$\#\left(S_6(\lambda)\right) = 16 - \sum_{\zeta \in C \cap \{\varphi_4(\zeta)=0\} \cap B_6(\lambda)} \mathrm{mult}_{\{\zeta\}}(C, \varphi_4)$$

by (5.1). Here, we have $E_6(\mathbb{I}_8) \subseteq C \cap \{\varphi_4(\zeta) = 0\} \cap B_6(\lambda)$ with $\mathrm{mult}_{E_6(\mathbb{I}_8)}(C, \varphi_4) = 4$.

What occurs in the difference "$16 - 4 = 12$"? It appears to be correct that $\#\left(S_6(\lambda)\right) = 12$; however this is not the case. In practice, the curve C intersects the lines $E_6(\mathbb{I}_1)$ and $E_6(\mathbb{I}_2)$. Precisely, the intersection points of the two curves C and $E_6(\mathbb{I}_1)$ are $E_6(\mathbb{I}_4)$ and $E_6(\mathbb{I}_5)$, while those of C and $E_6(\mathbb{I}_2)$ are $E_6(\mathbb{I}_6)$ and $E_6(\mathbb{I}_7)$; these four points do belong to the intersection $C \cap \{\varphi_4(\zeta) = 0\} \cap B_6(\lambda)$. Moreover as we will see in Theorem B, we have $\mathrm{mult}_{E_6(\mathbb{I}_\omega)}(C, \varphi_4) = \mathrm{mult}_{E_6(\mathbb{I}_\omega)}(\varphi_1, \ldots, \varphi_4) = 3$ for $4 \le \omega \le 7$. We thus have the equality $16 - (4 + 3 + 3 + 3 + 3) = 0$, which assures that $S_6(\lambda)$ is empty and that the intersection points of C and $\{\varphi_4(\zeta) = 0\}$ are $E_6(\mathbb{I}_\omega)$ for $4 \le \omega \le 8$, which does not cause any contradiction. To summarize, the family $\mathcal{C}(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ consists of $E_6(\mathbb{I}_\omega)$ for $\omega = 1, 2, 4, 5, 6, 7$ and 8, and the equality

$$4! - (1 \cdot 4 + 1 \cdot 4 + 3 + 3 + 3 + 3 + 4) = 0$$

implies that $S_6(\lambda)$ is an empty set.

As a conclusion of Example 5, we comment about the component $E_6(\mathbb{I}_3)$. The point $E_6(\mathbb{I}_3)$ may also appear as an element of $\mathcal{C}(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$. However by Theorem B below, we have $\mathrm{mult}_{E_6(\mathbb{I}_3)}(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = 0$, which means that in practice $E_6(\mathbb{I}_3)$ is not an element of $\mathcal{C}(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$.

By Example 5, we found that in order to count the number of the set $S_d(\lambda)$, we must also consider the "intersection multiplicities" of "components" which are proper subsets of $E_d(\mathbb{I})$ for some maximal $\mathbb{I} \in \mathfrak{I}(\lambda)$.

To state Theorem B, we need the following symbol:

Definition 6.7. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$ and $I \in \mathcal{I}(\lambda)$, we put $\lambda_I := (\lambda_i)_{i \in I}$.

Note that λ_I always belongs to $V_{\#(I)}$ by definition.

Theorem B. *Let λ be an element of V_d. Then*

(1) *we have* $\left\{C \in \mathcal{C}(\varphi_1, \ldots, \varphi_{d-2}) \;\middle|\; C \subseteq B_d(\lambda)\right\} \subseteq \left\{E_d(\mathbb{I}) \;\middle|\; \mathbb{I} \in \mathfrak{I}(\lambda)\right\}$.
(2) *For any $2 \le l \le d-1$ we have*

$$\left\{C \in \mathcal{C}(\varphi_1, \ldots, \varphi_{d-l}) \;\middle|\; \dim C > l-2\right\} \subseteq \left\{E_d(\mathbb{I}) \;\middle|\; \mathbb{I} \in \mathfrak{I}(\lambda)\right\}.$$

(3) *For any $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we have*

$$\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l}) = \prod_{u=1}^{l} \left(\left(\# (I_u) - 1 \right) \cdot \# \left(S_{\#(I_u)} (\lambda_{I_u}) \right) \right), \tag{6.3}$$

where the cardinality $\#\left(S_{\#(I_u)}(\lambda_{I_u})\right)$ is defined to be 1 if $\#(I_u)$ is equal to or smaller than 3.

By Proposition 5.3 and Theorem B, the variety $E_d(\mathbb{I})$ for $\mathbb{I} \in \mathfrak{I}(\lambda)$ is really an element of $\mathcal{C}(\varphi_1, \ldots, \varphi_{d-2})$, if and only if the right hand side of the equality (6.3) is strictly positive.

Remark 6.8. If an element $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$ is maximal, then $\mathfrak{I}(\lambda_{I_u})$ is empty for every u, which implies $\#\left(S_{\#(I_u)}(\lambda_{I_u})\right) = (\#(I_u) - 2)!$ by Definition 4.2, Lemmas 6.1 and 6.5. Thus Theorem A is a special case of Theorem B.

By Proposition 5.3 and Theorem B, we have the following:

Proposition C. *Let λ be an element of V_d. Then we have the equality*

$$\#\left(S_d(\lambda)\right) = (d-2)! - \sum_{\mathbb{I}\in\mathfrak{I}(\lambda)} \left(\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-\#(\mathbb{I})}) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-2} k \right). \tag{6.4}$$

Here, for $\mathbb{I} \in \mathfrak{I}(\lambda)$ with $\#(\mathbb{I}) = 2$, we assume that $\prod_{k=d-\#(\mathbb{I})+1}^{d-2} k = \prod_{k=d-1}^{d-2} k = 1$.

As we have seen in Theorem B and Proposition C, the cardinality $\#\left(S_d(\lambda)\right)$ is completely determined by the data $\mathfrak{I}(\lambda)$. Moreover it is practically computed only by hand, though the process of its computation may be rather long or complicated. To relieve the long computation, we give one more proposition.

Proposition D. *For $\lambda \in V_d$ and $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, the number $\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l})$ given in the equality (6.3) is also equal to*

$$\left(\prod_{u=1}^{l} \left(\#\left(I_u\right) - 1\right)!\right) - \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I},\ \mathbb{I}' \neq \mathbb{I}}} \left(\mathrm{mult}_{E_d(\mathbb{I}')}(\varphi_1, \ldots, \varphi_{d-\#(\mathbb{I}')}) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k \right) \right), \tag{6.5}$$

where $\chi_u\left(\mathbb{I}'\right)$ is the one defined in Main Theorem III. Here, if $\chi_u(\mathbb{I}') = 1$, then we assume that $\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k = \prod_{k=\#(I_u)}^{\#(I_u)-1} k = 1$.

Theorem A is just a corollary of Theorem B by Remark 6.8. However the proof of Theorem B is much harder than that of Theorem A. Therefore we prove Theorem A first in Section 7, and based on its proof we prove Theorem B in Section 8. Proposition D is also proved in Section 8.

7. Proof of Theorem A

In this section we prove Theorem A introduced in Section 6, together with preparing for the proof of Theorem B.

We fix our notation first, which is valid throughout Sections 7 and 8. For a given $\lambda \in V_d$ and $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, we put

$$\#(I_u) =: r_u + 1,\ \ (\zeta_i)_{i\in I_u} =: (\zeta_{u,0}, \zeta_{u,1}, \ldots, \zeta_{u,r_u}),$$
$$(\lambda_i)_{i\in I_u} =: (\lambda_{u,0}, \lambda_{u,1}, \ldots, \lambda_{u,r_u})$$

and $m_{u,i} := \frac{1}{1-\lambda_{u,i}}$. Moreover we assume $\zeta_{l,0} = \zeta_d = 0$. Then we have $\sum_{u=1}^{l}(r_u+1) = d$, $\sum_{i=0}^{r_u} m_{u,i} = 0$, $\varphi_k(\zeta) = \sum_{u=1}^{l}\sum_{i=0}^{r_u} m_{u,i}\zeta_{u,i}^k$ and

$$E_d(\mathbb{I}) = \left\{\zeta \in \mathbb{P}^{d-2} \;\middle|\; \zeta_{u,0} = \zeta_{u,1} = \cdots = \zeta_{u,r_u} \text{ for } 1 \le u \le l\right\} \cong \mathbb{P}^{l-2}.$$

Furthermore let $\alpha_1, \alpha_2, \ldots, \alpha_l$ be any mutually distinct complex numbers with $\alpha_l = 0$, and we denote by α the point $\zeta \in E_d(\mathbb{I})$ which satisfies $\zeta_{u,i} = \alpha_u$ for any u and i. In the following, we find $\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l})$ by cutting $E_d(\mathbb{I})$ at α by the plane $\mathcal{H}(\alpha) := \left\{\zeta \in \mathbb{P}^{d-2} \;\middle|\; \zeta_{u,0} = \alpha_u \text{ for } 1 \le u \le l\right\}$. We put $\xi_{u,i} := \zeta_{u,i} - \alpha_u$, $\xi_u := (\xi_{u,1}, \ldots, \xi_{u,r_u}) \in \mathbb{C}^{r_u}$, $\xi := (\xi_1, \ldots, \xi_l) \in \mathbb{C}^{d-l}$ and

$$\psi_k(\xi) := \varphi_k(\alpha + \xi) = \sum_{u=1}^{l} \left(m_{u,0}\alpha_u^k + \sum_{i=1}^{r_u} m_{u,i} \left(\alpha_u + \xi_{u,i}\right)^k \right). \tag{7.1}$$

Then ξ is a local coordinate system of $\mathcal{H}(\alpha)$ centered at α.

Proposition 7.1. *For any $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$ and for generic $\alpha \in E_d(\mathbb{I})$, we have*

$$\mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-l}) = \mathrm{mult}_0(\psi_1, \ldots, \psi_{d-l}). \tag{7.2}$$

Proof. Obvious by definition. □

In practice, the equality (7.2) always holds for any α if $\alpha_1, \ldots, \alpha_l$ are mutually distinct, which will be verified in Proposition 8.10.

We shall rewrite the equations $\psi_k(\xi) = 0$. Putting

$$p_{u,k}(\xi_u) = \sum_{i=1}^{r_u} m_{u,i}\xi_{u,i}^k$$

for each u and k, we have

$$\begin{aligned} \psi_k(\xi) &= \sum_{u=1}^{l} \left(\left(\sum_{i=0}^{r_u} m_{u,i} \right) \alpha_u^k + \sum_{i=1}^{r_u} \sum_{h=1}^{k} m_{u,i} \binom{k}{h} \alpha_u^{k-h} \xi_{u,i}^h \right) \\ &= \sum_{u=1}^{l} \sum_{h=1}^{k} \binom{k}{h} \alpha_u^{k-h} p_{u,h}(\xi_u), \end{aligned} \tag{7.3}$$

where $\binom{k}{h} = \frac{k(k-1)\cdots(k-h+1)}{h!}$ denotes the binomial coefficient. Hence $\psi_k(\xi)$ is a linear combination of $p_{u,h}(\xi_u)$ for $1 \le u \le l$ and $1 \le h \le k$.

Proposition 7.2. *The equations $\psi_k(\xi) = 0$ for $1 \le k \le d-l$ are equivalent to the equations*

$$p_{u,k}(\xi_u) = \sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} a_{u,k,v,h} p_{v,h}(\xi_v) \tag{7.4}$$

for $1 \le u \le l$ and $1 \le k \le r_u$, where the coefficients $a_{u,k,v,h}$ are some constants which depend only on $r_1, \ldots, r_l$ and $\alpha_1, \ldots, \alpha_l$.

Proof. It suffices to show the invertibility of the square matrix composed of the coefficients of $p_{u,h}(\xi_u)$ for $1 \le u \le l$ and $1 \le h \le r_u$ in the right hand side of the expressions (7.3). Proposition 7.2 is therefore reduced to the problem on linear algebra, whose proof is given in Lemma 7.8 at the end of this section. $\square$

By the aid of Propositions 7.1 and 7.2, we have reduced Theorem A to the following:

Proposition 7.3. *Suppose that an element $\mathbb{I} \in \mathfrak{I}(\lambda)$ is maximal. Then for any complex numbers $a_{u,k,v,h}$, the origin 0 is a discrete solution of the equations (7.4) for $1 \le u \le l$ and $1 \le k \le r_u$ with its intersection multiplicity $r_1! \cdots r_l!$.*

In the following, we prove Proposition 7.3.

Lemma 7.4. *Let $m_1, \dots, m_r$ be complex numbers such that $\sum_{i \in I} m_i \neq 0$ holds for any non-empty $I \subseteq \{1, \dots, r\}$. We put $p_k(\xi) := \sum_{i=1}^{r} m_i \xi_i^k$ for $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$. Then 0 is the only solution of the equations $p_k(\xi) = 0$ for $1 \le k \le r$ with its intersection multiplicity* $\mathrm{mult}_0(p_1, \dots, p_r) = r!$.

Proof. By the same argument as in the proof of Lemma 6.5, the existence of a solution other than 0 implies the equality $\sum_{i \in I} m_i = 0$ for some non-empty $I \subseteq \{1, \dots, r\}$; thus the contradiction assures the uniqueness of the solution.

