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Universality of the route to chaos: Exact analysis

Ken-ichi Okubo* and Ken Umeno*

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University
*E-mail: okubo.kenichi.65z@st.kyoto-u.ac.jp, umeno.ken.8z@kyoto-u.ac.jp

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The universality of the route to chaos is analytically proven for a countably infinite number of maps by proposing Super Generalized Boole (SGB) transformations. One of the routes to chaos, the intermittency route, was previously studied extensively by numerical methods. These researchers conjectured the universality in *Type 1* intermittency, namely that the critical exponent of the Lyapunov exponent in this type of intermittency is $\frac{1}{2}$. We prove their conjecture by showing that, for certain parameter ranges, the SGB transformations are *exact* and preserve the Cauchy distribution. Using the property of exactness, we prove that the critical exponent is $\frac{1}{2}$ for a countably infinite number of maps where *Type 1* intermittency occurs.

Subject Index A33

1. Universality in chaos

The route from stable states to chaotic (intermittent) states has attracted much attention from within the broader field of physics. This issue concerns the fundamental change of systems from a stable state to an unstable state, and it is an essential theme in analyzing the stability of physical systems. The route to chaos has also been studied theoretically and experimentally for systems such as Hamiltonian systems [1], map systems [2–7], coupled oscillators [8], the Belousov–Zhabotinskii reaction [9], Rayleigh–Benard convection [9], Couette–Taylor flow [9], a noise-induced system [10], a thermoacoustic system [11], and optomechanics [12–14]. Theoretical classifications of routes to chaos such as the intermittency route, the period-doubling route, and the frequency-locking route, have been proposed [9]. Frequently, these studies were motivated by attempts to discover the universality at the onset of chaos with respect to the critical exponent of the Lyapunov exponent, which is an indicator of chaos. The universality of the critical exponents in each route to chaos has been studied extensively by numerical simulations. In the case of the period-doubling route, Huberman and Rudnick [5] numerically estimated the critical exponent ν as $\nu = \frac{\log 2}{\log \delta}$, where δ represents the Feigenbaum constant. In terms of the intermittency route considered in this paper, Pomeau and Manneville [3,4] classified intermittency into three types and conjectured the universality of ν for each intermittent type. These authors [4] determined that *Type 1* intermittency occurs when tangent bifurcation appears, a finding that was subsequently confirmed by others [15]. For example, *Type 1* intermittency was observed in the Lorentz equation [4] and the logistic map [16]. Other researchers [17,18] found that *Type 1* intermittency occurs and superdiffusion is also observed in a climbing-sine map. Pomeau and Manneville [4] explained the reason why the critical exponent with the Lyapunov exponent is $\frac{1}{2}$ as follows. When *Type 1* intermittency occurs, a "channel" exists between $x_{n+1} = S(x_n)$, where S represents a map, and the bissectrix $x_{n+1} = x_n$. The function $S(x_n)$

is then expanded around a fixed point, such that this point can be assumed to be the original point, to obtain the following equation: $S(x_n) = x_n + x_n^2 + \varepsilon +$ (higher order terms). They estimated the number of iterative computations required to cross the channel to be of the order of $\varepsilon^{-\frac{1}{2}}$. Then they concluded that, in the vicinity of $\varepsilon = +0$, the Lyapunov exponent varies as $\varepsilon^{\frac{1}{2}}$ by using an elementary calculation and the numerical simulations. Subsequent to the work by Pomeau and Manneville, the critical exponent has been researched in various fields with relation to intermittency, such as the Billiard system [20,21], electronic circuits [22], plasma physics [23], and an intermittent map [24]. These studies, which were conducted by numerical simulations, suggest that their conjecture $\nu = \frac{1}{2}$ would be right.

However, in these studies, the critical exponent ν was either estimated numerically or an assumption about the ergodicity was necessary, without which the analytical formulae of the Lyapunov exponent λ could not be obtained.

On the other hand, the present authors [25] presented the analytical formula of the Lyapunov exponent λ as an explicit function in terms of the bifurcation parameter by showing the mixing property for the Generalized Boole (GB) transformation. With respect to the GB transformation, we proved that both *Type 1* and *Type 3* intermittency occur and that the conjecture by Pomeau and Manneville is correct.

In this study, we prove analytically that, for a countably infinite number of maps, it holds that

$$\lambda \sim b\,|\alpha - \alpha_{\mathrm{c}}|^{\nu}\,,\ \nu = \frac{1}{2},\ b > 0 \tag{1}$$

when *Type 1* intermittency occurs, where α and α_{c} represent a bifurcation parameter and the critical point, respectively. In order to prove this, we propose additional generalized maps, Super Generalized Boole (SGB) transformations, and show that there are parameter ranges in which the SGB transformations are *exact* (a stronger condition than ergodicity). This means that a countably infinite number of exact (ergodic) maps is obtained. Using this result, one can explicitly determine the analytical formulae of the Lyapunov exponents and critical exponents. The SGB transformations have advantages in their proven mathematical results, compared with the maps that were previously obtained [4]. Let us start with several definitions. We consider a set $Sing_S \in \mathbb{R}$ as consisting of the singular points for a map S when $S(x),\ x \in Sing_S$, cannot be defined.

We define two-parameterized one-dimensional maps, the Super Generalized Boole Transformations (SGB) $S_{K,\alpha} : \mathbb{R}\backslash B \to \mathbb{R}\backslash B$ as follows:

$$x_{n+1} = S_{K,\alpha}(x_n) \stackrel{\mathrm{def}}{=} \alpha K F_K(x_n), \tag{2}$$

where $\alpha > 0$, $K \in \mathbb{N}\backslash\{1\}$, $B \stackrel{\mathrm{def}}{=} \{x \in \mathbb{R} \mid \exists n \in \mathbb{Z}; S^n_{K,\alpha}(x) \in Sing_{S_{K,\alpha}}\}$, and the function F_K corresponds to the K-angle formula of the cot function defined in Appendix A. The GB transformation $T_{\alpha,\alpha}$ in Ref. [25] corresponds to the map $S_{2,\alpha}$. Figure 1 shows the return maps of $S_{3,\frac{1}{3}}$, $S_{4,\frac{1}{4}}$, and $S_{5,\frac{1}{5}}$.

2. Invariant density

In this section we prove that the SGB transformations preserve the Cauchy distribution for a certain condition. Solving the Perron–Frobenius equation, it is shown that the Cauchy distribution is an invariant distribution for certain parameter ranges.

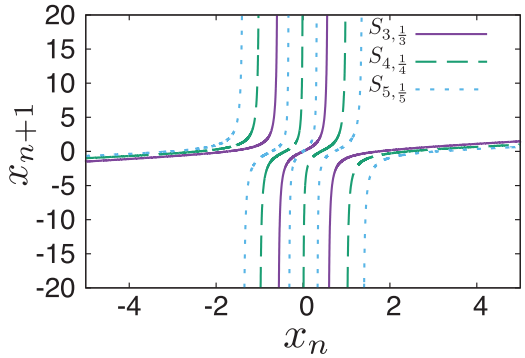

Fig. 1. Return maps of $S_{3,\frac{1}{3}}$, $S_{4,\frac{1}{4}}$, and $S_{5,\frac{1}{5}}$. The solid, broken, and dotted lines correspond to the form of $S_{3,\frac{1}{3}}$, $S_{4,\frac{1}{4}}$, and $S_{5,\frac{1}{5}}$, respectively. The explicit forms of these three maps are provided in the appendices.

First, assume that the variable x_n obeys the Cauchy distribution $f_n(x) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$ of which the scale parameter is $\gamma > 0$. According to Refs. [26,27], the map $S_{K,\alpha}$ is a K-to-one map (a non-injective map) as follows: $x_{n+1} = K\alpha \cot K\theta = K\alpha F_K(x_n(j))$, $x_n(j) = \cot\left(\theta + j\frac{\pi}{K}\right)$, $j = 1, 2, \ldots, K$. Then, according to Ref. [27] and solving the Perron–Frobenius equation, it is clear that the variable x_{n+1} obeys the density function $f_{n+1}(x) = \frac{1}{\pi}\frac{\alpha K G_K(\gamma)}{x^2+\alpha^2K^2G_K^2(\gamma)}$, where the function G_K corresponds to the K-angle formula of the coth function, as defined in Appendix A. Thus, the density about x changes from $f_n(x) = \frac{1}{\pi}\frac{\gamma}{x^2+\gamma^2}$ to $f_{n+1}(x) = \frac{1}{\pi}\frac{\alpha K G_K(\gamma)}{x^2+\alpha^2K^2G_K^2(\gamma)}$. The scale parameter $\gamma > 0$ is then transformed in one iterative step as $\gamma \longmapsto \alpha K G_K(\gamma)$. Now, for each K, let us obtain the fixed point $0 < \gamma_{K,\alpha} < \infty$, which satisfies the relation

$$\gamma_{K,\alpha} = \alpha K G_K(\gamma_{K,\alpha}), \tag{3}$$

and clarify the condition of α such that there exists a solution of Eq. (3). If there exists a fixed point, this means that the Cauchy distribution is the invariant density for the map $S_{K,\alpha}$. We approach this problem by defining Range A as follows.

Definition 1 When the parameters (K, α) satisfy a condition such as

$$\begin{cases} 0 < \alpha < 1 & \text{in the case of} \quad K = 2N, \\ \dfrac{1}{K^2} < \alpha < 1 & \text{in the case of} \quad K = 2N+1, \end{cases} \tag{4}$$

where $N \in \mathbb{N}$, we say that the parameters (K, α) are in Range A.