By Lemmas 6.1 and 6.5, the set of the common zeros of $p_1, \dots, p_{r-1}$ in $\mathbb{P}^{r-1}$ is discrete and has $(r-1)!$ points, whose intersection multiplicities are all 1. Hence the set of the common zeros of $p_1, \dots, p_{r-1}$ in $\mathbb{C}^r$ consists of $(r-1)!$ lines $\ell_1, \dots, \ell_{(r-1)!}$, all of which pass the origin. Moreover their intersection multiplicities $\mathrm{mult}_{\ell_i}(p_1, \dots, p_{r-1})$ are all 1. Since each line ℓ_i intersects the hypersurface $\{p_r(\xi) = 0\}$ only at the origin, the intersection multiplicity $\mathrm{mult}_0(\ell_i, p_r)$ is r for each i. We thus have the equality $\mathrm{mult}_0(p_1, \dots, p_r) = r \cdot (r-1)! = r!$. $\square$

The most important part in the proof of Proposition 7.3 is to reduce Proposition 7.3 to Lemma 7.4 by replacing all the coefficients $a_{u,k,v,h}$ by 0.

We denote by $A = (a_{u,k,v,h})$ an element of $\mathbb{C}^{(l-1)(d-l)^2}$, where the indices u, k, v, h range in $1 \le u \le l$, $1 \le k \le r_u$, $1 \le v \le l$ and $r_v + 1 \le h \le d - l$. We put

$$D_R := \left\{ A = (a_{u,k,v,h}) \in \mathbb{C}^{(l-1)(d-l)^2} \;\middle|\; |a_{u,k,v,h}| < R \text{ for any } u, k, v, h \right\}$$

and define the map $F : \mathbb{C}^{d-l} \times D_R \to \mathbb{C}^{d-l} \times D_R$ by

$$(\xi, A) \mapsto \left(\left(p_{u,k}(\xi_u) - \sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v) \right)_{u,k}, A \right),$$

where the indices u, k range in $1 \le u \le l$ and $1 \le k \le r_u$.

Proposition 7.5. *Suppose that an element $\mathbb{I} \in \mathfrak{I}(\lambda)$ is maximal. Then for any positive real number R and any open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$, there exist open neighborhoods U, W of 0 in $\mathbb{C}^{d-l}$ with $U \subseteq U_0$ such that the map*

$$(U \times D_R) \cap F^{-1}(W \times D_R) \xrightarrow{F} W \times D_R \tag{7.5}$$

is proper, and therefore a finite branched covering.

In the following, we prove Proposition 7.3 first under the assumption of Proposition 7.5, and secondly we prove Proposition 7.5.

Proof of Proposition 7.3. First for any given coefficients $a_{u,k,v,h}$, we take a positive real number R sufficiently large such that the ball D_R contains $A = (a_{u,k,v,h})$. Then the discreteness of the solution 0 is verified by the finiteness of the map (7.5). Secondly we take an open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$ sufficiently small such that the only solution of the equations (7.4) in U_0 is 0. Then the intersection multiplicity of the equations (7.4) at 0 is equal to the degree of the branched covering map (7.5), which is also equal to the intersection multiplicity of the equations (7.4) at 0 with all the coefficients $a_{u,k,v,h}$ equal to 0. Therefore it is $r_1! \cdots r_l!$ by Lemma 7.4, which completes the proof of Proposition 7.3. $\square$

Proof of Proposition 7.5. We put $|\xi_u| := \max_{1\le i\le r_u} |\xi_{u,i}|$, $Z_u := \left\{\xi_u \in \mathbb{C}^{r_u} \;\middle|\; |\xi_u| = 1\right\}$ and $\delta_u := \inf_{\xi_u\in Z_u} \max_{1\le k\le r_u} |p_{u,k}(\xi_u)|$ for each u. Then by the maximality of $\mathbb{I} \in \mathfrak{I}(\lambda)$ and Lemma 7.4, we have $\delta_u > 0$ for each u, which implies the inequality $\max_{1\le k\le r_u} |p_{u,k}(\xi_u)| \ge \delta_u |\xi_u|^{r_u}$ for any $\xi_u \in \mathbb{C}^{r_u}$ with $|\xi_u| \le 1$. Hence putting $\delta := \min_{1\le u\le l} \delta_u$ and $||\xi|| := \max_{1\le u\le l} |\xi_u|^{r_u}$, we have the inequality

$$\max_{u,k} |p_{u,k}(\xi_u)| \ge \delta \cdot ||\xi|| \tag{7.6}$$

for $||\xi|| \le 1$.

On the other hand, for any $A = (a_{u,k,v,h}) \in D_R$ and $\xi \in \mathbb{C}^{d-l}$ with $||\xi|| \le 1$, we have

$$\begin{aligned}\max_{u,k}\left|\sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v)\right| &\le \sum_{v,h} R\left(\sum_{i=1}^{r_v} |m_{v,i}|\right)|\xi_v|^h \\ &\le L \cdot ||\xi||^{1+\mu},\end{aligned} \tag{7.7}$$

where we put $L := R\sum_{v=1}^{l}(d-l-r_v)\left(\sum_{i=1}^{r_v}|m_{v,i}|\right)$ and $\mu := \frac{1}{\max_u r_u}$.

Therefore if we take $\xi \in \mathbb{C}^{d-l}$ with $||\xi|| \le \left(\frac{\delta}{2L}\right)^{1/\mu}$, then by the inequalities (7.6) and (7.7), we have

$$\max_{u,k}\left|p_{u,k}(\xi_u) - \sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v)\right|$$

$$\geq \max_{u,k} |p_{u,k}(\xi_u)| - \max_{u,k} \left| \sum_{v,h} a_{u,k,v,h} p_{v,h}(\xi_v) \right|$$
$$\geq \delta \cdot ||\xi|| - L \cdot ||\xi||^{1+\mu} \geq \delta \cdot ||\xi|| - L \cdot \frac{\delta}{2L} \cdot ||\xi|| = \frac{\delta}{2} \cdot ||\xi||.$$

We define a positive number ϵ sufficiently small such that the inequality $0 < \epsilon < \left(\frac{\delta}{2L}\right)^{1/\mu}$ holds and that the set $U := \left\{\xi \in \mathbb{C}^{d-l} \;\middle|\; ||\xi|| < \epsilon\right\}$ is included in U_0. Moreover we put

$$W := \left\{ \eta = (\eta_{u,k}) \in \mathbb{C}^{d-l} \;\middle|\; |\eta| = \max_{u,k} |\eta_{u,k}| < \tfrac{1}{2}\delta\epsilon \right\}.$$

Then we can easily verify that the map (7.5) is proper. Therefore by Lemma 7.6 below, the map (7.5) is a finite branched covering. □

Lemma 7.6. *Let U, V be connected open subsets of $\mathbb{C}^n$, and $f : U \to V$ a proper holomorphic map. Then $f : U \to V$ is a finite branched covering.*

Proof of Lemma 7.6. Note that there does not exist a compact analytic subset of $\mathbb{C}^n$ whose dimension is greater than or equal to 1. Since $K := \{z \in U \mid \det(Df)(z) = 0\}$ is an analytic subset of U with $K \neq U$, $f(K)$ is also an analytic subset of V by proper mapping theorem. Hence the map $U \setminus f^{-1} \circ f(K) \to V \setminus f(K)$ is proper and locally homeomorphic, and therefore is a covering space of finite degree, which implies that f is a finite branched covering. □

The rest of this section is devoted to Lemma 7.8 and its proof.

Definition 7.7. For non-negative integers n, b, k, h with $n > k$ and $b > h$, we denote by $A^{b,h}_{n,k}(\alpha)$ the $(n-k, b-h)$ matrix whose (i, j)-th entry is $\binom{i+k-1}{j+h-1}\alpha^{(i+k)-(j+h)}$ for each i and j. Moreover we put $A^b_{n,k}(\alpha) := A^{b,0}_{n,k}(\alpha)$ and $A^b_n(\alpha) := A^{b,0}_{n,0}(\alpha)$.

By definition, the matrix $A^{b,h}_{n,k}(\alpha)$ is obtained from the (n, b) matrix

$$A^b_n(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \alpha & 1 & 0 & 0 & \cdots & 0 \\ \alpha^2 & 2\alpha & 1 & 0 & \cdots & 0 \\ \alpha^3 & 3\alpha^2 & 3\alpha & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{n-1} & (n-1)\alpha^{n-2} & \binom{n-1}{2}\alpha^{n-3} & \binom{n-1}{3}\alpha^{n-4} & \cdots & \ddots \end{pmatrix}$$

by cutting off the upper k rows and the left h columns.

Lemma 7.8. *We put $r := r_1 + \cdots + r_l = d - l$, and denote by M the (r, r) square matrix defined by*

$$M = \left(A_{r+1,1}^{r_1+1,1}(\alpha_1), \ldots, A_{r+1,1}^{r_l+1,1}(\alpha_l) \right).$$

Then we have

$$\det M = \frac{r!}{r_1! \cdots r_l!} \cdot \prod_{1 \le v < u \le l} (\alpha_u - \alpha_v)^{r_v r_u}.$$

The matrix M defined above is the same as the square matrix composed of the coefficients of $p_{u,h}(\xi_u)$ for $1 \le u \le l$ and $1 \le h \le r_u$ in the right hand side of the expressions (7.3); hence Proposition 7.2 is reduced to Lemma 7.8.

To prove Lemma 7.8, we give a definition and a lemma.

Definition 7.9. For a positive integer b, we denote by X_b the (b, b) diagonal matrix whose (i, i)-th entry is i for $1 \le i \le b$, and by N_b the (b, b) nilpotent matrix whose $(i, i+1)$-th entry is 1 for $1 \le i \le b-1$ and whose other entries are 0, i.e.,

$$X_b = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix} \quad \text{and} \quad N_b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Lemma 7.10. *For positive integers n and b, we have the equalities*

$$A_{n+1,1}^{b+1,1}(\alpha) = X_n \cdot A_n^b(\alpha) \cdot {X_b}^{-1} \quad \textit{and} \quad A_n^n(\beta) \cdot A_n^b(\alpha) = A_n^b(\beta + \alpha).$$

Moreover for positive integers n, b, k with $n > k$ and a non-zero complex number α, we have the equality

$$A_{n,k}^b(\alpha) \cdot \sum_{h=0}^{b-1} \binom{-k}{h} \left(\alpha^{-1} N_b\right)^h = \alpha^k A_{n-k}^b(\alpha),$$

where $\left(\alpha^{-1} N_b\right)^0$ denotes the identity matrix of size (b, b).

Proof. The first equality is verified by $\binom{i}{j} = \binom{i-1}{j-1} \cdot \frac{i}{j}$, the second one by $\binom{i}{h}\binom{h}{j} = \binom{i}{j}\binom{i-j}{h-j}$ and $\sum_{h=0}^{k} \binom{k}{h} \alpha^h \beta^{k-h} = (\alpha + \beta)^k$, and the last one by the equality $\sum_{h=0}^{j} \binom{x}{h}\binom{y}{j-h} = \binom{x+y}{j}$. □

Proof of Lemma 7.8. By Lemma 7.10, we have $A_{r+1,1}^{r_u+1,1}(\alpha_u) = X_r \cdot A_r^{r_u}(\alpha_u) \cdot (X_{r_u})^{-1}$ for each $1 \le u \le l$. Hence putting $M' = \left(A_r^{r_1}(\alpha_1), \ldots, A_r^{r_l}(\alpha_l) \right)$, we have the equalities

$$\det M = \det X_r \cdot \det M' \cdot \prod_{u=1}^{l} \det \left(X_{r_u}\right)^{-1} = \frac{r!}{r_1! \cdots r_l!} \cdot \det M'.$$

Therefore to prove Lemma 7.8, we only need to show the equality

$$\det M' = \prod_{1\le v<u\le l} (\alpha_u - \alpha_v)^{r_v r_u}. \tag{7.8}$$

If there exist distinct indices u, v with $\alpha_u = \alpha_v$, then both hand sides of the equality (7.8) are clearly zero; hence we only need to consider the equality (7.8) in the case that $\alpha_1, \dots, \alpha_l$ are mutually distinct. Moreover if $l = 1$, the equality (7.8) trivially holds since $\det M' = 1$. In the following, we show the equality (7.8) by induction of l.

We put $r' = r_2 + \cdots + r_l$ and $\alpha'_u = \alpha_u - \alpha_1$ for $2 \le u \le l$. Then by Lemma 7.10, we have

$$A_r^r(-\alpha_1) \cdot M' = \left(A_r^{r_1}(0), A_r^{r_2}(\alpha'_2), \dots, A_r^{r_l}(\alpha'_l)\right) = \begin{pmatrix} I_{r_1} & * \\ O & \widetilde{M} \end{pmatrix},$$

where we put $\widetilde{M} = \left(A_{r,r_1}^{r_2}(\alpha'_2), \dots, A_{r,r_1}^{r_l}(\alpha'_l)\right)$, and I_{r_1} denotes the identity matrix of size (r_1, r_1). Moreover by Lemma 7.10, we have

$$A_{r,r_1}^{r_u}(\alpha'_u) \cdot \sum_{h=0}^{r_u-1} \binom{-r_1}{h} \left((\alpha'_u)^{-1} N_{r_u}\right)^h = (\alpha'_u)^{r_1} \cdot A_{r'}^{r_u}(\alpha'_u)$$

for each $2 \le u \le l$. Hence putting $M'' = \left(A_{r'}^{r_2}(\alpha'_2), \dots, A_{r'}^{r_l}(\alpha'_l)\right)$, we have the equalities

$$\det M' = \det \widetilde{M} = \det M'' \cdot \prod_{u=2}^{l} (\alpha'_u)^{r_1 r_u},$$

which completes the proof by induction of l. □

8. Proof of Theorem B

In this section we give the proofs of Theorem B and Proposition D introduced in Section 6, which are also the most crucial part in the proof of the main theorems. We first give a key estimate in Proposition 8.1, which is a substitute for the inequalities (7.6) and (7.7) in the case that $\mathbb{I} \in \mathfrak{I}(\lambda)$ is not necessarily maximal.