Then the following theorem holds.

Theorem I When the parameters (K, α) are in Range A, the SGB transformations $\{S_{K,\alpha}\}$ preserve the Cauchy distribution and the scale parameter can be chosen uniquely.

The proof of Theorem I is given in Appendix B. Although it has been proven that the map $S_{K,\alpha}$ preserves the Cauchy distribution and its scale parameter $\gamma_{K,\alpha}$ can be determined uniquely when the parameters (K, α) are in Range A, it is not straightforward to obtain the explicit form of fixed point $\gamma_{K,\alpha}$ for arbitrary K, because we have to solve the Kth-degree equations. From Theorem I, the condition that there exists only one solution of Eq. (3), which satisfied $0 < \gamma_{K,\alpha} < \infty$, is nothing but the Range A.

3. Exactness

According to Refs. [28–30], the Perron–Frobenius operator and the exactness are defined as follows.

Definition 2 (Perron–Frobenius operator) Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space and let f be a density function on $\mathcal{X}$. If a map $T : \mathcal{X} \to \mathcal{X}$ is a non-singular transformation, the unique operator $\mathcal{P} : L^1 \to L^1$ defined by

$$\int_A \mathcal{P}f(x)\mu(dx) = \int_{T^{-1}(A)} f(x)\mu(dx) \text{ for } A \in \mathcal{A} \tag{5}$$

is termed the Perron–Frobenius operator corresponding to T.

Definition 3 (Exactness) A map T on a phase space $\mathcal{X}$ with the Perron–Frobenius operator $\mathcal{P}_T$ and unique stationary density f_* is considered to be exact if and only if

$$\lim_{n\to\infty} \|\mathcal{P}_T^n f - f_*\|_{L^1} = 0 \tag{6}$$

for every initial density $f \in \mathcal{D}$, where $\mathcal{D}$ denotes the set of all densities on $\mathcal{X}$.
This definition is equivalent to the following:

$$\lim_{n\to\infty} \mu_*(T^n A) = 1,\ \forall A \in \mathcal{A},\ \mu_*(A) > 0, \tag{7}$$

where $\mathcal{A}$ denotes the σ-algebra and μ_* denotes the invariant measure corresponding to the invariant density f_*.

In terms of exactness, we obtain the following theorem.

Theorem II If the parameters (K, α) are in Range A, the SGB transformations $\{S_{K,\alpha}\}$ are exact.

The proof is given in Appendix C. From Theorems I and II, when the parameters (K, α) are in Range A, the map $S_{K,\alpha}$ preserves a certain Cauchy distribution f_*, and any initial density function f defined on $\mathbb{R}\backslash B$ converges to f_* as

$$\lim_{n\to\infty} \|\mathcal{P}_{S_{K,\alpha}}^n f - f_*\|_{L^1} = 0. \tag{8}$$

For example, $S_{3,\alpha}$, $S_{4,\alpha}$, and $S_{5,\alpha}$ are *exact* for $\frac{1}{9} < \alpha < 1$, $0 < \alpha < 1$, and $\frac{1}{25} < \alpha < 1$, respectively. According to Ref. [29], if the SGB transformations are exact, then the corresponding dynamical systems are mixing and ergodic. Therefore, the following corollary holds.

Corollary 3 Suppose that the parameters (K, α) are in Range A. Then the dynamical system $(\mathbb{R}\backslash B, S_{K,\alpha}, \mu_*)$ has the mixing property and it is ergodic where μ_* is the invariant measure corresponding to the invariant density f_*.

Using the property of exactness, one can obtain the *explicit* formula of the Lyapunov exponent such that

$$\lambda_{K,\alpha} = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|\frac{dS_{K,\alpha}}{dx}\right| \frac{\gamma_{K,\alpha}}{x^2 + \gamma_{K,\alpha}^2} dx. \tag{9}$$

Pesin's formula indicates that the Kolmogorov–Sinai entropy is equivalent to the Lyapunov exponent since the SGB transformations are one-dimensional maps with ergodic invariant measures which are absolutely continuous with respect to the Lebesgue measure.

By changing the variable from x to $\cot\theta$, one has that

$$\frac{dS_{K,\alpha}}{dx} = \alpha K^2 \frac{\sin^2\theta}{\sin^2(K\theta)}. \tag{10}$$

Using mathematical induction, one proves that for $K \geq 2$, the following relation is true:

$$K^2 \frac{\sin^2\theta}{\sin^2(K\theta)} \geq 1. \tag{11}$$

For $K = 2$, one sees that Eq. (11) is true by simple calculation, as

$$K^2 \frac{\sin^2\theta}{\sin^2(K\theta)} = \frac{1}{\cos^2\theta} \geq 1. \tag{12}$$

Assume that, for $K = n \geq 2$, Eq. (11) is true. Then, since it holds that

$$(n-1)^2 \leq n^2 + 2\frac{\sin(n\theta)}{\sin\theta}\cos(n\theta)\cos\theta + 1 \leq (n+1)^2, \tag{13}$$

one has that

$$\begin{aligned}(n+1)^2 \frac{\sin^2\theta}{\sin^2\{(n+1)\theta\}} &= \frac{(n+1)^2}{\frac{\sin^2(n\theta)}{\sin^2\theta}\cos^2\theta + 2\frac{\sin(n\theta)}{\sin\theta}\cos(n\theta)\cos\theta + \cos^2\theta} \\ &\geq \frac{(n+1)^2}{n^2 + 1 + 2\frac{\sin(n\theta)}{\sin\theta}\cos(n\theta)\cos\theta} \\ &\geq 1.\end{aligned} \tag{14}$$

Thus, for $K \geq 2$, Eq. (11) holds. Therefore, for $\alpha > 1$ and $K \geq 2$, it holds that

$$\frac{dS_{K,\alpha}}{dx} > 1. \tag{15}$$

For $\alpha > 1$, changing the variable by $z_n = 1/x_n$ enables the map $\widetilde{S}_{K,\alpha}$ to be obtained, defined as

$$\begin{cases} \widetilde{S}_{2N,\alpha}(z) \stackrel{\text{def}}{=} \dfrac{\displaystyle\sum_{i=0}^{N-1}(-1)^i {}_{2N}C_{2N-2i-1} z^{2N-2i-1}}{\alpha K \displaystyle\sum_{i=0}^{N}(-1)^i {}_{2N}C_{2N-2i} z^{2N-2i}}, \\[2ex] \widetilde{S}_{2N+1,\alpha}(z) \stackrel{\text{def}}{=} \dfrac{\displaystyle\sum_{i=0}^{N}(-1)^i {}_{2N+1}C_{2N-2i+1} z^{2N-2i+1}}{\alpha K \displaystyle\sum_{i=0}^{N}(-1)^i {}_{2N+1}C_{2N-2i} z^{2N-2i}}. \end{cases} \tag{16}$$

Then one has that $\left|\frac{d\widetilde{S}_{K,\alpha}}{dz}(0)\right| = \frac{1}{\alpha} < 1$, so that for any K, the point $z = 0$ is an attracting point for $\alpha > 1$. Then, since for $\alpha > 1$, Eq. (15) holds, almost all orbits approach infinity. Thus, the Lyapunov exponent $\lambda_{K,\alpha}$ for $\alpha > 1$ is derived from the inclination at the infinite point as

$$\lambda_{K,\alpha} = \log\alpha \ \forall K \in \mathbb{N}\backslash\{1\}. \tag{17}$$

4. Scaling behavior

At $(K,\alpha) = (\bar{N},1)$, $(2N,0)$, and $\left(2N+1, \frac{1}{(2N+1)^2}\right)$, for any $\bar{N} \in \mathbb{N}\backslash\{1\}$ and $N \in \mathbb{N}$, one has that

$$\begin{aligned} &\gamma_{K,\alpha} = \infty \ \text{ for } \alpha = 1, K = \bar{N},\ \bar{N} \in \mathbb{N}\backslash\{1\}, \\ &\gamma_{K,\alpha} = 0 \begin{cases} \text{for} \quad \alpha = 0 \quad \text{in the case of} \quad K = 2N, \quad N \in \mathbb{N}, \\ \text{for} \quad \alpha = \frac{1}{K^2} \quad \text{in the case of} \quad K = 2N+1, \quad N \in \mathbb{N}. \end{cases} \end{aligned} \tag{18}$$

Then the Lyapunov exponent converges to zero at $(K,\alpha) = (\bar{N},1)$, $(2N,0)$, and $\left(2N+1, \frac{1}{(2N+1)^2}\right)$. A discussion of the critical phenomena requires the critical points to be defined as $\alpha_{c_1} = 1$, $\alpha_{c_2} = \frac{1}{(2N+1)^2}$, and $\alpha_{c_3} = 0$ and the critical exponents ν_1, ν_2, and ν_3 corresponding to α_{c_i}, $i = 1,2,3$ are defined. In terms of the scaling behavior of the Lyapunov exponents $\lambda \sim b\left|\alpha - \alpha_{c_i}\right|^{\nu_i}$, $b > 0$, $i = 1,2,3$, the following theorem holds.

Theorem III Suppose that the parameters (K,α) are in Range A.

- For any $K \in \mathbb{N}\backslash\{1\}$, it holds that $\nu_1 = \frac{1}{2}$ as $\alpha \to 1-0$.
- For any $K \in \mathbb{N}\backslash\{1\}$, it holds that $\nu_1 = 1$ as $\alpha \to 1+0$.
- For $K = 2N+1$, it holds that $\nu_2 = \frac{1}{2}$ as $\alpha \to \frac{1}{K^2}+0$.