Proposition 8.1. *Let r be a positive integer, and $m_1, \dots, m_r$ non-zero complex numbers with $\sum_{i=1}^r m_i \ne 0$. We put $m = (m_1, \dots, m_r)$,*

$$p_k(\xi) := \sum_{i=1}^{r} m_i \xi_i^k, \quad B(m) := \left\{\xi \in \mathbb{C}^r \;\middle|\; p_k(\xi) = 0 \text{ for } 1 \le k \le r\right\},$$

and $|\xi| := \max_{1\le i\le r} |\xi_i|$ for $\xi = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$. Then

(1) *for each positive integer h, there exists a positive real number L_h such that the inequality*

$$|p_h(\xi)| \leq L_h \cdot \max_{1 \leq k \leq r} |p_k(\xi)| \tag{8.1}$$

holds for any $\xi \in \mathbb{C}^r$ with $|\xi| = 1$.

(2) *There exist an open neighborhood O of $B(m) \cap \{\xi \in \mathbb{C}^r \mid |\xi| = 1\}$ in $\mathbb{C}^r$ and a positive real number L' such that the inequality*

$$|p_r(\xi)| \leq L' \cdot \max_{1 \leq k \leq r-1} |p_k(\xi)| \tag{8.2}$$

holds for any $\xi \in O$.

Proof. We put $m_0 := -\sum_{i=1}^{r} m_i$,

$$\mathfrak{I}(m) := \left\{ \{I_1, \ldots, I_l\} \;\middle|\; \begin{array}{c} I_1 \amalg \cdots \amalg I_l = \{0, \ldots, r\}, \quad l \geq 1 \\ I_u \neq \emptyset \text{ and } \sum_{i \in I_u} m_i = 0 \text{ for } 1 \leq u \leq l \end{array} \right\},$$

$$E(\mathbb{I}) := \left\{ (\xi_1, \ldots, \xi_r) \in \mathbb{C}^r \;\middle|\; \text{If } i, j \in I \in \mathbb{I}, \text{ then } \xi_i = \xi_j \right\}$$

for each $\mathbb{I} \in \mathfrak{I}(m)$, and

$$\mathbb{I}(\xi) := \left\{ I \subseteq \{0, 1, \ldots, r\} \;\middle|\; \begin{array}{c} I \neq \emptyset. \quad \text{If } i, j \in I, \text{ then } \xi_i = \xi_j. \\ \text{If } i \in I \text{ and } j \in \{0, 1, \ldots, r\} \setminus I, \text{ then } \xi_i \neq \xi_j. \end{array} \right\}$$

for each $\xi \in B(m)$, where we are assuming $\xi_0 = 0$. Then we have the equality

$$B(m) = \bigcup_{\mathbb{I} \in \mathfrak{I}(m)} E(\mathbb{I}), \tag{8.3}$$

and we also have $\mathbb{I}(\xi) \in \mathfrak{I}(m)$ and $\xi \in E(\mathbb{I}(\xi))$ for each $\xi \in B(m)$ by the same argument as the proof of Lemma 6.5. Note that in this setting, the set $\mathfrak{I}(m)$ always contains the element $\mathbb{I}_0 := \{\{0, \ldots, r\}\}$, and that the equalities $E(\mathbb{I}_0) = \{0\}$ and $\mathbb{I}(0) = \mathbb{I}_0$ hold.

We make use of the following auxiliary lemmas:

Lemma 8.2. *There exists an open neighborhood O of $B(m) \cap \{\xi \in \mathbb{C}^r \mid |\xi| = 1\}$ in $\mathbb{C}^r$ such that for each positive integer h, there exists a positive real number L'_h such that the inequality*

$$|p_h(\xi)| \leq L'_h \cdot \max_{1 \leq k \leq r-1} |p_k(\xi)| \tag{8.4}$$

holds for any $\xi \in O$.

Lemma 8.3. *Let α be a point in $B(m) \setminus \{0\}$. Then there exists an open neighborhood O_α of α in $\mathbb{C}^r$ such that for each positive integer h, there exists a positive real number $L_{\alpha,h}$ such that the inequality*

$$|p_h(\xi)| \leq L_{\alpha,h} \cdot \max_{1 \leq k \leq r+1-\#(\mathbb{I}(\alpha))} |p_k(\xi)| \tag{8.5}$$

holds for any $\xi \in O_\alpha$.

Note that the implications

"Proposition 8.1 $\Longrightarrow$ Lemma 8.2 $\Longrightarrow$ The assertion (2) in Proposition 8.1"

are clear. In the following, we prove Lemmas 8.2, 8.3 and the assertion (1) in Proposition 8.1 simultaneously by induction. To make the induction work well, we define the "depth" of a point $\alpha \in B(m)$ by

$$\tau_m(\alpha) := \max \left\{ \nu \;\middle|\; \begin{array}{l} \mathbb{I}(\alpha) =: \mathbb{I}_1 \precneqq \mathbb{I}_2 \precneqq \cdots \precneqq \mathbb{I}_\nu \\ \mathbb{I}_\omega \in \mathfrak{I}(m) \ \text{ for } 1 \leq \omega \leq \nu \end{array} \right\},$$

where the symbol $\mathbb{I} \precneqq \mathbb{I}'$ for $\mathbb{I}, \mathbb{I}' \in \mathfrak{I}(m)$ denotes that $\mathbb{I}'$ is a refinement of $\mathbb{I}$ with $\mathbb{I} \neq \mathbb{I}'$. Note that the inequality $\tau_m(0) > \tau_m(\alpha)$ holds for any $\alpha \in B(m) \setminus \{0\}$ and that the equality $\tau_m(0) = 1$ holds if and only if $B(m) = \{0\}$.

We consider the following assertions for each non-negative integer ν:

$(1)_\nu$ if $\tau_m(0) \leq \nu + 1$, then the assertion (1) in Proposition 8.1 holds.
$(2)_\nu$ If $\tau_m(0) \leq \nu + 1$, then Lemma 8.2 holds.
$(3)_\nu$ If $\tau_m(\alpha) \leq \nu$, then Lemma 8.3 holds.

Note that the assertion $(2)_0$ trivially holds since $\tau_m(0) \leq 1$ implies $B(m) = \{0\}$. In the following, we show the implications

$$(1)_{\nu-1} \Rightarrow (3)_\nu \Rightarrow (2)_\nu \Rightarrow (1)_\nu$$

for each ν, which will complete the proofs of Lemmas 8.2, 8.3 and Proposition 8.1. We put

$$Z := \{\xi \in \mathbb{C}^r \mid |\xi| = 1\}.$$

Proof of the implication $(3)_\nu \Rightarrow (2)_\nu$. We suppose $(3)_\nu$ and prove $(2)_\nu$. When $\tau_m(0) \leq \nu + 1$, the inequality $\tau_m(\alpha) \leq \nu$ holds for any $\alpha \in Z \cap B(m)$. Hence by the assumption $(3)_\nu$, we can choose, for each $\alpha \in Z \cap B(m)$, an open neighborhood O_α of α and a positive real number $L_{\alpha,h}$ for each $h \in \mathbb{N}$ such that the inequality (8.5) holds for any $\xi \in O_\alpha$. Since $Z \cap B(m)$ is compact, there exist finite number of open neighborhoods $O_{\alpha_1}, \ldots, O_{\alpha_\mu}$

which cover $Z \cap B(m)$. On the other hand, since $\#(\mathbb{I}(\alpha)) \geq 2$ for any $\alpha \in Z \cap B(m)$, we always have $r + 1 - \#(\mathbb{I}(\alpha)) \leq r - 1$. Therefore, putting $O := \bigcup_{1 \leq \omega \leq \mu} O_{\alpha_\omega}$ and $L'_h := \max_{1 \leq \omega \leq \mu} L_{\alpha_\omega, h}$ for each h, we have, by the inequality (8.5), the inequality (8.4) for any $\xi \in O$. $\square$

Proof of the implication $(2)_\nu \Rightarrow (1)_\nu$. We suppose $(2)_\nu$ and verify $(1)_\nu$. The set $Z \setminus O$ is compact and does not have common zeros of $p_1, \ldots, p_r$. Hence the infimum $\inf_{\xi \in Z \setminus O} \max_{1 \leq k \leq r} |p_k(\xi)|$ is positive, which assures the existence of a positive real number L_h for each $h \in \mathbb{N}$ satisfying the inequality (8.1) for any $\xi \in Z \setminus O$. Replacing the maximum of L_h and L'_h by L_h, we have the inequality (8.1) for any $\xi \in Z$. $\square$

In the rest of the proof, we suppose $(1)_{\nu-1}$ and prove $(3)_\nu$. We fix $\alpha \in B(m) \setminus \{0\}$ with $\tau_m(\alpha) \leq \nu$, put $\mathbb{I}(\alpha) =: \{I_1, \ldots, I_l\}$, and denote by α_u^0 the i-th coordinate of α for $i \in I_u$. Note that $\alpha_1^0, \ldots, \alpha_l^0$ are mutually distinct. We put $\#(I_u) = r_u + 1$, $(\xi_i)_{i \in I_u} = (\xi_{u,0}, \xi_{u,1}, \ldots, \xi_{u,r_u})$, $(m_i)_{i \in I_u} = (m_{u,0}, m_{u,1}, \ldots, m_{u,r_u})$, $m(I_u) = (m_{u,1}, \ldots, m_{u,r_u})$, $x_{u,i} = \xi_{u,i} - \xi_{u,0}$, $\alpha_u = \xi_{u,0}$, $x_u = (x_{u,1}, \ldots, x_{u,r_u})$, $x = (x_1, \ldots, x_l)$, $|x_u| = \max_{1 \leq i \leq r_u} |x_{u,i}|$ and $|x| = \max_{1 \leq u \leq l} |x_u|$. We may assume $\alpha_l = \alpha_l^0 = \xi_{l,0} = \xi_0 = 0$. We may also consider the coordinates $(\alpha_1, \ldots, \alpha_{l-1}, x)$ as a local coordinate system around α in $\mathbb{C}^r$. Note that the point $(\alpha_1, \ldots, \alpha_{l-1}, x)$ coincides with α if and only if $x = 0$ and $\alpha_u = \alpha_u^0$ for $1 \leq u \leq l-1$, and that the point $(\alpha_1, \ldots, \alpha_{l-1}, x)$ belongs to $E(\mathbb{I}(\alpha))$ if and only if $x = 0$. Furthermore we put

$$\theta_{u,k}(x_u) = \sum_{i=1}^{r_u} m_{u,i} x_{u,i}^k$$

for $1 \leq u \leq l$ and $k \in \mathbb{N}$.

Then we have the equality

$$p_k(\xi) = \sum_{u=1}^{l} \sum_{h=1}^{k} \binom{k}{h} \alpha_u^{k-h} \theta_{u,h}(x_u) \tag{8.6}$$

by the same computation as in the equalities (7.3). Moreover by Lemma 7.8, the equalities (8.6) for $1 \leq k \leq r+1-l$ are equivalent in some neighborhood of α to the equalities

$$\theta_{u,k}(x_u) = \sum_{h=1}^{r+1-l} b_{u,k,h} p_h(\xi) + \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} a_{u,k,v,h} \theta_{v,h}(x_v) \tag{8.7}$$

for $1 \leq u \leq l$ and $1 \leq k \leq r_u$, where the coefficients $b_{u,k,h}$ and $a_{u,k,v,h}$ depend only on $r_1, \ldots, r_l$ and $\alpha_1, \ldots, \alpha_{l-1}$. Moreover its dependence is continuous on the domain where $\alpha_1, \ldots, \alpha_{l-1}$ and 0 are mutually distinct. Therefore taking a small open neighborhood Δ of $(\alpha_1^0, \ldots, \alpha_{l-1}^0)$ in $\mathbb{C}^{l-1}$ and a sufficiently large real number R, we may assume that the inequalities

$$|\alpha_u| \leq R, \quad |b_{u,k,h}| \leq R \quad \text{and} \quad |a_{u,k,v,h}| \leq R$$

hold for all u, k, v, h and for any $(\alpha_1, \ldots, \alpha_{l-1}) \in \Delta$.