The proof appears in Appendix D.

We discuss the Floquet multipliers in the cases of $(K,\alpha) = (\bar{N},\alpha_{c_1})$ and $(2N+1,\alpha_{c_2})$ for $\bar{N} \in \mathbb{N}\backslash\{1\}$, $N \in \mathbb{N}$. By changing the variable by $x = \cot\theta$, the derivative of the map $S_{K,\alpha}$ is rewritten as $\frac{dS_{K,\alpha}}{dx} = \alpha K^2 \frac{\sin^2\theta}{\sin^2 K\theta}$.

(i) In the case of $(\bar{N},\alpha_{c_1})$, the derivatives at the infinite point are denoted as

$$\lim_{x\to+\infty} \frac{dS_{\bar{N},1}}{dx} = \lim_{\theta\to+0} \frac{(\bar{N}\theta)^2}{\sin^2\bar{N}\theta} \frac{\sin^2\theta}{\theta^2} = 1. \tag{19}$$

Thus, the Floquet multiplier χ for $(K,\alpha) = (\bar{N},\alpha_{c_1})$, $\bar{N} \in \mathbb{N}\backslash\{1\}$ is unity.

(ii) In the case of $(2N+1,\alpha_{c_2})$, the original point is the fixed point. Then, the derivative at the original point is denoted as

$$\frac{dS_{K,\frac{1}{K^2}}}{dx}(0) = \frac{\sin^2(\frac{\pi}{2})}{\sin^2\left\{(2N+1)\frac{\pi}{2}\right\}} = 1. \tag{20}$$

Thus, at $(K,\alpha) = (2N+1,\alpha_{c_2})$, it holds that $\chi = 1$.

By considering (i) and (ii), it is clear that in the case of $K = 2N+1$, *Type 1* intermittency occurs at $\alpha = \alpha_{c_1}$ and α_{c_2}, and that in the case of $K = 2N$, *Type 1* intermittency occurs at $\alpha = \alpha_{c_1}$. Therefore, according to the above results and Theorem C, it has been proven that for a countably infinite number of *exact* maps, the *universal* scaling behavior of Eq. (1) holds where *Type 1* intermittency occurs. In Appendix D.1, the Floquet multipliers corresponding to $K = 3$, 4, and 5 are illustrated.

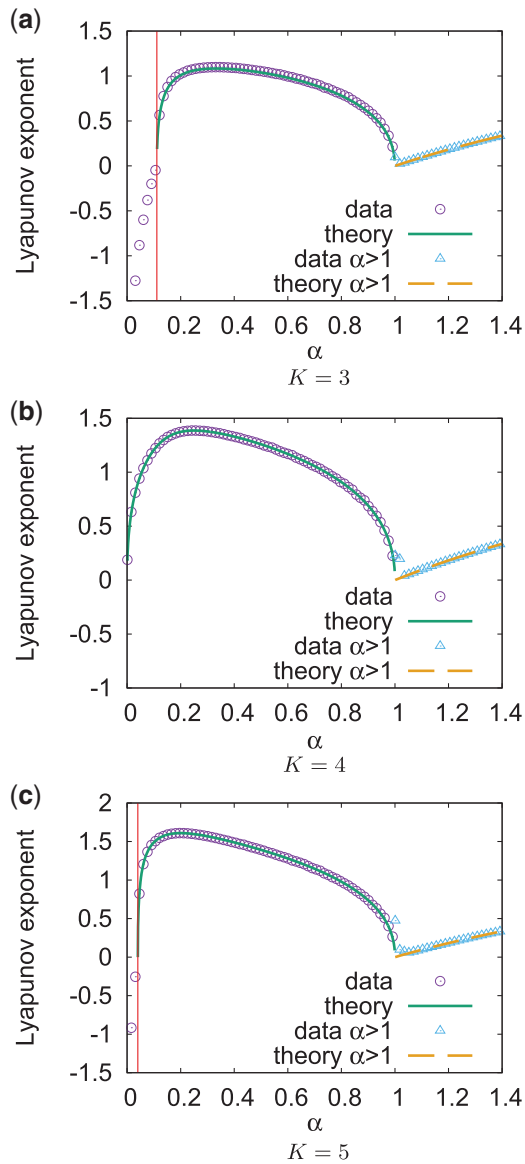

Fig. 2. Relations between the Lyapunov exponents of the SGB transformations and α for $K = 3$, 4, and 5. Circles and triangles represent the numerical results for $\alpha < 1$ and $\alpha > 1$, respectively. Solid and broken lines represent the analytical results for $\alpha < 1$ and $\alpha > 1$, respectively. The initial point is $x_0 = 5\sqrt{7}$. The number of iterative steps is 1×10^5 for $\alpha < 1$ and 200 for $\alpha > 1$. The vertical line corresponds to $\alpha = \frac{1}{9}, \frac{1}{25}$, respectively.

5. The cases $K = 3$, 4, and 5

This section provides examples corresponding to $K = 3$, 4, and 5. The solutions of Eq. (3), which satisfy $0 < \gamma_{K,\alpha} < \infty$, are uniquely determined as follows:

$$\begin{aligned}
\gamma_{3,\alpha} &= \sqrt{\frac{9\alpha - 1}{3 - 3\alpha}}, \\
\gamma_{4,\alpha} &= \sqrt{\frac{6\alpha - 1 + \sqrt{32\alpha^2 - 8\alpha + 1}}{2(1 - \alpha)}}, \\
\gamma_{5,\alpha} &= \sqrt{\frac{-5(1 - 5\alpha) + \sqrt{20(25\alpha^2 - 6\alpha + 1)}}{5(1 - \alpha)}}.
\end{aligned} \tag{21}$$

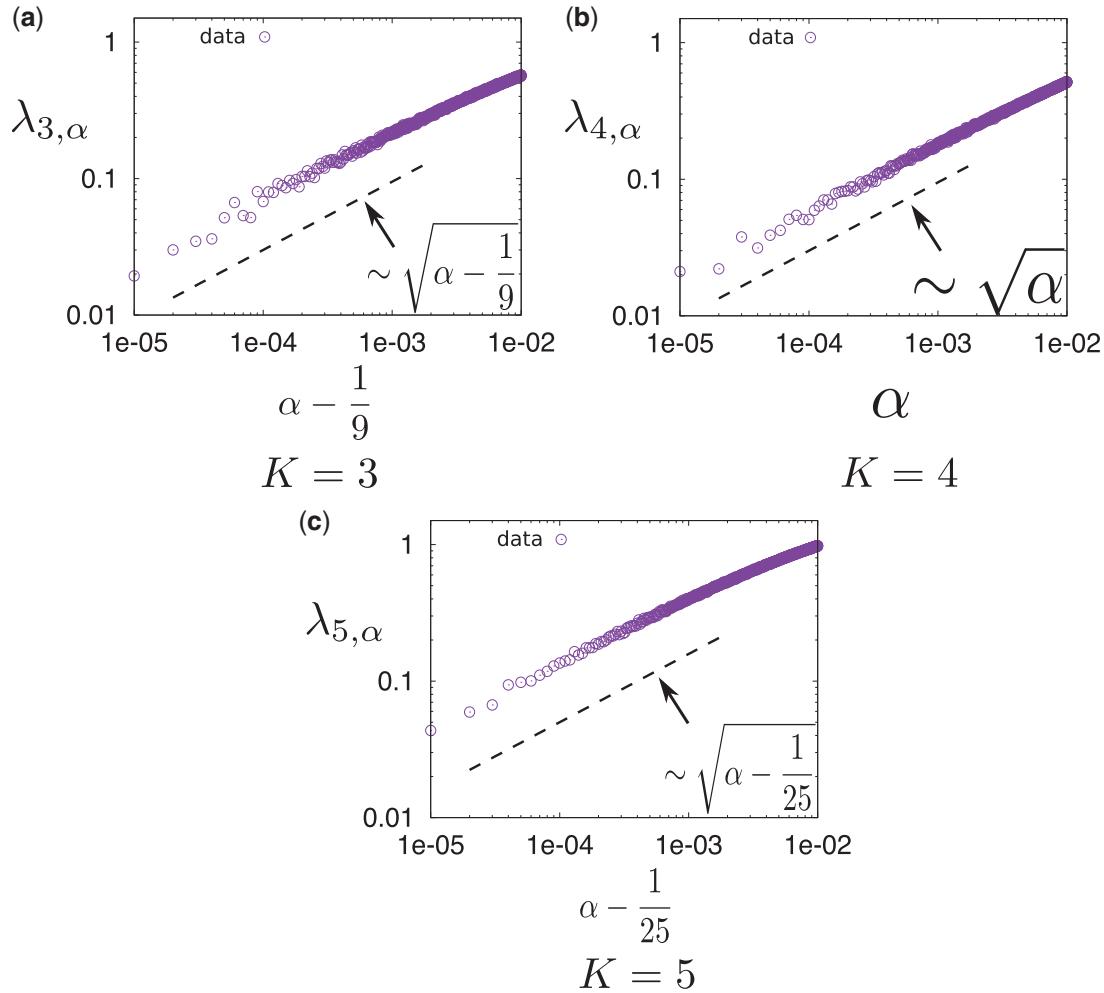

Fig. 3. Scaling behavior of the Lyapunov exponents of the SGB transformation in the cases of $K = 3$, 4, and 5. Circles represent the numerically simulated values and broken lines represent the order. The initial point is $x_0 = 5\sqrt{7}$. The number of iterative computations is 2×10^5.