On the other hand, since $\tau_m(\alpha) \leq \nu$, we always have $\tau_{m(I_u)}(0) \leq \nu$ for any u. Hence by the assumption $(1)_{\nu-1}$, there exists, for each u and for each positive integer h, a positive real number $L_{u,h}$ such that the inequality

$$|\theta_{u,h}(x_u)| \leq L_{u,h} \cdot \max_{1 \leq k \leq r_u} |\theta_{u,k}(x_u)|$$

holds for any $x_u \in \mathbb{C}^{r_u}$ with $|x_u| = 1$. Hence by the homogeneity of $\theta_{u,k}(x_u)$, the inequality

$$|\theta_{u,h}(x_u)| \leq L_{u,h} \cdot \max_{1 \leq k \leq r_u} |\theta_{u,k}(x_u)| \cdot |x_u|$$

holds for $h \geq r_u + 1$ and for any $x_u \in \mathbb{C}^{r_u}$ with $|x_u| \leq 1$. Therefore by the equality (8.7), we have the following for $(\alpha_1, \ldots, \alpha_{l-1}) \in \Delta$ and $|x| \leq 1$:

$$\begin{aligned}
\max_{u,k} \left| \sum_{h=1}^{r+1-l} b_{u,k,h} p_h(\xi) \right| &\geq \max_{u,k} |\theta_{u,k}(x_u)| - \max_{u,k} \left| \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} a_{u,k,v,h} \theta_{v,h}(x_v) \right| \\
&\geq \left(1 - R \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} L_{v,h} \cdot |x| \right) \max_{u,k} |\theta_{u,k}(x_u)| .
\end{aligned}$$

Hence putting

$$J := \max \left\{ 1,\ 2R \sum_{v=1}^{l} \sum_{h=r_v+1}^{r+1-l} L_{v,h} \right\}, \quad L := 2R(r+1-l)$$

$$\text{and} \quad O_\alpha := \left\{ (\alpha_1, \ldots, \alpha_{l-1}, x) \in \mathbb{C}^r \;\middle|\; (\alpha_1, \ldots, \alpha_{l-1}) \in \Delta, |x| < 1/J \right\},$$

we have, for any $\xi = (\alpha_1, \ldots, \alpha_{l-1}, x) \in O_\alpha$, the inequality

$$\max_{u,k} |\theta_{u,k}(x_u)| \leq 2 \max_{u,k} \left| \sum_{h=1}^{r+1-l} b_{u,k,h} p_h(\xi) \right| \leq L \cdot \max_{1 \leq k \leq r+1-l} |p_k(\xi)|. \tag{8.8}$$

On the other hand, by the equality (8.6), we have, for each positive integer h, the inequalities

$$|p_h(\xi)| \leq \sum_{u=1}^{l} \sum_{k=1}^{h} \binom{h}{k} R^{h-k} L_{u,k} \cdot \max_{1 \leq k \leq r_u} |\theta_{u,k}(x_u)| \leq L_h \cdot \max_{u,k} |\theta_{u,k}(x_u)| \tag{8.9}$$

for any $(\alpha_1, \ldots, \alpha_{l-1}, x) \in O_\alpha$, where we put $L_h := \sum_{u=1}^{l} \sum_{k=1}^{h} \binom{h}{k} R^{h-k} L_{u,k}$. Therefore by the inequalities (8.8) and (8.9), we have

$$|p_h(\xi)| \leq L_h L \cdot \max_{1\leq k\leq r+1-l} |p_k(\xi)|$$

for any $\xi = (\alpha_1, \ldots, \alpha_{l-1}, x) \in O_\alpha$ and for each h. Thus the assertion $(3)_\nu$ is proved, which completes the proof of Lemmas 8.2, 8.3 and Proposition 8.1. □

In the rest of this section, the notation follows that in Section 7. Therefore λ is an element of V_d, and $\mathbb{I} = \{I_1, \ldots, I_l\}$ an element of $\mathfrak{I}(\lambda)$, which are fixed throughout the rest of this section. Moreover the notation r_u, $\zeta_{u,i}$, $\lambda_{u,i}$, $m_{u,i}$, α_u, α, $\xi_{u,i}$, ξ_u, ξ, $\psi_k(\xi)$, $p_{u,k}(\xi_u)$, $A = (a_{u,k,v,h})$, D_R and the map F is the same as in Section 7. Note that Propositions 7.1 and 7.2 are valid for non-maximal $\mathbb{I} \in \mathfrak{I}(\lambda)$

We give a proposition next which is the most important part in the proof of Theorem B, whose proof is essentially based on Proposition 8.1.

Proposition 8.4. *For any positive real numbers R and $1 > \epsilon > 0$, and for any open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$, there exist open neighborhoods U, W of 0 in $\mathbb{C}^{d-l}$ with $U \subseteq U_0$ such that the map*

$$(U \times D_R) \cap F^{-1}(W_\epsilon \times D_R) \xrightarrow{F} W_\epsilon \times D_R \tag{8.10}$$

is proper, and therefore a finite branched covering, where

$$W_\epsilon := W \cap \Xi_\epsilon \quad \textit{and} \quad \Xi_\epsilon := \left\{ \eta = (\eta_{u,k}) \in \mathbb{C}^{d-l} \;\middle|\; \min_{1\leq u\leq l} |\eta_{u,r_u}| > \epsilon \cdot \max_{u,k} |\eta_{u,k}| \right\}.$$

Proof. Remember that the map $F : \mathbb{C}^{d-l} \times D_R \to \mathbb{C}^{d-l} \times D_R$ is defined by $F(\xi, A) = (\eta, A)$, where $\xi = (\xi_{u,i})$, $\eta = (\eta_{u,k})$, $A = (a_{u,k,v,h})$ and

$$\eta_{u,k} = p_{u,k}(\xi_u) - \sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} a_{u,k,v,h} p_{v,h}(\xi_v)$$

for $1 \leq u \leq l$ and $1 \leq k \leq r_u$. We put

$$|\xi_u| := \max_{1\leq i\leq r_u} |\xi_{u,i}|, \quad |\xi| := \max_{1\leq u\leq l} |\xi_u|, \quad |\eta| := \max_{u,k} |\eta_{u,k}|,$$
$$\widetilde{B}_u(\lambda_{I_u}) := \left\{ \xi_u \in \mathbb{C}^{r_u} \;\middle|\; p_{u,k}(\xi_u) = 0 \text{ for } 1 \leq k \leq r_u \right\} \text{ and}$$
$$Z_u := \left\{ \xi_u \in \mathbb{C}^{r_u} \;\middle|\; |\xi_u| = 1 \right\}.$$

By the assertion (1) in Proposition 8.1, there exists a positive real number $L_{u,h}$ for each u and h such that the inequality

$$\left|p_{u,h}(\xi_u)\right| \leq L_{u,h} \cdot \max_{1\leq k\leq r_u} \left|p_{u,k}(\xi_u)\right|$$

holds for any $\xi_u \in Z_u$. Hence by the homogeneity of $p_{u,k}(\xi_u)$, we have

$$\left|p_{u,h}(\xi_u)\right| \leq L_{u,h} \cdot \max_{1\leq k\leq r_u} \left|p_{u,k}(\xi_u)\right| \cdot |\xi_u| \tag{8.11}$$

for any $\xi_u \in \mathbb{C}^{r_u}$ with $|\xi_u| \leq 1$ and for each $h \geq r_u + 1$.

On the other hand, by the assertion (2) in Proposition 8.1, there exist an open neighborhood O_u of $\widetilde{B}_u(\lambda_{I_u}) \cap Z_u$ in $\mathbb{C}^{r_u}$ and a positive real number L'_u for each u such that the inequality

$$\left|p_{u,r_u}(\xi_u)\right| \leq L'_u \cdot \max_{1\leq k\leq r_u-1} \left|p_{u,k}(\xi_u)\right|$$

holds for any $\xi_u \in O_u$. We put

$$\Omega_u := \left\{(t\xi_{u,1}, \dots, t\xi_{u,r_u}) \in \mathbb{C}^{r_u} \;\middle|\; t \in \mathbb{R}, \quad t > 0, \quad (\xi_{u,1}, \dots, \xi_{u,r_u}) \in O_u \cap Z_u\right\}$$

for each u and

$$\Omega := \left\{\xi = (\xi_1, \dots, \xi_l) \in \mathbb{C}^{d-l} \;\middle|\; \xi_u \in \Omega_u \text{ holds for some } 1 \leq u \leq l\right\}.$$

Then Ω_u is an open neighborhood of $\widetilde{B}_u(\lambda_{I_u}) \setminus \{0\}$ in $\mathbb{C}^{r_u} \setminus \{0\}$, and Ω is an open set in $\mathbb{C}^{d-l}$. Moreover for $\xi_u \in \mathbb{C}^{r_u} \setminus \{0\}$, the point $\xi_u/|\xi_u|$ belongs to the set $O_u \cap Z_u = \Omega_u \cap Z_u$ if and only if $\xi_u \in \Omega_u$. Hence by the homogeneity of $p_{u,k}(\xi_u)$, we have the inequality

$$\left|p_{u,r_u}(\xi_u)\right| \leq L'_u \cdot \max_{1\leq k\leq r_u-1} \left|p_{u,k}(\xi_u)\right| \cdot |\xi_u| \tag{8.12}$$

for any $\xi_u \in \Omega_u$ with $|\xi_u| \leq 1$.

For the simplicity of notation, we put

$$L := \max_{1\leq u\leq l} \left(\max_{r_u+1\leq h\leq d-l} L_{u,h}\right) \quad \text{and} \quad L' := \max_{1\leq u\leq l} L'_u.$$

For any positive real numbers R and $1 > \epsilon > 0$, and for any open neighborhood U_0 of 0 in $\mathbb{C}^{d-l}$, we take a positive real number δ such that the inequality

$$0 < \delta < \min\left\{1, \; \frac{\epsilon}{3(l-1)(d-l)RL}, \; \frac{\epsilon}{3L'}\right\}$$

holds and that the set

$$U := \left\{\xi \in \mathbb{C}^{d-l} \;\middle|\; |\xi| < \delta\right\}$$

is included in U_0.

Then for any $A = (a_{u,k,v,h}) \in D_R$ and $\xi \in U$, we have

$$\max_{u,k} \left|\sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} a_{u,k,v,h} p_{v,h}(\xi_v)\right| \leq \sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} R \cdot L_{v,h} \cdot |\xi_v| \cdot \max_{1\leq k\leq r_v} \left|p_{v,k}(\xi_v)\right|$$

$$\leq \frac{\epsilon}{3} \cdot \max_{u,k} \bigl| p_{u,k}(\xi_u) \bigr|$$

by the inequality (8.11), which implies

$$\begin{aligned} |\eta| = \max_{u,k} |\eta_{u,k}| &\geq \max_{u,k} \bigl| p_{u,k}(\xi_u) \bigr| - \max_{u,k} \left| \sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} a_{u,k,v,h} p_{v,h}(\xi_v) \right| \\ &\geq \frac{2}{3} \max_{u,k} \bigl| p_{u,k}(\xi_u) \bigr|. \end{aligned} \tag{8.13}$$

On the other hand, for $A = (a_{u,k,v,h}) \in D_R$ and $\xi \in U \cap \Omega$, we have $\xi_u \in \Omega_u$ for some u, which implies

$$\begin{aligned} |\eta_{u,r_u}| &\leq \bigl| p_{u,r_u}(\xi_u) \bigr| + \left| \sum_{v=1}^{l} \sum_{h=r_v+1}^{d-l} a_{u,r_u,v,h} p_{v,h}(\xi_v) \right| \\ &\leq L'_u \cdot \max_{1 \leq k \leq r_u - 1} \bigl| p_{u,k}(\xi_u) \bigr| \cdot |\xi_u| + \frac{\epsilon}{3} \cdot \max_{u,k} \bigl| p_{u,k}(\xi_u) \bigr| \\ &\leq \frac{2\epsilon}{3} \cdot \max_{u,k} \bigl| p_{u,k}(\xi_u) \bigr| \leq \epsilon \cdot |\eta| \end{aligned}$$

by the inequality (8.12). Therefore we have

Lemma 8.5. *For $(\xi, A) \in (U \cap \Omega) \times D_R$, we have $F(\xi, A) \notin \Xi_\epsilon \times D_R$.*

We put

$$\mu_u := \min_{\xi_u \in Z_u \setminus \Omega_u} \max_{1 \leq k \leq r_u} \bigl| p_{u,k}(\xi_u) \bigr| \quad \text{and} \quad \mu := \min_{1 \leq u \leq l} \mu_u.$$

Then μ is positive by the compactness of $Z_u \setminus \Omega_u$ for each u. Moreover by the homogeneity of $p_{u,k}(\xi_u)$, we have the inequality

$$\max_{1 \leq k \leq r_u} \bigl| p_{u,k}(\xi_u) \bigr| \geq \mu_u |\xi_u|^{r_u} \tag{8.14}$$

for any $\xi_u \in \mathbb{C}^{r_u} \setminus \Omega_u$ with $|\xi_u| \leq 1$. We put $r := \max_u r_u$.