The above discussion indicates that *Type 1* intermittency occurs in the case of $K = 3$, 4, and 5. The Lyapunov exponents in the cases of $K = 3$, 4, and 5 are given as follows:

$$\lambda_{3,\alpha} = \log\left|\frac{1}{\alpha}\left(\frac{3(1-\alpha)}{8}\right)^2\left[1+\sqrt{\frac{9\alpha-1}{3-3\alpha}}\right]^4\right|,$$

$$\lambda_{4,\alpha} = \log\left|\frac{\alpha(1+\gamma_{4,\alpha})^6}{\gamma_{4,\alpha}^2(1+\gamma_{4,\alpha}^2)^2}\right|,$$

$$\lambda_{5,\alpha} = \log\left|\frac{25}{256\alpha}\frac{(1-\alpha)^4}{(\sqrt{125\alpha^2-30\alpha+5}+11\alpha-1)^2}|1+\gamma_{5,\alpha}|^8\right|. \tag{22}$$

Figures 2(a), (b), and (c) show the Lyapunov exponents against α in the case of $K = 3$, 4, and 5, respectively. The numerical simulations are seen to be exactly consistent with the obtained analytical formulae except for the vicinities of α_{c_i} for $i = 1, 2, 3$. Because $\frac{\partial \lambda_{K,\alpha}}{\partial \alpha} = \pm\infty$ holds at the critical points, the parameter dependence of the Lyapunov exponent is observed to diverge at the critical points. This means that obtaining the true value of the Lyapunov exponent by numerical simulation

would be computationally difficult. Figure 3 shows the scaling behavior of the Lyapunov exponent. These results show that $\nu_2 = \frac{1}{2}$ and $\nu_3 = \frac{1}{2}$ in the cases of $K = 3$, 4, and 5.

6. Conclusion

This work is the first example in which the conjecture by Pomeau and Manneville expressed in Eq. (1) is analytically proven to be true for a countably infinite number of maps (the proposed Super Generalized Boole transformations). This research has presented the theoretical picture of the stable–unstable transition for intermittent maps. In the course of providing the proof, we have shown that the SGB transformations preserve the unique Cauchy distribution, and also proved that SGB transformations are *exact* and that any initial density function converges to the invariant Cauchy distribution when the parameters (K, α) are in Range A.

By applying the property of exactness, it becomes possible to obtain the *analytical* formulae of the Lyapunov exponents for the SGB transformations. In these transformations, the Lyapunov exponents $\lambda_{K,\alpha}$ are equivalent to the Kolmogorov–Sinai entropy as applied to Pesin's theorem. Using the analytical formulae of the Lyapunov exponents, we confirmed that for $K = 3$, 4, and 5, the derivative $\frac{\partial \lambda_{K,\alpha}}{\partial \alpha}$ diverges at the critical points and we obtained $\nu_1 = \nu_2 = \nu_3 = \frac{1}{2}$ for $K = 3$, 4, and 5. We also proved the critical exponents for a countably infinite number of maps. Thus, we have proved the universality of the route to chaos for a large class of chaotic systems.

In future, we plan to clarify the scaling relation between the critical exponent ν and the other critical exponents, and this would enable us to obtain a new perspective of chaos in physics.

Acknowledgements

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Appendix A. Definitions

In the following appendices it is shown that

- the SGB transformations preserve the Cauchy distribution for certain parameter ranges, and the Cauchy distribution is determined uniquely;
- the SGB transformations are exact for the parameter ranges; and
- the critical exponent of the Lyapunov exponent is $\nu = \frac{1}{2}$ for all $K \in \mathbb{N}\backslash\{1\}$ when *Type 1* intermittency occurs.

Let us begin with definitions. The definitions of F_K, G_K, and $S_{K,\alpha}$ are written as follows.

Definition A.1 Let $F_K : \mathbb{R}\backslash A \to \mathbb{R}\backslash A$ be a map denoted as [27]

$$F_K(\cot\theta) \stackrel{\text{def}}{=} \cot K\theta, \tag{A.1}$$

where $K \in \mathbb{N}\backslash\{1\}$ and $A \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid \exists\, n \in \mathbb{Z}; F_K^n(x) \in Sing_{F_K}\}$.

Definition A.2 Let $G_K : \mathbb{R} \to \mathbb{R}$ be a map denoted as [27]

$$G_K(\coth\theta) \stackrel{\text{def}}{=} \coth K\theta, \tag{A.2}$$

where $K \in \mathbb{N}\backslash\{1\}$.

Definition A.3 (Super Generalized Boole Transformation) The map $S_{K,\alpha} : \mathbb{R}\backslash B \to \mathbb{R}\backslash B$ denoted as

$$x_{n+1} = S_{K,\alpha}(x_n) \stackrel{\text{def}}{=} \alpha K F_K(x_n) \tag{A.3}$$

is referred to as the Super Generalized Boole Transformation, where $\alpha > 0$, $K \in \mathbb{N}\backslash\{1\}$, and $B \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid \exists n \in \mathbb{Z}; S^n_{K,\alpha}(x) \in Sing_{S_{K,\alpha}}\}$.

For example, $S_{3,\alpha}$, $S_{4,\alpha}$, and $S_{5,\alpha}$ are as follows:

$$S_{3,\alpha}(x_n) = 3\alpha \frac{x_n^3 - 3x_n}{3x_n^2 - 1}, \tag{A.4}$$

$$S_{4,\alpha}(x_n) = 4\alpha \frac{x_n^4 - 6x_n^2 + 1}{4x_n^3 - 4x_n}, \tag{A.5}$$

$$S_{5,\alpha}(x_n) = 5\alpha \frac{x_n^5 - 10x_n^3 + 5x_n}{5x_n^4 - 10x_n^2 + 1}. \tag{A.6}$$

The derivative of $S_{K,\alpha}$ with respect to x is denoted as

$$S'_{2N,\alpha}(x) = (2N)^2\alpha \frac{(1+x^2)^{2N-1}}{\left[\sum_{r=0}^{N-1}(-1)^r {}_{2N}\mathrm{C}_{2r+1} x^{2N-2r-1}\right]^2} > 0,$$

$$S'_{2N+1,\alpha}(x) = (2N+1)^2\alpha \frac{(1+x^2)^{2N}}{\left[\sum_{r=0}^{N}(-1)^r {}_{2N+1}\mathrm{C}_{2r+1} x^{2(N-r)}\right]^2} > 0.$$

Appendix B. Existence of the invariant density function

In this section we prove that

- $K = 2N$, $N \in \mathbb{N}$, and $0 < \alpha < 1 \Longrightarrow$ the map $S_{2N,\alpha}$ preserves the Cauchy distribution and can be determined uniquely;
- $K = 2N+1$, $N \in \mathbb{N}$, and $\frac{1}{(2N+1)^2} < \alpha < 1 \Longrightarrow$ the map $S_{2N+1,\alpha}$ preserves the Cauchy distribution and can be determined uniquely.

According to Ref. [27], the map $S_{K,\alpha}$ is a K-to-one map as follows:

$$x_{n+1} = K\alpha \cot K\theta = K\alpha F_K(x_n(j)),\ j = 1, 2, \ldots, K,$$
$$x_n(j) = \cot\left(\theta + j\frac{\pi}{K}\right),\ j = 1, 2, \ldots, K.$$

If the variable x_n obeys the Cauchy distribution

$$f_n(x) = \frac{1}{\pi}\frac{\gamma}{x^2 + \gamma^2},$$

then according to the Perron–Frobenius equation and Ref. [27], it holds that

$$f_{n+1}(x_{n+1})|dx_{n+1}| = f_n(x_n(1))|dx_n(1)| + f_n(x_n(2))|dx_n(2)| + \cdots + f_n(x_n(K))|dx_n(K)|,$$
$$f_{n+1}(x_{n+1}) = \frac{1}{\pi}\frac{\alpha K G_K(\gamma)}{x_{n+1}^2 + \alpha^2 K^2 G_K^2(\gamma)}. \tag{B.1}$$

Then the scale parameter γ is transformed in a single iterative computation as

$$\gamma \longmapsto \alpha K G_K(\gamma). \tag{B.2}$$

Now, for each K let us obtain the fixed point $\gamma_{K,\alpha}$ which satisfies

$$\gamma_{K,\alpha} = \alpha K G_K(\gamma_{K,\alpha}). \tag{B.3}$$

If we discover a real and positive solution of Eq. (B.3), it constitutes evidence that the map $S_{K,\alpha}$ preserves the Cauchy distribution.

The map $G_K(x)$ is denoted as

$$G_{2N}(x) = \frac{\displaystyle\sum_{k=0}^{N} {}_{2N}C_{2k} x^{2(N-k)}}{\displaystyle\sum_{k=0}^{N-1} {}_{2N}C_{2k+1} x^{2(N-k)-1}}, \tag{B.4}$$

$$G_{2N+1}(x) = \frac{\displaystyle\sum_{k=0}^{N} {}_{2N+1}C_{2k} x^{2(N-k)+1}}{\displaystyle\sum_{k=0}^{N} {}_{2N+1}C_{2k+1} x^{2(N-k)}}. \tag{B.5}$$

If one changes the variable from x to $\coth y$, one has that

$$\begin{aligned} G_{2N}(x) &= \coth(2Ny), \\ G_{2N+1}(x) &= \coth\{(2N+1)y\}. \end{aligned} \tag{B.6}$$

If one changes the variable from x to $\tanh y$, one has that

$$\begin{aligned} G_{2N}(x) &= \frac{1}{\tanh(2Ny)}, \\ G_{2N+1}(x) &= \tanh\{(2N+1)y\}. \end{aligned} \tag{B.7}$$

Because at $\alpha = \frac{1}{K}$ the SGB transformation $S_{K,\alpha}$ is equivalent to the K-angle formula of the cot function, it is obvious that the fixed point of the scaling parameter $\gamma_{K,\frac{1}{K}}$ is unity by simple calculation. In the following, we discuss the case in which $\alpha \neq \frac{1}{K}$. The following lemmas hold.

Lemma B.1 For $\frac{1}{K} < \alpha < 1$, fix α and there is only one solution that satisfies Eq. (B.3) in the range $\gamma_{K,\alpha} > 1$, and for $0 < \alpha < \frac{1}{K}$ there is no solution in the range $\gamma_{K,\alpha} > 1$.

Proof. The goal is to identify the condition of (K, α) in which there exists a solution $\gamma_{K,\alpha}$ satisfying Eq. (B.3) for $\gamma_{K,\alpha} > 1$. In the beginning, for simplicity, we change the form of Eq. (B.3) by changing the variable from $\gamma_{K,\alpha} > 1$ to $\coth y$ as Eq. (B.6). One has that

$$\coth y = \alpha K \coth(Ky). \tag{B.8}$$

In the following, one identifies the condition (K, α) in which there exists a solution $y^*_{K,\alpha} > 0$ ($\coth y^*_{K,\alpha} > 1$) satisfying $\coth y^*_{K,\alpha} = \alpha K \coth(K y^*_{K,\alpha})$. A function $g(y)$ is defined to be

$$g(y) = \coth y - \alpha K \coth(Ky). \tag{B.9}$$

In the range where $y > 0$, since the function $\coth y$ decreases monotonically, it holds that

$$\coth y > \coth(Ky). \tag{B.10}$$

Then, if the condition $\frac{1}{K} < \alpha < 1$ is satisfied, one has that

$$g'(y) = (1 - \coth^2 y) - \alpha K^2 \left\{1 - \coth^2(Ky)\right\} < 0. \tag{B.11}$$

Thus, the function $g(y)$ decreases monotonically. Let us discuss the value at $y = 0$. In the limit of $y \to +0$, it holds from Eqs. (B.4), (B.5), and (B.6) that

$$\lim_{y\to+0} \frac{\coth\{Ky\}}{\coth y} = \begin{cases} \displaystyle\lim_{y\to+0} \frac{{}_{2N}C_0 \coth^{2N} y}{{}_{2N}C_1 \coth^{2N} y} = \frac{1}{2N} \text{ for } K = 2N, \\ \displaystyle\lim_{y\to+0} \frac{{}_{2N+1}C_0 \coth^{2N} y}{{}_{2N+1}C_1 \coth^{2N} y} = \frac{1}{2N+1} \text{ for } K = 2N+1. \end{cases}$$

Thus, one has that

$$\lim_{y\to+0} g(y) = \begin{cases} \displaystyle\lim_{y\to+0} \coth y \times \left(1 - \alpha\frac{K}{2N}\right) = +\infty \text{ for } K = 2N, \\ \displaystyle\lim_{y\to+0} \coth y \times \left(1 - \alpha\frac{K}{2N+1}\right) = +\infty \text{ for } K = 2N+1. \end{cases} \tag{B.12}$$

One also has that

$$\lim_{y\to\infty} g(y) = 1 - \alpha K < 0. \tag{B.13}$$

Thus, from Eqs. (B.11), (B.12), and (B.13), it is proven that there is only one solution that satisfies $g(y) = 0$. Therefore, for $\frac{1}{K} < \alpha < 1$, a solution exists that satisfies Eq. (B.3) in the range $\gamma_{K,\alpha} > 1$.

In the case of $0 < \alpha < \frac{1}{K}$, it holds that, for any $y > 0$,

$$g(y) > 0. \tag{B.14}$$

Then, there is no solution that satisfies $\gamma_{K,\alpha} > 1$. □

Lemma B.2 In the case of $K = 2N$ for $0 < \alpha < \frac{1}{K}$, assign a constant value to α. There is only one solution that satisfies Eq. (B.3) in the range $0 < \gamma_{K,\alpha} < 1$, and for $\frac{1}{K} < \alpha$ there is no solution in the range $0 < \gamma_{K,\alpha} < 1$.

Proof. The goal is to identify the condition of (K, α) for which there exists a solution $\gamma_{K,\alpha}$ satisfying Eq. (B.3) for $0 < \gamma_{K,\alpha} < 1$. In the beginning, for simplicity, change the form of Eq. (B.3) by changing the variable from $0 < \gamma_{K,\alpha} < 1$ to $\tanh y$ as Eq. (B.7). One has that

$$\tanh y = \frac{\alpha(2N)}{\tanh(2Ny)}. \tag{B.15}$$

In the following, one identifies the condition (K, α) for which there exists a solution $y^*_{K,\alpha} > 0$ $(0 < \tanh y^*_{K,\alpha} < 1)$ satisfying $\tanh y^*_{K,\alpha} = \frac{\alpha(2N)}{\tanh(2Ny^*_{K,\alpha})}$. A function $h_{2N}(y)$ is defined to be

$$h_{2N}(y) = \tanh y - \frac{\alpha(2N)}{\tanh(2Ny)}. \tag{B.16}$$

The derivative of $h_{2N}(y)$ is

$$h'_{2N}(y) = 1 - \tanh^2 y + \alpha(2N)^2 \frac{1 - \tanh^2(2Ny)}{\tanh^2(2Ny)} > 0. \tag{B.17}$$

One has that

$$\begin{aligned} h_{2N}(0) &= -\infty < 0, \\ h_{2N}(\infty) &= 1 - \alpha K > 0 \text{ for } 0 < \alpha < \tfrac{1}{K}. \end{aligned} \tag{B.18}$$

From Eqs. (B.17) and (B.18), it is clear that in the case of $K = 2N$ for constant α that satisfies $0 < \alpha < \frac{1}{K}$, there is only one solution that satisfies Eq. (B.3) in the range $0 < \gamma_{K,\alpha} < 1$.

In the case of $\frac{1}{2N} < \alpha$, it holds that, for all $y > 0$,

$$h_{2N}(y) < 0. \tag{B.19}$$

Therefore, there is no solution that satisfies $0 < \gamma_{K,\alpha} < 1$. □

Lemma B.3 In the case of $K = 2N + 1$ for $\frac{1}{K^2} < \alpha < \frac{1}{K}$, assign a constant value to α. There is only one solution that satisfies Eq. (B.3) in the range $0 < \gamma_{K,\alpha} < 1$, and for $\frac{1}{K} < \alpha$ there is no solution in the range $0 < \gamma_{K,\alpha} < 1$.

Proof. The goal is to identify the condition of (K, α) for which there exists a solution $\gamma_{K,\alpha}$ satisfying Eq. (B.3) for $0 < \gamma_{K,\alpha} < 1$. In the beginning, for simplicity, change the form of Eq. (B.3) by changing the variable from $0 < \gamma_{K,\alpha} < 1$ to $\tanh y$ as Eq. (B.7). One has that

$$\tanh y = \alpha(2N + 1) \tanh\{(2N + 1)y\}. \tag{B.20}$$

In the following, one identifies the condition (K, α) for which there exists a solution $y^*_{K,\alpha} > 0$ satisfying $\tanh y^*_{K,\alpha} = \alpha(2N + 1) \tanh\left\{(2N + 1)y^*_{K,\alpha}\right\}$. A function $h_{2N+1}(y)$ is defined to be

$$h_{2N+1}(y) = \tanh y - \alpha(2N + 1) \tanh\{(2N + 1)y\}. \tag{B.21}$$

It holds that

$$\begin{aligned} h_{2N+1}(0) &= 0, \\ h_{2N+1}(\infty) &= 1 - \alpha(2N + 1) > 0 \text{ for } \frac{1}{(2N + 1)^2} < \alpha < \frac{1}{(2N + 1)}. \end{aligned} \tag{B.22}$$

The derivative of $h_{2N+1}(y)$ is

$$\begin{aligned} h'_{2N+1}(y) &= 1 - \alpha(2N + 1)^2 + \alpha(2N + 1)^2 \tanh^2\{(2N + 1)y\} - \tanh^2 y, \\ h'_{2N+1}(0) &= 1 - \alpha(2N + 1)^2 < 0 \text{ for } \frac{1}{(2N + 1)^2} < \alpha < \frac{1}{2N + 1}. \end{aligned} \tag{B.23}$$

The derivative $h'_{2N+1}(y)$ is also expressed using cosh functions as follows:

$$h'_{2N+1}(y) = \frac{\cosh^2\{(2N + 1)y\}\left[1 - \alpha(2N + 1)^2 \frac{\cosh^2 y}{\cosh^2\{(2N+1)y\}}\right]}{\cosh^2 y \cosh^2\{(2N + 1)y\}}. \tag{B.24}$$

The function $J(y) = \frac{\cosh y}{\cosh\{(2N+1)y\}} \geq 0$ decreases monotonically since the derivative is

$$J'(y) = \frac{-\sinh(2Ny) - 2N\cosh y \sinh\{(2N+1)y\}}{\cosh^2\{(2N+1)y\}} < 0. \tag{B.25}$$

Considering the fact that

$$\lim_{y\to\infty} \frac{\cosh(y)}{\cosh(2N+1)y} = 0, \tag{B.26}$$

part of $h'_{2N+1}(y)$, $\left[1 - \alpha(2N+1)\frac{\cosh^2 y}{\cosh^2\{(2N+1)y\}}\right]$, increases monotonically and there is a unique point y_* at which the sign of $h'_{2N+1}(y)$ changes from minus to plus. Therefore, there is a unique point $0 < y_{**} < \infty$ at which it holds that $h_{2N+1}(y_{**}) = 0$.