Lemma 8.6. *For $(\xi, A) \in (U \setminus \Omega) \times D_R$, we have $|\eta| \geq \frac{2}{3}\mu|\xi|^r$.*

Proof. For $\xi \in U \setminus \Omega$, we have $\xi_u \notin \Omega_u$ for any u. Hence for $(\xi, A) \in (U \setminus \Omega) \times D_R$, by the inequalities (8.13) and (8.14), we have

$$|\eta| \geq \tfrac{2}{3} \max_{u,k} \bigl| p_{u,k}(\xi_u) \bigr| \geq \tfrac{2}{3} \max_{1 \leq u \leq l} \mu_u |\xi_u|^{r_u} \geq \tfrac{2}{3} \mu |\xi|^r. \qquad \square$$

We put

$$W := \left\{ \eta = (\eta_{u,k}) \in \mathbb{C}^{d-l} \;\middle|\; |\eta| < \tfrac{2}{3}\mu \cdot \delta^r \right\}.$$

Then Lemma 8.5 implies the inclusion relation

$$(U \times D_R) \cap F^{-1}(W_\epsilon \times D_R) \subseteq (U \setminus \Omega) \times D_R.$$

Therefore for any $(\xi, A) \in (U \times D_R) \cap F^{-1}(W_\epsilon \times D_R)$, we have the inequality $|\eta| \geq \frac{2}{3}\mu|\xi|^r$ by Lemma 8.6, which assures that the map (8.10) is proper. Hence by Lemma 7.6 the map (8.10) is a finite branched covering. □

Proposition 8.7. *The degree of the branched covering map (8.10) defined in Proposition 8.4 is equal to the right hand side of the equality (6.3) in Theorem B.*

Proof. To consider the map F on

$$\widetilde{W}' := \left\{ (\eta, 0) \in W_\epsilon \times D_R \;\middle|\; \eta_{u,k} = 0 \;\text{ for } 1 \leq u \leq l \text{ and } 1 \leq k \leq r_u - 1 \right\},$$

we define the map $F_u : \mathbb{C}^{r_u} \to \mathbb{C}^{r_u}$ by $F_u(\xi_u) = (p_{u,1}(\xi_u), \ldots, p_{u,r_u}(\xi_u))$, and put

$$X_u := \left\{ \xi_u \in \mathbb{C}^{r_u} \;\middle|\; \begin{array}{l} p_{u,k}(\xi_u) = 0 \;\text{ for } 1 \leq k \leq r_u - 1 \\ p_{u,r_u}(\xi_u) \neq 0 \end{array} \right\}$$

for each u. We consider the following two lemmas:

Lemma 8.8. *The Jacobian of the map F_u is not zero at any point of X_u.*

Lemma 8.9. *The degree of the map $p_{u,r_u}|_{X_u} : X_u \to \mathbb{C}^*$ is $r_u \cdot \#(S_{r_u+1}(\lambda_{I_u}))$, where we define $\#(S_{r_u+1}(\lambda_{I_u})) = 1$ if $r_u \leq 2$.*

Lemma 8.8 assures that the branched covering map (8.10) is unbranched on some neighborhood of $\widetilde{W}'$ in $W_\epsilon \times D_R$, that X_u is a smooth Riemann surface, and that the map $p_{u,r_u}|_{X_u} : X_u \to \mathbb{C}^*$ is unbranched. Therefore the degree of the map (8.10) is equal to that of the map $(U \times D_R) \cap F^{-1}(\widetilde{W}') \xrightarrow{F} \widetilde{W}'$, which is also equal to $\prod_{1 \leq u \leq l} \deg(p_{u,r_u}|_{X_u})$; hence Lemmas 8.8 and 8.9 imply the proposition.

We show Lemma 8.8 first. Since $p_{u,k}(\xi_u) = \sum_{i=1}^{r_u} m_{u,i}\xi_{u,i}^k$, we have

$$\det(dF_u)(\xi_u) = r_u! \cdot \prod_{i=1}^{r_u} m_{u,i} \cdot \prod_{1 \leq i < j \leq r_u} (\xi_{u,j} - \xi_{u,i})$$

by a similar computation to the proof of Lemma 6.1. Hence the Jacobian is not equal to zero if and only if $\xi_{u,1}, \ldots, \xi_{u,r_u}$ are mutually distinct. On the other hand, by a similar argument to the proof of Lemma 6.5, we find that for a common zero $\xi_u = (\xi_{u,1}, \dots, \xi_{u,r_u})$ of $p_{u,1}, \dots, p_{u,r_u-1}$, the inequality $p_{u,r_u}(\xi_u) \neq 0$ holds if and only if $0, \xi_{u,1}, \dots, \xi_{u,r_u}$ are mutually distinct. Hence for any $\xi_u \in X_u$, the Jacobian $\det(dF_u)(\xi_u)$ is not zero, which completes the proof of Lemma 8.8.

We show Lemma 8.9 next. Since $p_{u,k}(\xi_u)$ is homogeneous for any u and k, the Riemann surface X_u is invariant under the action of $\mathbb{C}^*$; hence the set

$$\left\{(\xi_{u,1} : \dots : \xi_{u,r_u}) \in \mathbb{P}^{r_u-1} \;\middle|\; (\xi_{u,1}, \dots, \xi_{u,r_u}) \in X_u\right\}$$

is well-defined and is equal to $S_{r_u+1}(\lambda_{I_u})$ by definition. Therefore X_u consists of $\#\left(S_{r_u+1}(\lambda_{I_u})\right)$ components, each of which is biholomorphic to $\mathbb{C}^*$. Moreover on each component of X_u, the degree of the map p_{u,r_u} is $\deg p_{u,r_u} = r_u$, which completes the proofs of Lemma 8.9 and the proposition. $\square$

On the basis of Propositions 8.4 and 8.7, we prove the following:

Proposition 8.10. *Let $\psi_k(\xi)$ be the expression defined in the equality (7.1). Then the number*

$$\mathrm{mult}_0(\psi_1, \dots, \psi_{d-l})$$

is equal to the right hand side of the equality (6.3) in Theorem B.

Proof. We define the map $\Psi : \mathbb{C}^{d-l} \to \mathbb{C}^{d-l}$ by $\Psi(\xi) := (\psi_k(\xi))_{1 \le k \le d-l}$, and put

$$Y := \left\{\xi \in \mathbb{C}^{d-l} \;\middle|\; \psi_1(\xi) = \dots = \psi_{d-l-1}(\xi) = 0,\ \psi_{d-l}(\xi) \neq 0\right\}.$$

We denote by $M_{(r_1,\dots,r_l)}$ the square matrix M defined in Lemma 7.8.

Lemma 8.11. *For any open neighborhood $\widetilde{U}'$ of 0 in $\mathbb{C}^{d-l}$, there exist open neighborhoods U', W' of 0 with $U' \subset \widetilde{U}'$ and $W' \subset \mathbb{C}$ such that $Y \cap U'$ is a smooth Riemann surface, that the map*

$$Y \cap U' \cap \psi_{d-l}^{-1}(W' \setminus \{0\}) \overset{\psi_{d-l}}{\to} W' \setminus \{0\} \tag{8.15}$$

is an unbranched covering, and that the number $\mathrm{mult}_0(\psi_1, \dots, \psi_{d-l})$ is equal to the degree of the map (8.15).

Proof. First we shall check that $\det(d\Psi)(\xi) \neq 0$ holds for any $\xi \in Y \cap U'$, if we take U' sufficiently small. By a similar argument to the proof of Lemma 6.1, the equality $\det(d\Psi)(\xi) = 0$ holds for $\xi \in U'$ if and only if $\alpha_u + \xi_{u,i} = \alpha_v + \xi_{v,j}$ holds for some u, i, v and j with $(u, i) \neq (v, j)$, which is equivalent to the condition that $\xi_{u,i} = \xi_{u,j}$ holds for some u, i and j with $i \neq j$ if we take U' sufficiently small. Suppose for instance that $\xi_{1,1} = \xi_{1,2}$

holds for some $\xi \in Y \cap U'$. Then putting $\Psi'(\xi) := (\psi_k(\xi))_{1\le k\le d-l-1}$, considering the map $M^{-1}_{(r_1-1,r_2,\dots,r_l)} \circ \Psi'$, and keeping in mind the inequalities (8.13), we have $p_{u,k}(\xi) = 0$ for any u and k, which contradicts $\psi_{d-l}(\xi) \neq 0$. Therefore we have $\det(d\Psi)(\xi) \neq 0$ for any $\xi \in Y \cap U'$, which assures that $Y \cap U'$ is a smooth Riemann surface, and that the map (8.15) is an unbranched covering if we take W' sufficiently small. Moreover since $\det(d\Psi)(\xi) \neq 0$ for any $\xi \in Y \cap U'$, we have $\mathrm{mult}_{Y'}(\psi_1, \dots, \psi_{d-l-1}) = 1$ for any connected component Y' of $Y \cap U'$; hence we have $\mathrm{mult}_0(\psi_1, \dots, \psi_{d-l}) = \mathrm{mult}_0(\overline{Y} \cap U', \psi_{d-l})$ by definition, where $\overline{Y} \cap U'$ is the closure of $Y \cap U'$ in U'. Since $\mathrm{mult}_0(\overline{Y} \cap U', \psi_{d-l})$ is clearly equal to the degree of the covering map (8.15), all the assertions in Lemma 8.11 are verified. $\square$

We proceed the proof of the proposition. It is clear that there exists $A = (a_{u,k,v,h}) \in \mathbb{C}^{(l-1)(d-l)^2}$ such that the equality $F(\xi, A) = (M^{-1}_{(r_1,\dots,r_l)} \circ \Psi(\xi), A)$ holds for any $\xi \in \mathbb{C}^{d-l}$. Let e be the $(d-l, 1)$ column vector whose $(d-l)$-th entry is 1 and whose other entries are 0. Moreover we put $M^{-1}_{(r_1,\dots,r_l)} e =: \eta = (\eta_{u,k})_{1\le u\le l, 1\le k\le r_u}$. Then the equality $Y \times \{A\} = F^{-1}(\mathbb{C}\eta \setminus \{0\}, A)$ holds, and the map $F|_{Y\times\{A\}}$ is equal to the map $M^{-1}_{(r_1,\dots,r_l)} \circ \Psi|_Y$. Hence, if we can show $\eta_{u,r_u} \neq 0$ for $1 \le u \le l$, then we have $(\mathbb{C}\eta \setminus \{0\}) \cap W \subseteq W_\epsilon$ for some ϵ, which assures that the degree of the covering map (8.15) is equal to that of the branched covering map (8.10); thus the proposition will be verified by Proposition 8.7 and Lemma 8.11.

We show $\eta_{u,r_u} \neq 0$ for $1 \le u \le l$. Suppose $\eta_{l,r_l} = 0$ for instance, and put $\eta' = {}^t(\eta_{1,1}, \dots, \eta_{l,r_l-1}) \in \mathbb{C}^{d-l-1}$ so that the equality $\eta = {}^t({}^t\eta', 0)$ holds. Then by the equality $e = M_{(r_1,\dots,r_l)}\eta$, we have $0 = M_{(r_1,\dots,r_{l-1},r_l-1)}\eta'$. Since $M_{(r_1,\dots,r_{l-1},r_l-1)}$ is invertible, we have $\eta' = 0$, which implies $\eta = 0$ and the contradiction $e = M_{(r_1,\dots,r_l)}0 = 0$. Therefore $\eta_{u,r_u} \neq 0$ holds for any $1 \le u \le l$, which completes the proof of the proposition. $\square$

We complete the proof of Theorem B.

Proof of Theorem B. Remember the definition of $\mathbb{I}(\alpha) \in \mathfrak{I}(\lambda)$ for $\alpha \in B_d(\lambda)$ in the proof of Lemma 6.5. By Lemma 8.3, we can easily verify that for any $\alpha \in B_d(\lambda)$ there exists an open neighborhood O_α of α in $\mathbb{P}^{d-2}$ such that the equality

$$\left\{\zeta \in O_\alpha \;\middle|\; \varphi_k(\zeta) = 0 \;\text{ for } 1 \le k \le d - \#\left(\mathbb{I}(\alpha)\right)\right\} = B_d(\lambda) \cap O_\alpha$$

holds, which implies the first two assertions in Theorem B. On the other hand, the last assertion in Theorem B is the direct consequence of Propositions 7.1 and 8.10. $\square$

At the end of this section, we prove Proposition D.

Proof of Proposition D. For the brevity of notation, we put

$$\mathfrak{I}'(\lambda) := \mathfrak{I}(\lambda) \cup \left\{\{\{1, \dots, d\}\}\right\} \quad \text{for } \lambda \in V_d,$$

$$e_{\mathbb{I}}(\lambda) := \mathrm{mult}_{E_d(\mathbb{I})}(\varphi_1, \ldots, \varphi_{d-\#(\mathbb{I})}) \quad \text{for each } \mathbb{I} \in \mathfrak{I}(\lambda), \text{ and}$$
$$e_{\{\{1,\ldots,d\}\}}(\lambda) := (d-1) \cdot \# \left(S_d(\lambda)\right).$$

Note that $\{\{1,\ldots,d\}\}$ is the only minimum element of $\mathfrak{I}'(\lambda)$ with respect to the partial order $\prec$.