The above discussion indicates that, in the case of $K = 2N+1$ and $\frac{1}{K^2} < \alpha < \frac{1}{K}$, there is only one solution that satisfies Eq. (B.3) in the range $0 < \gamma_{K,\alpha} < 1$.

In the case of $\frac{1}{K} < \alpha$, since it holds that, for all $y > 0$,

$$h_{2N+1}(y) < 0, \tag{B.27}$$

there is no solution in the range $0 < \gamma_{K,\alpha} < 1$. □

From Lemmas B.1, B.2, and B.3, the following lemmas hold.

Lemma B.4 Consider the case of $K = 2N$. For $\frac{1}{K} \leq \alpha < 1$ there is a unique solution of Eq. (B.3), and the solution $\gamma_{K,\alpha}$ is in the range $\gamma_{K,\alpha} \geq 1$. For $0 < \alpha < \frac{1}{K}$ there is a unique solution of Eq. (B.3), and the solution $\gamma_{K,\alpha}$ is in the range $0 < \gamma_{K,\alpha} < 1$.

Lemma B.5 Consider the case of $K = 2N+1$. For $\frac{1}{K} \leq \alpha < 1$ there is a unique solution of Eq. (B.3), and the solution $\gamma_{K,\alpha}$ is in the range $\gamma_{K,\alpha} \geq 1$. For $\frac{1}{K^2} < \alpha < \frac{1}{K}$ there is a unique solution of Eq. (B.3), and the solution $\gamma_{K,\alpha}$ is in the range $0 < \gamma_{K,\alpha} < 1$.

Range A is defined as follows:

Definition B.6 When the parameters (K, α) satisfy a condition such as

$$\begin{cases} 0 < \alpha < 1 & \text{in the case of} \quad K = 2N, \\ \dfrac{1}{K^2} < \alpha < 1 & \text{in the case of} \quad K = 2N+1, \end{cases} \tag{B.28}$$

where $N \in \mathbb{N}$, we say that the parameters (K, α) are in Range A.

From Lemmas B.4 and B.5, this theorem holds.

Theorem I When the parameters (K, α) are in Range A, the SGB transformations $\{S_{K,\alpha}\}$ preserve the Cauchy distribution and the scale parameter can be chosen uniquely.

Appendix C. Exactness

According to Refs. [28–30], the Perron–Frobenius operator and the exactness are defined as follows.

Definition C.1 (Perron–Frobenius operator) Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space and let f be a density function on $\mathcal{X}$. If a map $T : \mathcal{X} \to \mathcal{X}$ is a non-singular transformation, the unique operator $\mathcal{P} : L^1 \to L^1$ defined by

$$\int_A \mathcal{P}f(x)\mu(dx) = \int_{T^{-1}(A)} f(x)\mu(dx) \text{ for } A \in \mathcal{A} \tag{C.1}$$

is termed the Perron–Frobenius operator corresponding to T.

Definition C.2 (Exactness) A map T on a phase space $\mathcal{X}$ with the Perron–Frobenius operator $\mathcal{P}_T$ and unique stationary density f_* is considered to be exact if and only if

$$\lim_{n\to\infty} \|\mathcal{P}_T^n f - f_*\|_{L^1} = 0 \tag{C.2}$$

for every initial density $f \in \mathcal{D}$, where $\mathcal{D}$ denotes the set of all densities on $\mathcal{X}$.

This definition is equivalent to the following:

$$\lim_{n\to\infty} \mu_*(T^n A) = 1, \ \forall A \in \mathcal{A}, \ \mu_*(A) > 0, \tag{C.3}$$

where $\mathcal{A}$ denotes the σ-algebra and μ_* denotes the invariant measure corresponding to the invariant density f_*.

Theorem II When the parameters (K, α) are in Range A, the SGB transformations $\{S_{K,\alpha}\}$ are exact.

Proof. This proof is based on one in a previous paper of ours, Ref. [25]. For the map $S_{K,\alpha}$ defined by Eq. (A.3), substituting $\cot(\pi\theta_n)$ into x_n, one has the map $\bar{S}_{K,\alpha} : [0, 1) \to [0, 1)$ such that

$$\begin{aligned} \cot(\pi\theta_{n+1}) &= \alpha K \cot(\pi K\theta_n), \\ \theta_{n+1} = \bar{S}_{K,\alpha}(\theta_n) &= \frac{1}{\pi}\cot^{-1}\{\alpha K \cot(\pi K\theta_n)\}. \end{aligned} \tag{C.4}$$

The derivative of $\bar{S}_{K,\alpha}$ with respect to θ is:

$$\bar{S}'_{K,\alpha}(\theta) = \frac{\alpha K^2\left\{1 + \cot^2(\pi K\theta)\right\}}{\alpha^2 K^2 \cot^2(\pi K\theta) + 1} > 0 \ \text{ for } \ 0 < \alpha < 1. \tag{C.5}$$

Then, $\bar{S}_{K,\alpha}$ increases monotonically. Because it holds that

$$\frac{1}{\pi}\cot^{-1}\{\alpha K \cot(\pi K\theta_n)\} = \frac{1}{\pi}\cot^{-1}\left[\alpha K \cot\left\{\pi K\left(\theta_n + \frac{j}{K}\right)\right\}\right], \ j = 1, 2 \ldots, K, \tag{C.6}$$

the form of $\bar{S}_{K,\alpha}$ has translational symmetry and it can be constructed by shifting the form on $\left[0, \frac{1}{K}\right)$. That is, the map $\bar{S}_{K,\alpha}$ is a K-to-one map, and on any interval $\left[j, \frac{j}{K}\right)$, $j = 0, \ldots, K-1$, the form of the map $\bar{S}_{K,\alpha}$ is the same as that on the interval $\left[0, \frac{1}{K}\right)$. Then, by operating $\bar{S}_{K,\alpha}^{-1}$, the measure on $[0, 1)$ is divided into K equivalently. We obtain intervals $\{I_{j,n}\}$ defined below by operating $\bar{S}_{K,\alpha}^{-n}$ into $[0, 1)$. The interval $I_{j,n} \subset [0, 1)$ is defined to be

$$\begin{aligned} I_{j,n} &\stackrel{\text{def}}{=} \left[\eta_{j,n}, \eta_{j+1,n}\right),\ \eta_{j,n} < \eta_{j+1,n},\ 0 \le j \le K^n - 1, \\ \eta_{0,n} &= 0 \text{ and } \eta_{K^n,n} = 1, \\ \bar{S}_{K,\alpha}^{n}(I_{j,n}) &= [0, 1), \\ \mu(I_{j,n}) &= \frac{1}{K^n}. \end{aligned} \tag{C.7}$$

For any non-zero measure subset $C \subset [0, 1)$, the set C includes a union of intervals $\bigcup_{j',n'} I_{j',n'}$. Then, for an invariant density f_* and associated measure μ_* [30], it holds that

$$1 \ge \lim_{n\to\infty} \mu_*(\bar{S}_{K,\alpha}^{n}(C)) \ge \lim_{n\to\infty} \mu_* \left(\bar{S}_{K,\alpha}^{n} \left\{ \bigcup_{j',n'} I_{j',n'} \right\} \right) = 1. \tag{C.8}$$

Therefore, the map $\bar{S}_{K,\alpha}$ on a phase space $[0, 1)$, is exact. Owing to the topological conjugacy, the map $S_{K,\alpha}$ is also exact. □

Appendix D. Scaling behavior

First, the case of $\frac{1}{K} < \alpha < 1 (\gamma_{K,\alpha} > 1)$ is discussed.

Lemma D.1 In the case of $K = 2N$, $\gamma_{2N,\alpha}$ behaves as

$$\frac{1}{\gamma_{2N,\alpha}} = O(\sqrt{1-\alpha}) \tag{D.1}$$

in the limit of $\gamma_{2N,\alpha} \to \infty$.

Proof. In the case of $K = 2N$, according to Eq. (B.8), Eq. (B.3) is rewritten as

$$\alpha = \frac{\coth y}{2N \coth(2Ny)}. \tag{D.2}$$

Then, one has that

$$\begin{aligned} 1 - \alpha &= \frac{1}{2N} \cdot \frac{2N \sum_{k=0}^{N} {}_{2N}C_{2k} \cdot \gamma_{2N,\alpha}^{-2k} - \sum_{k=0}^{N-1} {}_{2N}C_{2k+1} \cdot \gamma_{2N,\alpha}^{-2k}}{\sum_{k=0}^{N} {}_{2N}C_{2k} \cdot \gamma_{2N,\alpha}^{-2k}}, \\ &= \frac{1}{2N} \cdot \frac{2N \cdot \gamma_{2N,\alpha}^{-2N} + \sum_{k=1}^{N-1} \frac{2k(2N+1)}{2k+1} {}_{2N}C_{2k} \cdot \gamma_{2N,\alpha}^{-2k}}{\sum_{k=0}^{N} {}_{2N}C_{2k} \cdot \gamma_{2N,\alpha}^{-2k}}. \end{aligned} \tag{D.3}$$

In the limit of $\gamma_{2N,\alpha} \to \infty$, it holds that

$$\begin{aligned} 1-\alpha &\sim \frac{4N^2-1}{3} \cdot \gamma_{2N,\alpha}^{-2}, \\ \therefore \frac{1}{\gamma_{2N,\alpha}} &= O\left(\sqrt{1-\alpha}\right). \end{aligned} \tag{D.4}$$

□

Lemma D.2 In the case of $K = 2N+1$, $\gamma_{2N+1,\alpha}$ behaves as

$$\frac{1}{\gamma_{2N+1,\alpha}} = O(\sqrt{1-\alpha}) \tag{D.5}$$

in the limit of $\gamma_{2N+1,\alpha} \to \infty$.

Proof. In the case of $K = 2N+1$, according to Eq. (B.8), Eq. (B.3) is rewritten as

$$\alpha = \frac{\coth y}{(2N+1)\coth\{(2N+1)y\}}. \tag{D.6}$$

Then one has that

$$1-\alpha = \frac{1}{2N+1} \cdot \frac{\displaystyle\sum_{k=1}^{N} \frac{4k(N+1)}{2k+1} {}_{2N+1}C_{2k} \cdot \gamma_{2N+1,\alpha}^{-2k}}{\displaystyle\sum_{k=0}^{N} {}_{2N+1}C_{2k} \cdot \gamma_{2N+1,\alpha}^{-2k}}. \tag{D.7}$$

In the limit of $\gamma_{2N+1,\alpha} \to \infty$, it holds that

$$\begin{aligned} 1-\alpha &\sim \frac{4N(N+1)}{3} \cdot \gamma_{2N+1,\alpha}^{-2}, \\ \therefore \frac{1}{\gamma_{2N+1,\alpha}} &= O(\sqrt{1-\alpha}). \end{aligned} \tag{D.8}$$

□

Lemma D.3 In the case of $K = 2N$, $\gamma_{2N,\alpha}$ behaves as

$$\gamma_{2N,\alpha} \sim \sqrt{\alpha} \tag{D.9}$$

in the limit of $\gamma_{2N,\alpha} \to 0$.

Proof. We discuss the case of $K = 2N$ and $0 < \alpha < \frac{1}{K}$. According to Eq. (B.15), Eq. (B.3) is rewritten as

$$\alpha = \frac{1}{2N} \tanh y \cdot \tanh(2Ny). \tag{D.10}$$

Then one has that

$$\alpha = \frac{1}{2N} \cdot \frac{\displaystyle\sum_{k=0}^{N-1} {}_{2N}C_{2k+1} \gamma_{2N,\alpha}^{2k+2}}{\displaystyle\sum_{k=0}^{N} {}_{2N}C_{2k} \gamma_{2N,\alpha}^{2k}}. \tag{D.11}$$

In the limit of $\gamma_{2N,\alpha} \to 0$, it holds that

$$\begin{aligned} \alpha &\sim \frac{1}{2N} \cdot 2N \cdot \gamma_{2N,\alpha}^2 = \gamma_{2N,\alpha}^2, \\ \therefore \gamma_{2N,\alpha} &\sim \sqrt{\alpha}. \end{aligned} \tag{D.12}$$

□

Lemma D.4 In the case of $K = 2N + 1$, $\gamma_{2N+1,\alpha}$ behaves as

$$\gamma_{2N+1,\alpha} = O\left(\sqrt{\alpha - \frac{1}{(2N+1)^2}}\right) \tag{D.13}$$

in the limit of $\gamma_{2N+1,\alpha} \to 0$.

Proof. We discuss the case of $K = 2N + 1$ and $\frac{1}{K^2} < \alpha < \frac{1}{K}$. According to Eq. (B.20), Eq. (B.3) is rewritten as

$$\alpha = \frac{\tanh y}{(2N+1)\tanh\{(2N+1)y\}}. \tag{D.14}$$

Then one has that

$$\alpha - \frac{1}{(2N+1)^2} = \frac{\displaystyle\sum_{k=1}^{N} \frac{4k(N+1)}{4k+1} {}_{2N+1}C_{2k} \cdot \gamma_{2N+1,\alpha}^{2k}}{(2N+1)^2 \displaystyle\sum_{k=0}^{N} {}_{2N+1}C_{2k+1} \cdot \gamma_{2N+1,\alpha}^{2k}}. \tag{D.15}$$

In the limit of $\gamma_{2N+1,\alpha} \to 0$, it holds that

$$\begin{aligned} \alpha - \frac{1}{(2N+1)^2} &\sim \frac{4N(N+1)\gamma_{2N+1,\alpha}^2}{5(2N+1)^2}, \\ \therefore \gamma_{2N+1,\alpha} &= O\left(\sqrt{\alpha - \tfrac{1}{(2N+1)^2}}\right). \end{aligned} \tag{D.16}$$

□

From Lemmas D.1, D.2, D.3, and D.4, one knows that there are relations between the parameter α and the scaling parameter $\gamma_{K,\alpha}$. For all α in Range A, the Lyapunov exponent is denoted as

$$\lambda_{K,\alpha} = \frac{1}{\pi}\int_{\mathbb{R}\setminus A} \log\left|S'_{K,\alpha}(x)\right| \frac{\gamma_{K,\alpha}}{\gamma_{K,\alpha}^2 + x^2}dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|S'_{K,\alpha}(x)\right| \frac{\gamma_{K,\alpha}}{\gamma_{K,\alpha}^2 + x^2}dx < \infty, \tag{D.17}$$

$$= \begin{cases} \dfrac{\gamma_{K,\alpha}}{\pi}\displaystyle\int_{-\infty}^{\infty} \log\left|S'_{K,\alpha}(x)\right| \frac{1}{\gamma_{K,\alpha}^2 + x^2}dx & \text{for } 0 < \gamma_{K,\alpha} < 1, \\ \dfrac{1}{\gamma_{K,\alpha}\pi}\displaystyle\int_{-\infty}^{\infty} \log\left|S'_{K,\alpha}(x)\right| \frac{1}{1 + (x/\gamma_{K,\alpha})^2}dx & \text{for } 1 \le \gamma_{K,\alpha}. \end{cases} \tag{D.18}$$

Define an invariant density $\widehat{f}_{K,\alpha}(x)$ on $[0,1)$ for $\bar{S}_{K,\alpha}$. From simple calculation, it is obvious that $\widehat{f}_{K,\frac{1}{K}}(x)$ is the uniform distribution. According to Ref. [29], define the entropy $H_K(\alpha) = H[\widehat{f}_{K,\alpha}] = -\int_{[0,1)}\widehat{f}_{K,\alpha}(x)\log\widehat{f}_{K,\alpha}(x)\mu(dx)$; then, from Proposition 9.1.1 in Ref. [29], one has the maximal entropy for the constant density $\widehat{f}_{K,\frac{1}{K}}$. Since the dynamical system is ergodic when the parameters (K,α) are in Range A, almost all orbits do not converge and one sees that $-\infty < \lambda_{K,\alpha}$. Thus it holds that, for all α in Range A,

$$-\infty < \lambda_{K,\alpha} \le \lambda_{K,\frac{1}{K}} = \log|K| < \infty. \tag{D.19}$$

Accordingly, define functions $\phi_1(\theta,\gamma_{K,\alpha})$ and $\phi_2(\theta,1/\gamma_{K,\alpha})$ to be

$$\begin{aligned} \phi_1(\theta,\gamma_{K,\alpha}) &= \log\left|\frac{\alpha K^2\sin^2\theta}{\sin^2 K\theta}\right| \frac{\gamma_{K,\alpha}}{\gamma_{K,\alpha}^2\sin^2\theta + \cos^2\theta}, \\ \phi_2(\theta,1/\gamma_{K,\alpha}) &= \log\left|\frac{\alpha K^2\sin^2\theta}{\sin^2 K\theta}\right| \frac{1/\gamma_{K,\alpha}}{\sin^2\theta + (\cos\theta/\gamma_{K,\alpha})^2}, \end{aligned} \tag{D.20}$$

and also define a set of points $\{a_n\}_{n=1}^{K-1}$ such that at $\theta = a_n \in (0,\pi]$, the function $\log\left|\frac{\alpha K^2\sin^2\theta}{\sin^2 K\theta}\right|$ is not continuous.

By changing the variable from x to $\cot\theta$, the Lyapunov exponent $\lambda_{K,\alpha}$ is rewritten as

$$\lambda_{K,\alpha} = \frac{1}{\pi}\int_0^{\pi} \phi_1(\theta,\gamma_{K,\alpha})d\theta. \tag{D.21}$$

Theorem III Suppose that the parameters (K,α) are in Range A.

- For any $K \in \mathbb{N}\setminus\{1\}$, it holds that $\nu_1 = \frac{1}{2}$ as $\alpha \to 1-0$.
- For any $K \in \mathbb{N}\setminus\{1\}$, it holds that $\nu_1 = 1$ as $\alpha \to 1+0$.
- For $K = 2N+1$, it holds that $\nu_2 = \frac{1}{2}$ as $\alpha \to \frac{1}{K^2}+0$.