Under the notation above, the equality (6.4) in Proposition C is equivalent to the equality

$$(d-1)! = \sum_{\mathbb{I} \in \mathfrak{I}'(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#(\mathbb{I})+1}^{d-1} k \right), \tag{8.16}$$

whereas the equality (6.3) in Theorem B is rewritten in the form

$$e_{\mathbb{I}}(\lambda) = \prod_{u=1}^{l} e_{\{I_u\}}\left(\lambda_{I_u}\right) = \prod_{I \in \mathbb{I}} e_{\{I\}}\left(\lambda_I\right), \tag{8.17}$$

where $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, and $\{I\}$ denotes the minimum element of the set $\mathfrak{I}'\left(\lambda_I\right)$ for each $I \in \mathcal{I}(\lambda)$. On the other hand, Proposition D is rewritten in the form

$$\prod_{u=1}^{l} \left(\#\left(I_u\right) - 1\right)! = \sum_{\mathbb{I}' \in \mathfrak{I}(\lambda),\ \mathbb{I}' \succ \mathbb{I}} \left(e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k \right) \right) \tag{8.18}$$

for $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$, where $\chi_u(\mathbb{I}')$ is the one defined in Main Theorem III. Note that $\mathbb{I} \succ \mathbb{I}$ holds for any $\mathbb{I} \in \mathfrak{I}'(\lambda)$. To complete the proof of Proposition D, we only need to derive the equality (8.18) from the equalities (8.16) and (8.17).

Note that for $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}'(\lambda)$, we have

$$\left\{ \mathbb{I}' \in \mathfrak{I}'(\lambda) \,\middle|\, \mathbb{I}' \succ \mathbb{I} \right\} = \left\{ \mathbb{I}_1 \cup \cdots \cup \mathbb{I}_l \,\middle|\, \mathbb{I}_u \in \mathfrak{I}'\left(\lambda_{I_u}\right) \quad \text{for } 1 \leq u \leq l \right\}$$

by definition. Hence we have the following equalities for $\mathbb{I} = \{I_1, \ldots, I_l\} \in \mathfrak{I}(\lambda)$ from the equalities (8.16) and (8.17):

$$\begin{aligned}
\prod_{u=1}^{l} \left(\#\left(I_u\right) - 1\right)! &= \prod_{u=1}^{l} \left(\sum_{\mathbb{I}_u \in \mathfrak{I}'(\lambda_{I_u})} \left(e_{\mathbb{I}_u}\left(\lambda_{I_u}\right) \cdot \prod_{k=\#(I_u)-\#(\mathbb{I}_u)+1}^{\#(I_u)-1} k \right) \right) \\
&= \sum_{\mathbb{I}_1 \in \mathfrak{I}'(\lambda_{I_1})} \cdots \sum_{\mathbb{I}_l \in \mathfrak{I}'(\lambda_{I_l})} \prod_{u=1}^{l} \left(\prod_{I'_u \in \mathbb{I}_u} e_{\{I'_u\}}\left(\lambda_{I'_u}\right) \cdot \prod_{k=\#(I_u)-\#(\mathbb{I}_u)+1}^{\#(I_u)-1} k \right) \\
&= \sum_{\mathbb{I}_1 \in \mathfrak{I}'(\lambda_{I_1})} \cdots \sum_{\mathbb{I}_l \in \mathfrak{I}'(\lambda_{I_l})} \left(e_{\mathbb{I}_1 \cup \cdots \cup \mathbb{I}_l}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\#(\mathbb{I}_u)+1}^{\#(I_u)-1} k \right) \right)
\end{aligned}$$

$$= \sum_{\mathbb{I}' \in \mathfrak{I}(\lambda),\ \mathbb{I}' \succ \mathbb{I}} \left(e_{\mathbb{I}'}(\lambda) \cdot \prod_{u=1}^{l} \left(\prod_{k=\#(I_u)-\chi_u(\mathbb{I}')+1}^{\#(I_u)-1} k \right) \right).$$

The equality (8.18) is thus obtained, which completes the proof of Proposition D. □

9. Relation between the sets $S_d(\lambda)$ and $\Phi_d^{-1}(\bar{\lambda})$

In this section we state the explicit relation between the cardinalities $\#(S_d(\lambda))$ and $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$. Let λ be an element of V_d, which is fixed throughout this section. Remember the definitions of $K_1, \dots, K_q, \kappa_1, \dots, \kappa_q, g_1, \dots, g_q$ defined in Definition 1.7, and $\mathfrak{S}(\mathcal{K}(\lambda))$ defined in Definition 4.2. We put

$$\Sigma_d(\lambda) := \left\{ (\zeta_1 : \cdots : \zeta_d) \in \mathbb{P}^{d-1} \;\middle|\; \begin{array}{c} \sum_{i=1}^{d} \zeta_i = 0 \\ \sum_{i=1}^{d} m_i \zeta_i^k = 0 \quad \text{for} \quad 1 \le k \le d-2 \\ \zeta_1, \dots, \zeta_d \text{ are mutually distinct} \end{array} \right\}.$$

Proposition 9.1. *The bijection $\tilde{\imath} : \Sigma_d(\lambda) \to S_d(\lambda)$ is defined by*

$$(\zeta_1 : \cdots : \zeta_d) \mapsto (\zeta_1 - \zeta_d : \cdots : \zeta_{d-1} - \zeta_d).$$

The group $\mathfrak{S}(\mathcal{K}(\lambda))$ acts on $\Sigma_d(\lambda)$ by the permutation of the homogeneous coordinates. Moreover the actions of $\mathfrak{S}(\mathcal{K}(\lambda))$ on $S_d(\lambda)$ and $\Sigma_d(\lambda)$ commute with the map $\tilde{\imath}$; hence we have the bijection $\Sigma_d(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda)) \overset{\cong}{\to} \Phi_d^{-1}\left(\bar{\lambda}\right)$.

Proof. The bijectivity of the map $\iota(\lambda)$ in Proposition 4.7 implies the proposition. □

Lemma 9.2. *Let $\zeta = (\zeta_1 : \cdots : \zeta_d)$ be an element of $\Sigma_d(\lambda)$ and suppose that there exists a non-identity permutation $\sigma \in \mathfrak{S}(\mathcal{K}(\lambda))$ with $\sigma \cdot \zeta = \zeta$. Then there exists a unique suffix i with $\zeta_i = 0$. Moreover if $i \in K_w$, then the fixing subgroup $\left\{\sigma \in \mathfrak{S}(\mathcal{K}(\lambda)) \;\middle|\; \sigma \cdot \zeta = \zeta\right\}$ of ζ is a cyclic group whose order divides g_w.*

Proof. For any $\sigma \in \mathfrak{S}(\mathcal{K}(\lambda))$ with $\sigma \cdot \zeta = \zeta$, there exists a non-zero complex number a satisfying $\zeta_{\sigma^{-1}(i)} = a\zeta_i$ for $1 \le i \le d$, which induces the injective group homomorphism

$$\mathfrak{S}(\zeta) := \left\{\sigma \in \mathfrak{S}(\mathcal{K}(\lambda)) \;\middle|\; \sigma \cdot \zeta = \zeta\right\} \ni \sigma \overset{\mathfrak{a}}{\mapsto} a \in \left\{a \in \mathbb{C}^* \;\middle|\; |a| = 1\right\}.$$

In the following, we fix non-identity $\sigma \in \mathfrak{S}(\zeta)$, and denote by t the order of σ. Then $a = \mathfrak{a}(\sigma)$ is a primitive t-th radical root of 1. Moreover the cardinality $\#\left(\left\{\sigma^s(i) \;\middle|\; s \in \mathbb{Z}\right\}\right)$ is equal to 1 or t according as ζ_i is equal to 0 or not.

Suppose that $\zeta_i \neq 0$ holds for any i. Then t is a common divisor of $\kappa_1, \dots, \kappa_q$. We may assume

$$m = (\underbrace{m_1, \dots, m_1}_{t}, \dots, \underbrace{m_{d/t}, \dots, m_{d/t}}_{t})$$

and

$$\zeta = (\zeta_1 : a\zeta_1 : \cdots : a^{t-1}\zeta_1 : \cdots : \zeta_{d/t} : a\zeta_{d/t} : \cdots : a^{t-1}\zeta_{d/t}).$$

Under the above notation, the equations $\varphi_k(\zeta) = 0$ for $1 \leq k \leq d-2$ are equivalent to the equations $\sum_{i=1}^{d/t} m_i \zeta_i^{tk} = 0$ for $1 \leq k \leq \frac{d}{t} - 1$, which implies $m_i = 0$ for any i by the mutual distinctness of $0, \zeta_1^t, \ldots, \zeta_{d/t}^t$. We thus obtain contradiction, which assures the existence of i with $\zeta_i = 0$.

Next we suppose $\zeta_i = 0$ and $i \in K_w$. Then for any $\sigma \in \mathfrak{S}(\zeta)$, the order t of σ is a common divisor of $\kappa_1, \ldots, \kappa_{w-1}$, $\kappa_w - 1$, $\kappa_{w+1}, \ldots, \kappa_q$, i.e., a divisor of g_w. Therefore $\mathfrak{S}(\zeta)$ is isomorphic to a subgroup of $\left\{a \in \mathbb{C}^* \mid a^{g_w} = 1\right\}$ by the map $\mathfrak{a}$, which completes the proof. $\square$

Remember the definitions of $d[t]$ and $\lambda[t]$ in Definition 1.7. In the following, the symbol $a|b$ denotes that a divides b for positive integers a and b.

Theorem E. *If we put $s_d(\lambda) := \#\left(S_d(\lambda)\right) = \#\left(\Sigma_d(\lambda)\right)$ for $\lambda \in V_d$, then the third and fourth steps in Main Theorem III hold.*

Proof. For each $t \in \bigcup_{1 \leq w \leq q} \left\{t \mid t|g_w\right\}$, we put

$$\Theta_t(\lambda) := \left\{C \in \Sigma_d(\lambda)/\mathfrak{S}\left(\mathcal{K}(\lambda)\right) \;\middle|\; \#(C) = \frac{\#\left(\mathfrak{S}\left(\mathcal{K}(\lambda)\right)\right)}{t}\right\}$$

and $c_t(\lambda) := \#\left(\Theta_t(\lambda)\right)$. Then by Proposition 9.1 and Lemma 9.2, we have

$$\Phi_d^{-1}\left(\bar{\lambda}\right) \overset{\cong}{\leftarrow} \Sigma_d(\lambda)/\mathfrak{S}\left(\mathcal{K}(\lambda)\right) = \left(\coprod_{w=1}^{q}\left(\coprod_{t|g_w,\ t\geq 2}\Theta_t(\lambda)\right)\right)\coprod \Theta_1(\lambda),$$

which implies the equalities (1.5) and (1.6). Hence to complete the proof, we only need to show the equalities (1.4) for each t with $t \geq 2$. In the rest of the proof, we fix $1 \leq w \leq q$.

For each t with $t|g_w$ and $t \geq 2$, we define the group $\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right)$ to be isomorphic to $\mathfrak{S}_{\frac{\kappa_1}{t}} \times \cdots \times \mathfrak{S}_{\frac{\kappa_w - 1}{t}} \times \cdots \times \mathfrak{S}_{\frac{\kappa_q}{t}}$. Then $\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right)$ naturally acts on $S_{d[t]}(\lambda[t])$, and we have $\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right) \subseteq \mathfrak{S}\left(\mathcal{K}\left(\lambda[t]\right)\right)$. Note that in some cases the equality $\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right) = \mathfrak{S}\left(\mathcal{K}\left(\lambda[t]\right)\right)$ does not hold, e.g., $\lambda[2]$ in Example 3 in Section 2. For each divisor b of $\frac{g_w}{t}$, we put

$$\Theta_b'(\lambda[t]) := \left\{C' \in S_{d[t]}(\lambda[t])/\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right) \;\middle|\; \#(C') = \frac{\#\left(\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right)\right)}{b}\right\}.$$

Then we have

$$S_{d[t]}(\lambda[t])/\mathfrak{S}\left(\mathcal{K}'\left(\lambda[t]\right)\right) = \coprod_{b|(g_w/t)} \Theta'_b(\lambda[t]) \tag{9.1}$$

by a similar argument to the proof of Lemma 9.2.

Let t, b be positive integers with $t|b$, $b|g_w$ and $t \geq 2$, and a a primitive b-th radical root of 1. Then a point

$$\left(\zeta_1 : a\zeta_1 : \cdots : a^{b-1}\zeta_1 : \cdots : \zeta_{d[b]-1} : a\zeta_{d[b]-1} : \cdots : a^{b-1}\zeta_{d[b]-1} : 0\right) \in \mathbb{P}^{d-1}$$

represents an element of $\Theta_b(\lambda)$ if and only if

$$\left(\zeta_1^t : a^t\zeta_1^t : \cdots : a^{t((b/t)-1)}\zeta_1^t : \cdots : \zeta_{d[b]-1}^t : a^t\zeta_{d[b]-1}^t : \cdots : a^{t((b/t)-1)}\zeta_{d[b]-1}^t\right) \in \mathbb{P}^{d[t]-2}$$

represents an element of $\Theta'_{b/t}(\lambda[t])$, which gives the bijection between $\Theta_b(\lambda)$ and $\Theta'_{b/t}(\lambda[t])$. The bijection and the equality (9.1) imply the equalities (1.4), which completes the proof of the theorem. □

10. Completion of the proof

In Propositions 4.8, 4.10, 6.2 and 6.6, we had already proved the assertions (5), (7), (1) and (4) in Main Theorem I. In this section we complete the rest of the proofs of the main theorems.

Proposition 10.1. *Main Theorem III and the assertion (2) in Main Theorem I hold.*

Proof. These two are the direct consequences of Theorem B, Propositions C, D and Theorem E. □

Proposition 10.2. *Main Theorem II and the assertion (3) in Main Theorem I hold.*

Proof. In the following, we always identify V_d with

$$\left\{ (m_1, \ldots, m_d) \in (\mathbb{C}^*)^d \;\middle|\; \sum_{i=1}^{d} m_i = 0 \right\}$$

by the correspondence $m_i = \frac{1}{1-\lambda_i}$, and define the following spaces:

$$\begin{aligned}
\mathrm{MP}'_d &:= \Phi_d^{-1}(\widetilde{V}_d), \\
\mathcal{X}_d &:= \left\{ (\zeta_1, \ldots, \zeta_d, \rho) \in \mathbb{C}^d \times \mathbb{C}^* \;\middle|\; \zeta_1, \ldots, \zeta_d \text{ are mutually distinct} \right\}, \\
\widetilde{\mathcal{X}}_d &:= \mathcal{X}_d/\mathrm{Aut}(\mathbb{C}), \\
(\mathcal{P}\mathcal{X})_d &:= \left\{ (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \;\middle|\; \zeta_1, \ldots, \zeta_d \text{ are mutually distinct} \right\},
\end{aligned}$$

$$
\begin{aligned}
(\widetilde{\mathcal{P}\mathcal{X}})_d &:= (\mathcal{P}\mathcal{X})_d/\mathrm{Aut}(\mathbb{C}),\\
(\mathcal{P}\mathcal{V})_d &:= \left\{(m_1 : \cdots : m_d) \in \mathbb{P}^{d-1} \;\middle|\; \sum_{i=1}^{d} m_i = 0,\ m_i \neq 0 \text{ for } 1 \leq i \leq d\right\},\\
\mathcal{Y}_d &:= \left\{((\zeta,\rho),m) \in \widetilde{\mathcal{X}}_d \times V_d \;\middle|\; \sum_{i=1}^{d} m_i\zeta_i^k = \begin{cases} 0 & (1 \leq k \leq d-2) \\ -\frac{1}{\rho} & (k = d-1) \end{cases}\right\},\\
(\mathcal{P}\mathcal{Y})_d &:= \left\{(\zeta,m) \in (\widetilde{\mathcal{P}\mathcal{X}})_d \times (\mathcal{P}\mathcal{V})_d \;\middle|\; \sum_{i=1}^{d} m_i\zeta_i^k = 0 \text{ for } 1 \leq k \leq d-2\right\},
\end{aligned}
$$

where the actions of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{X}_d$ and $(\mathcal{P}\mathcal{X})_d$ are defined by

$$
\gamma \cdot (\zeta_1, \ldots, \zeta_d, \rho) = \left(\gamma(\zeta_1), \ldots, \gamma(\zeta_d), a^{-d+1}\rho\right) \quad \text{and} \quad \gamma \cdot (\zeta_1, \ldots, \zeta_d) = (\gamma(\zeta_1), \ldots, \gamma(\zeta_d))
$$

for $\gamma(z) = az + b \in \mathrm{Aut}(\mathbb{C})$, $(\zeta_1, \ldots, \zeta_d, \rho) \in \mathcal{X}_d$ and $(\zeta_1, \ldots, \zeta_d) \in (\mathcal{P}\mathcal{X})_d$. Then we have the commutative diagram

$$
\begin{tikzcd}
(\widetilde{\mathcal{P}\mathcal{X}})_d & \widetilde{\mathcal{X}}_d \arrow[l, "/\mathbb{C}^*"'] \arrow[dr, "/\mathfrak{S}_d"] & \\
(\mathcal{P}\mathcal{Y})_d \arrow[u, "\cong"'] \arrow[d, "\widetilde{\Phi}'_d"'] & \mathcal{Y}_d \arrow[l, "/\mathbb{C}^*"] \arrow[r, "/\mathfrak{S}_d"'] \arrow[u, "\cong"'] \arrow[d, "\Phi'_d"'] & \mathrm{MP}'_d \arrow[d, "\Phi_d"'] \\
(\mathcal{P}\mathcal{V})_d & V_d \arrow[l, "P"', "/\mathbb{C}^*"] \arrow[r, "pr", "/\mathfrak{S}_d"'] & \widetilde{V}_d,
\end{tikzcd}
$$

where each map is defined to be the natural projection except for the maps Φ_d and

$$
\widetilde{\mathcal{X}}_d \ni (\zeta_1, \ldots, \zeta_d, \rho) \mapsto z + \rho(z-\zeta_1)\cdots(z-\zeta_d) \in \mathrm{MP}'_d.
$$

Here, the first projection maps $\mathcal{Y}_d \to \widetilde{\mathcal{X}}_d$ and $(\mathcal{P}\mathcal{Y})_d \to (\widetilde{\mathcal{P}\mathcal{X}})_d$ are isomorphisms. The d-th symmetric group $\mathfrak{S}_d$ acts on $\widetilde{\mathcal{X}}_d$, $\mathcal{Y}_d$ and V_d by the permutation of coordinates. These actions of $\mathfrak{S}_d$ commute with the projection maps $\mathcal{Y}_d \overset{\cong}{\to} \widetilde{\mathcal{X}}_d$ and $\Phi'_d : \mathcal{Y}_d \to V_d$. Moreover we have the natural isomorphisms $\mathcal{Y}_d/\mathfrak{S}_d \cong \widetilde{\mathcal{X}}_d/\mathfrak{S}_d \cong \mathrm{MP}'_d$ and $V_d/\mathfrak{S}_d \cong \widetilde{V}_d$. On the other hand, the multiplicative group $\mathbb{C}^*$ acts on $\widetilde{\mathcal{X}}_d$, $\mathcal{Y}_d$ and V_d by $a \cdot (\zeta, \rho) = (\zeta, a^{-1}\rho)$ and $a \cdot (m_1, \ldots, m_d) = (am_1, \ldots, am_d)$ for $a \in \mathbb{C}^*$, $(\zeta, \rho) \in \widetilde{\mathcal{X}}_d$ and $(m_1, \ldots, m_d) \in V_d$. These actions of $\mathbb{C}^*$ are free, commute with the actions of $\mathfrak{S}_d$, and also commute with the projection maps $\mathcal{Y}_d \overset{\cong}{\to} \widetilde{\mathcal{X}}_d$ and $\Phi'_d : \mathcal{Y}_d \to V_d$. We have the natural isomorphisms $\widetilde{\mathcal{X}}_d/\mathbb{C}^* \cong (\widetilde{\mathcal{P}\mathcal{X}})_d \cong (\mathcal{P}\mathcal{Y})_d \cong \mathcal{Y}_d/\mathbb{C}^*$ and $V_d/\mathbb{C}^* \cong (\mathcal{P}\mathcal{V})_d$.

Therefore to analyze the fiber structure of the map $\Phi_d|_{\mathrm{MP}'_d}$, we only need to consider the second projection map $\widetilde{\Phi}'_d : (\mathcal{P}\mathcal{Y})_d \to (\mathcal{P}\mathcal{V})_d$ and the actions of $\mathfrak{S}_d$ on $\mathcal{Y}_d$ and V_d, most of which had however already been examined since we can make the following identifications as usual:

$$
\begin{aligned}
(\widetilde{\mathcal{PX}})_d &= \left\{ (\zeta_1 : \cdots : \zeta_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \zeta_1, \ldots, \zeta_{d-1}, 0 \text{ are mutually distinct} \right\}, \\
(\mathcal{PV})_d &= \left\{ (m_1 : \cdots : m_{d-1}) \in \mathbb{P}^{d-2} \;\middle|\; \sum_{i=1}^{d-1} m_i \neq 0,\ m_i \neq 0 \text{ for } 1 \leq i \leq d-1 \right\}, \\
(\mathcal{PY})_d &= \left\{ (\zeta, m) \in (\widetilde{\mathcal{PX}})_d \times (\mathcal{PV})_d \;\middle|\; \sum_{i=1}^{d-1} m_i \zeta_i^k = 0 \text{ for } 1 \leq k \leq d-2 \right\}.
\end{aligned}
$$

Especially, we have $(\widetilde{\Phi}'_d)^{-1}(P(\lambda)) = S_d(\lambda)$ for any $\lambda \in V_d$.

For each $(\mathcal{I}, \mathcal{K}) \in \left\{ (\mathcal{I}(\lambda), \mathcal{K}(\lambda)) \;\middle|\; \lambda \in V_d \right\}$, we put

$$
\begin{aligned}
\overline{V(\mathcal{I}, \mathcal{K})} &:= \left\{ \lambda \in V_d \,\middle|\, \mathcal{I}(\lambda) \supseteq \mathcal{I},\ \mathcal{K}(\lambda) \supseteq \mathcal{K} \right\}, \\
V(\mathcal{I}, \mathcal{K}) &:= \left\{ \lambda \in V_d \,\middle|\, \mathcal{I}(\lambda) = \mathcal{I},\ \mathcal{K}(\lambda) = \mathcal{K} \right\}, \\
V(\mathcal{I}, *) &:= \left\{ \lambda \in V_d \,\middle|\, \mathcal{I}(\lambda) = \mathcal{I} \right\}, \\
V(*, \mathcal{K}) &:= \left\{ \lambda \in V_d \,\middle|\, \mathcal{K}(\lambda) = \mathcal{K} \right\}
\end{aligned}
$$

and $\mathcal{PV}(\mathcal{I}, *) := P(V(\mathcal{I}, *))$. Remember that $\widetilde{V}(\mathcal{I}, \mathcal{K}) = pr(V(\mathcal{I}, \mathcal{K}))$, $\widetilde{V}(\mathcal{I}, *) = pr(V(\mathcal{I}, *))$ and $\widetilde{V}(*, \mathcal{K}) = pr(V(*, \mathcal{K}))$ hold by the definition in Main Theorem II. Note that $V(\mathcal{I}, \mathcal{K})$ is a Zariski open subset of $\overline{V(\mathcal{I}, \mathcal{K})}$.

First, we show the assertion (3) in Main Theorem I. Let λ_0, λ' be elements of V_d with $\mathcal{I}(\lambda_0) \subseteq \mathcal{I}(\lambda')$ and $\mathcal{K}(\lambda_0) \subseteq \mathcal{K}(\lambda')$. Then we have $\lambda' \in \overline{V(\mathcal{I}(\lambda_0), \mathcal{K}(\lambda_0))}$ and $\mathfrak{S}(\mathcal{K}(\lambda_0)) \subseteq \mathfrak{S}(\mathcal{K}(\lambda'))$. By Lemma 6.1 and Implicit function theorem, the second projection map $\widetilde{\Phi}'_d$ is locally homeomorphic, which implies that the map Φ'_d is also a local homeomorphism. We put $(\Phi'_d)^{-1}(\lambda') = \{\zeta(1), \ldots, \zeta(s_d(\lambda'))\}$. Then there exist an open neighborhood U of λ' in $\overline{V(\mathcal{I}(\lambda_0), \mathcal{K}(\lambda_0))}$ and holomorphic sections $\tau_j : U \to \mathcal{Y}_d$ for $1 \leq j \leq s_d(\lambda')$ such that $\Phi'_d \circ \tau_j = id_U$ and $\tau_j(\lambda') = \zeta(j)$. Moreover the action of $\mathfrak{S}(\mathcal{K}(\lambda_0))$ on $(\Phi'_d)^{-1}(\lambda')$ is naturally extended to the action of $\mathfrak{S}(\mathcal{K}(\lambda_0))$ on $\left\{ \tau_j(\lambda) \,\middle|\, 1 \leq j \leq s_d(\lambda') \right\}$ for any $\lambda \in U$. Hence we have $\#\left(\Phi_d^{-1}(\bar{\lambda}_0)\right) \geq \#\left(\Phi_d^{-1}(\bar{\lambda}')\right)$, which completes the proof of the assertion (3) in Main Theorem I.

Let us prove next the assertion (2) in Main Theorem II. Since the map Φ'_d is locally homeomorphic and since the map $pr|_{V(*, \mathcal{K})} : V(*, \mathcal{K}) \to \widetilde{V}(*, \mathcal{K})$ is an unbranched covering for each $\mathcal{K} \in \left\{ \mathcal{K}(\lambda) \,\middle|\, \lambda \in V_d \right\}$, the map $\Phi_d|_{\Phi_d^{-1}(\widetilde{V}(*, \mathcal{K}))} : \Phi_d^{-1}\left(\widetilde{V}(*, \mathcal{K})\right) \to \widetilde{V}(*, \mathcal{K})$ is a local homeomorphism, which verifies the assertion (2b) in Main Theorem II. For each $\mathcal{I} \in \left\{ \mathcal{I}(\lambda) \,\middle|\, \lambda \in V_d \right\}$, the cardinality of $(\widetilde{\Phi}'_d)^{-1}(m)$ does not depend on the choice of $m \in \mathcal{PV}(\mathcal{I}, *)$, which assures that the map $(\widetilde{\Phi}'_d)^{-1}(\mathcal{PV}(\mathcal{I}, *)) \overset{\widetilde{\Phi}'_d}{\to} \mathcal{PV}(\mathcal{I}, *)$ is an unbranched covering. Hence the map $(\Phi'_d)^{-1}(V(\mathcal{I}, *)) \overset{\Phi'_d}{\to} V(\mathcal{I}, *)$ is also an unbranched covering. Therefore since the map $V(\mathcal{I}, *) \overset{pr}{\to} \widetilde{V}(\mathcal{I}, *)$ is proper, the map $\Phi_d^{-1}\left(\widetilde{V}(\mathcal{I}, *)\right) \overset{\Phi_d}{\to} \widetilde{V}(\mathcal{I}, *)$ is also proper, which verifies the assertion (2a) in Main Theorem II. The assertions (2a) and (2b) imply the assertion (2c); thus we have completed the proof of the assertion (2) in Main Theorem II.

Finally, we prove the assertion (1) in Main Theorem II. In the following, we consider V_d as an open dense subset of the vector space $\mathbb{C}^{d-1} = \left\{(m_1, \dots, m_d) \in \mathbb{C}^d \;\middle|\; \sum_{i=1}^d m_i = 0\right\}$ with the standard inner product. We take $\lambda \in V_d$, and put $\mathcal{I}(\lambda) =: \mathcal{I}$ and $\mathcal{K}(\lambda) =: \mathcal{K}$, which are fixed in the rest of the proof. We denote by $H(\lambda)$ the orthogonal complement of the linear subspace spanned by $V(\mathcal{I}, \mathcal{K})$ in $\mathbb{C}^{d-1}$. Then the space $H(\lambda)$ is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$. Hence we can take an arbitrarily small open neighborhood $H_\epsilon(\lambda)$ of 0 in $H(\lambda)$ which is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$. Moreover we denote by $U(\lambda)$ a sufficiently small open neighborhood of λ in $V(\mathcal{I}, \mathcal{K})$. Then the map $H_\epsilon(\lambda) \times U(\lambda) \ni (h, m) \mapsto h + m \in V_d$ defines a local coordinate system around λ in V_d. Hereafter, we identify $(h, m) \in H_\epsilon(\lambda) \times U(\lambda)$ with $h + m \in V_d$.

Since $H_\epsilon(\lambda)$ and $U(\lambda)$ are sufficiently small, we have $\mathcal{I}(h, m) \subseteq \mathcal{I}(\lambda)$ and $\mathcal{K}(h, m) \subseteq \mathcal{K}(\lambda)$ for any $(h, m) \in H_\epsilon(\lambda) \times U(\lambda)$. Moreover $\mathcal{I}(h, m)$ and $\mathcal{K}(h, m)$ do not depend on the choice of $m \in U(\lambda)$. Hence, for each $h \in H_\epsilon(\lambda)$ and for each connected component Y of $(\Phi'_d)^{-1}(\{h\} \times U(\lambda))$, the map $\Phi'_d|_Y : Y \to \{h\} \times U(\lambda)$ is a homeomorphism. Therefore we have the natural isomorphism $(\Phi'_d)^{-1}(H_\epsilon(\lambda) \times U(\lambda)) \to (\Phi'_d)^{-1}(H_\epsilon(\lambda) \times \{\lambda\}) \times U(\lambda)$ which commutes with the projection maps onto $H_\epsilon(\lambda) \times U(\lambda)$.

For each $m \in U(\lambda)$, the space $H_\epsilon(\lambda) \times \{m\}$ is invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$ with a fixed point $(0, m)$. Moreover we have the natural isomorphism $(H_\epsilon(\lambda)/\mathfrak{S}(\mathcal{K}(\lambda))) \times U(\lambda) \cong (H_\epsilon(\lambda) \times U(\lambda))/\mathfrak{S}(\mathcal{K}(\lambda)) \cong pr(H_\epsilon(\lambda) \times U(\lambda))$. Hence $(\Phi'_d)^{-1}(H_\epsilon(\lambda) \times U(\lambda))$ is also invariant under the action of $\mathfrak{S}(\mathcal{K}(\lambda))$, and its action commutes with the isomorphism $(\Phi'_d)^{-1}(H_\epsilon(\lambda) \times U(\lambda)) \to (\Phi'_d)^{-1}(H_\epsilon(\lambda) \times \{\lambda\}) \times U(\lambda)$. Therefore we have the isomorphism

$$\Phi_d^{-1}(pr(H_\epsilon(\lambda) \times U(\lambda))) \cong \Phi_d^{-1}(pr(H_\epsilon(\lambda) \times \{\lambda\})) \times U(\lambda)$$

which commutes with the projection maps onto $pr(H_\epsilon(\lambda) \times U(\lambda))$. Hence for each $\lambda \in V(\mathcal{I}, \mathcal{K})$,

$$\left\{\lambda' \in V(\mathcal{I}, \mathcal{K}) \;\middle|\; \begin{array}{c} \text{the pair } \lambda, \lambda' \text{ satisfies the condition} \\ \text{in the assertion (1) in Main Theorem II} \end{array}\right\}$$

is an open subset of $V(\mathcal{I}, \mathcal{K})$ containing λ. Since $V(\mathcal{I}, \mathcal{K})$ is connected, the assertion (1) in Main Theorem II holds. □

Proposition 10.3. *The assertion (6) in Main Theorem I holds.*

Proof. The set $\Phi_d^{-1}(\bar{\lambda})$ is empty if and only if the set $S_d(\lambda)$ is empty by Proposition 4.3. On the other hand, the cardinality $\#(S_d(\lambda))$ is completely determined and is computed by $\mathfrak{I}(\lambda)$. Hence to show the assertion (6) in Main Theorem I, we only need to check all the possible cases of $\mathfrak{I}(\lambda)$. However this may be hard for $d = 6$ or 7, and we shall relieve it a little.

By a similar argument to the proof of the assertion (3) in Main Theorem I, we can verify that for $\lambda, \lambda' \in V_d$, the inequality $\#(S_d(\lambda)) \geq \#(S_d(\lambda'))$ holds if $\mathfrak{I}(\lambda) \subseteq \mathfrak{I}(\lambda')$. Hence putting

$$\widetilde{\mathfrak{I}}_d := \left\{ \mathfrak{I}(\lambda) \;\middle|\; \begin{array}{l} \lambda \in V_d \text{ does not satisfy the assumption} \\ \text{in the assertion (5) in Main Theorem I} \end{array} \right\},$$

we only need to show the inequality $\#(S_d(\lambda)) > 0$ for any *maximal* $\mathfrak{I}(\lambda)$ in $\widetilde{\mathfrak{I}}_d$. On the other hand, $\mathfrak{S}_d$ naturally acts on $\widetilde{\mathfrak{I}}_d$ by $\sigma \cdot \mathfrak{I}(\lambda) := \mathfrak{I}(\sigma \cdot \lambda)$, and the cardinality $\#(S_d(\lambda))$ is determined only by the equivalence class of $\mathfrak{I}(\lambda)$. Moreover the inclusion relation in $\widetilde{\mathfrak{I}}_d$ naturally induces the partial order in $\widetilde{\mathfrak{I}}_d/\mathfrak{S}_d$. Hence it suffices to show $\#(S_d(\lambda)) > 0$ for any maximal $\overline{\mathfrak{I}(\lambda)}$ in $\widetilde{\mathfrak{I}}_d/\mathfrak{S}_d$. In the following, we shall consider the cases of $d = 4, 5, 6$ and 7 individually.

In the case $d = 4$, the family $\widetilde{\mathfrak{I}}_4/\mathfrak{S}_4$ consists of the equivalence class of the empty set and that of $\big\{\{\{1,2\},\{3,4\}\}\big\}$. Hence the unique maximal element of $\widetilde{\mathfrak{I}}_4/\mathfrak{S}_4$ is represented by $\big\{\{\{1,2\},\{3,4\}\}\big\}$, which is obtained from $\lambda = (\lambda_1, \ldots \lambda_4) \in V_4$ with $(1-\lambda_1)^{-1} : \cdots : (1-\lambda_4)^{-1} = 1 : -1 : a : -a$, where $a \neq 0, \pm 1$. For such $\lambda \in V_4$, we have $\#(S_4(\lambda)) = 1$.

Let us consider the case $d = 5$ next. If $\mathfrak{I}(\lambda) \in \widetilde{\mathfrak{I}}_5$ have only one element, then $\mathfrak{I}(\lambda)$ lies in the equivalence class of $\big\{\{\{1,2\},\{3,4,5\}\}\big\}$. If $\mathfrak{I}(\lambda) \in \widetilde{\mathfrak{I}}_5$ have exactly two elements, then it must be in the equivalence class of $\big\{\{\{1,2\},\{3,4,5\}\},\ \{\{1,3\},\{2,4,5\}\}\big\}$ since any $(1-\lambda_i)^{-1}$ is not equal to 0. By a similar argument, if $\mathfrak{I}(\lambda)$ have at least three elements, then it must be in the equivalence class of $\big\{\{\{1,2\},\{3,4,5\}\},\{\{1,3\},\{2,4,5\}\},\{\{1,4\},\{2,3,5\}\}\big\}$. However this is obtained only from $\lambda = (\lambda_1, \ldots \lambda_5) \in V_5$ with $(1-\lambda_1)^{-1} : \cdots : (1-\lambda_5)^{-1} = -1 : 1 : 1 : 1 : -2$; hence it is not in $\widetilde{\mathfrak{I}}_5$ by definition. Therefore the maximal element of $\widetilde{\mathfrak{I}}_5/\mathfrak{S}_5$ is also unique and is represented by $\big\{\{\{1,2\},\{3,4,5\}\},\ \{\{1,3\},\{2,4,5\}\}\big\}$, which is obtained from $\lambda = (\lambda_1, \ldots \lambda_5) \in V_5$ with

$$(1-\lambda_1)^{-1} : \cdots : (1-\lambda_5)^{-1} = -1 : 1 : 1 : a : (-a-1),$$

where $a \neq 0, \pm 1, -2$. For such $\lambda \in V_5$, we have $\#(S_5(\lambda)) = 2$.

In the case $d = 6$ or 7, we only give the list of $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$ which generate all the maximal $\overline{\mathfrak{I}(\lambda)}$ in $\widetilde{\mathfrak{I}}_d/\mathfrak{S}_d$.

In the case $d = 6$, there are six maximal elements in $\widetilde{\mathfrak{I}}_6/\mathfrak{S}_6$, and they are obtained from $\lambda = (\lambda_1, \ldots \lambda_6) \in V_6$ such that $(1-\lambda_1)^{-1} : \cdots : (1-\lambda_6)^{-1}$ is equal to either of the followings:

$$\begin{aligned}
&1 : 1 : -1 : -1 : a : -a, \text{ where } a \neq 0, \pm 1, \pm 2, \\
&1 : -1 : a : -a : (a+1) : -(a+1), \text{ where } a \neq 0, -1/2, \pm 1, -2, \\
&1 : 1 : 1 : -1 : a : -(a+2), \text{ where } a \neq 0, \pm 1, -2, -3, \\
&1 : 1 : a : a : -(a+1) : -(a+1), \text{ where } a \neq 0, -1/2, \pm 1, -2, \\
&1 : 1 : 1 : 2 : -2 : -3 \qquad \text{or} \qquad 1 : 1 : 3 : -1 : -2 : -2.
\end{aligned}$$

In the case $d = 7$, there are 27 maximal elements in $\widetilde{\mathfrak{I}}_7/\mathfrak{S}_7$, and they are obtained from $\lambda = (\lambda_1, \dots \lambda_7) \in V_7$ such that $(1-\lambda_1)^{-1} : \cdots : (1-\lambda_7)^{-1}$ is equal to either of the followings:

$$1:1:1:-1:-1:a:-(a+1), \text{ where } a \neq 0, \pm 1, \pm 2, -3,$$
$$1:1:1:1:-1:a:-(a+3), \text{ where } a \neq 0, \pm 1, -2, -3, -4,$$
$$1:1:2:2:-1:-1:-4, \quad 1:1:1:3:-1:-2:-3, \quad 1:1:2:2:-1:-2:-3,$$
$$1:1:2:3:-1:-2:-4, \quad 1:1:2:3:-2:-2:-3, \quad 1:2:2:3:-1:-3:-4,$$
$$1:1:1:4:-1:-3:-3, \quad 1:2:2:2:-1:-1:-5, \quad 1:1:3:3:-1:-2:-5,$$
$$1:1:1:2:2:-2:-5, \quad 1:1:2:5:-2:-3:-4, \quad 1:1:1:2:3:-3:-5,$$
$$1:1:2:4:-2:-3:-3, \quad 1:2:2:4:-1:-3:-5, \quad 1:1:1:1:3:-3:-4,$$
$$1:1:1:1:2:-2:-4, \quad 1:2:2:2:-1:-3:-3, \quad 1:1:1:1:2:-3:-3,$$
$$1:1:1:2:2:-3:-4, \quad 1:1:1:3:-2:-2:-2, \quad 1:1:1:4:-2:-2:-3,$$
$$1:1:1:5:-2:-3:-3, \quad 1:1:3:3:-2:-2:-4, \quad 1:1:3:4:-2:-2:-5 \quad \text{or}$$
$$1:1:2:2:3:-4:-5.$$

We can verify the inequality $\#(S_d(\lambda)) > 0$ for every $\lambda \in V_d$ listed above, which completes the proof of the proposition. $\square$

To summarize the above mentioned, we have completed the proof of the main theorems.

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