Proof. The integrand in Eq. (D.21) is continuous in $(a_n, a_{n+1}]$ for $0 \le n \le K-1$, where $a_0 = 0$ and $a_K = \pi$. The derivative of $\phi_1(\theta,\gamma_{K,\alpha})$ with respect to $\gamma_{K,\alpha}$ is

$$\begin{aligned} \frac{\partial\phi_1}{\partial\gamma_{K,\alpha}} &= \frac{1}{\alpha}\frac{\partial\alpha}{\partial\gamma_{K,\alpha}}\frac{\gamma_{K,\alpha}}{\gamma_{K,\alpha}^2\sin^2\theta + \cos^2\theta} + \log|\alpha|\frac{-\gamma_{K,\alpha}^2\sin^2\theta + \cos^2\theta}{(\gamma_{K,\alpha}^2\sin^2\theta + \cos^2\theta)^2} \\ &\quad + \log\left|\frac{K^2\sin^2\theta}{\sin^2 K\theta}\right| \frac{-\gamma_{K,\alpha}^2\sin^2\theta + \cos^2\theta}{(\gamma_{K,\alpha}^2\sin^2\theta + \cos^2\theta)^2}. \end{aligned} \tag{D.22}$$

The derivative is continuous on each interval $(a_n, a_{n+1}]$.

(i) In the limit of $\alpha \to 1-0$ ($\gamma_{K,\alpha} \to \infty$) for any K, change the variable by $z = \frac{1}{\gamma_{K,\alpha}}$ and denote the function $\phi_2(\theta, z)$ as

$$\phi_2(\theta, z) = \log \left| \frac{\alpha K^2 \sin^2\theta}{\sin^2 K\theta} \right| \frac{z}{\sin^2\theta + z^2\cos^2\theta}. \tag{D.23}$$

It holds that

$$\begin{aligned} \left|\lambda_{K,\alpha}\right| &< \infty, \\ \lambda_{K,\alpha} &= \frac{1}{\pi}\int_0^\pi \phi_2(\theta, z) d\theta \\ &= \frac{1}{\pi}\sum_{n=0}^{K-1}\int_{a_n}^{a_{n+1}} d\theta \left[\phi_2(\theta, 0) + \frac{\partial\phi_2}{\partial z}(\theta, 0) z + O(z^2)\right] \\ &= \frac{z}{\pi}\sum_{n=0}^{K-1}\int_{a_n}^{a_{n+1}} d\theta \left[\frac{\partial\phi_2}{\partial z}(\theta, 0) + O(z)\right]. \end{aligned} \tag{D.24}$$

Since $\lambda_{K,\alpha}$ and z are finite, $\frac{1}{\pi}\sum_{n=0}^{K-1}\int_{a_n}^{a_{n+1}} d\theta \left[\frac{\partial\phi_2}{\partial z}(\theta, 0) + O(z)\right]$ is also finite. In addition, $\frac{\partial\phi_2}{\partial z}(\theta, 0)$ does not depend on z. Therefore, in the limit of $z \to +0$ ($\gamma_{K,\alpha} \to \infty$, $\alpha \to 1-0$), it holds that

$$\lambda_{K,\alpha} = O(z) = O\left(\sqrt{1-\alpha}\right). \tag{D.25}$$

(ii) In the case of $K = 2N+1$, according to Eq. (D.16) there is a relation $\alpha \sim \frac{4N(N+1)}{5(2N+1)^2}\gamma_{K,\alpha}^2 + \frac{1}{(2N+1)^2}$ in the limit of $\gamma_{K,\alpha} \to +0$ ($\alpha \to \frac{1}{K^2} + 0$). Then it holds that

$$\begin{aligned} \left|\lambda_{K,\alpha}\right| &< \infty, \\ \lambda_{K,\alpha} &= \frac{1}{\pi}\int_0^\pi \phi_1(\theta, \gamma_{K,\alpha}) d\theta \\ &= \frac{1}{\pi}\sum_{n=0}^{K-1}\int_{a_n}^{a_{n+1}} d\theta \left[\phi_1(\theta, 0) + \frac{\partial\phi_1}{\partial\gamma_{K,\alpha}}(\theta, 0)\gamma_{K,\alpha} + O(\gamma_{K,\alpha}^2)\right] \\ &= \frac{\gamma_{K,\alpha}}{\pi}\sum_{n=0}^{K-1}\int_{a_n}^{a_{n+1}} d\theta \left[\frac{\partial\phi_1}{\partial\gamma_{K,\alpha}}(\theta, 0) + O(\gamma_{K,\alpha})\right]. \end{aligned} \tag{D.26}$$

Thus, it also holds that $\frac{1}{\pi}\sum_{n=0}^{K-1}\int_{a_n}^{a_{n+1}} d\theta \left[\frac{\partial\phi_1}{\partial\gamma_{K,\alpha}}(\theta, 0) + O(\gamma_{K,\alpha})\right]$ is finite and $\frac{\partial\phi_1}{\partial\gamma_{K,\alpha}}(\theta, 0)$ does not depend on $\gamma_{K,\alpha}$. Therefore, in the limit of $\gamma_{K,\alpha} \to +0$ ($\alpha \to \frac{1}{K^2} + 0$), it holds that

$$\lambda_{K,\alpha} = O(\gamma_{K,\alpha}) = O\left(\sqrt{\alpha - \frac{1}{K^2}}\right). \tag{D.27}$$

(iii) In the limit of $\alpha \to 1+0$ for any K, the Lyapunov exponent is given, for $\alpha > 1$, by

$$\begin{aligned} \lambda_{K,\alpha} &= \log\alpha = \log\{1 + (\alpha - 1)\}, \\ &= (\alpha - 1) - \tfrac{1}{2}(\alpha - 1)^2 + \tfrac{1}{3}(\alpha - 1)^3 - \cdots. \end{aligned} \tag{D.28}$$

Therefore, it holds that

$$\lambda_{K,\alpha} = O(\alpha - 1). \tag{D.29}$$

From Eqs. (D.25), (D.27), and (D.29), it is clear that for any K, the critical exponent of the Lyapunov exponent ν_1 is $\frac{1}{2}$ in the limit of $\alpha \to 1-0$, that for $K = 2N+1$, $\nu_2 = \frac{1}{2}$ in the limit of $\alpha \to \frac{1}{K^2}+0$, and that for any K, $\nu_1 = 1$ in the limit of $\alpha \to 1+0$. □

The above discussion proves that the derivatives of the Lyapunov exponent with respect to parameter α diverge at the critical points, which means that the parameter dependence of the Lyapunov exponent diverges at the critical points. This result implies the difficulty of calculating the Lyapunov exponent near the critical points.

Appendix D.1. Scaling behavior for $K = 3$, 4, and 5

The solutions of Eq. (B.3) that satisfy $0 < \gamma_{K,\alpha} < \infty$ in the cases of $K = 3$, 4, and 5 are determined uniquely when the parameters (K, α) are in Range A as follows:

$$\begin{aligned}
\gamma_{3,\alpha} &= \sqrt{\frac{9\alpha-1}{3-3\alpha}}, \\
\gamma_{4,\alpha} &= \sqrt{\frac{6\alpha-1+\sqrt{32\alpha^2-8\alpha+1}}{2(1-\alpha)}}, \\
\gamma_{5,\alpha} &= \sqrt{\frac{-5(1-5\alpha)+\sqrt{20(25\alpha^2-6\alpha+1)}}{5(1-\alpha)}}.
\end{aligned} \tag{D.30}$$

The analytical formulae of the Lyapunov exponent can be determined from Eq. (D.17) as follows:

$$\begin{aligned}
\lambda_{3,\alpha} &= \log\left|\frac{1}{\alpha}\left(\frac{3(1-\alpha)}{8}\right)^2\left[1+\sqrt{\frac{9\alpha-1}{3-3\alpha}}\right]^4\right|, \\
\lambda_{4,\alpha} &= \log\left|\frac{\alpha(1+\gamma_4)^6}{\gamma_4^2(1+\gamma_4^2)^2}\right|, \\
\lambda_{5,\alpha} &= \log\left|\frac{25}{256\alpha}\frac{(1-\alpha)^4}{(\sqrt{125\alpha^2-30\alpha+5}+11\alpha-1)^2}|1+\gamma_5|^8\right|.
\end{aligned} \tag{D.31}$$

In the case of $K = 3$, Eq. (D.31) converges to zero in the limit of $\alpha \to \frac{1}{9}+0$ and $\alpha \to 1-0$, and the derivative of the Lyapunov exponent $\frac{\partial\lambda_{3,\alpha}}{\partial\alpha}$ diverges at $\alpha = \frac{1}{9}, 1$. When the parameter α is close to $\frac{1}{9}$, the Lyapunov exponent increases as follows:

$$\begin{aligned}
\lambda_{3,\alpha} &= -\log\left|1+9\left(\alpha-\frac{1}{9}\right)\right| + 2\log\left|1-\frac{9}{8}\left(\alpha-\frac{1}{9}\right)\right| + 4\log\left|1+\sqrt{\frac{3\left(\alpha-\frac{1}{9}\right)}{1-\alpha}}\right| \\
&= O\left(-9\left(\alpha-\frac{1}{9}\right)-\frac{9}{4}\left(\alpha-\frac{1}{9}\right)+4\sqrt{\frac{3\left(\alpha-\frac{1}{9}\right)}{1-\alpha}}\right) \\
&\sim 6\sqrt{\frac{3}{2}}\sqrt{\alpha-\frac{1}{9}}.
\end{aligned} \tag{D.32}$$

Then, the critical exponent ν_2 of the Lyapunov exponent at $\alpha = \frac{1}{9}$ is $\frac{1}{2}$. In the case of $\alpha \lesssim 1$, the Lyapunov exponent $\lambda_{3,\alpha}$ behaves as follows:

$$\begin{aligned}\lambda_{3,\alpha} &= 2\log\left|1 - \frac{9}{8}(1-\alpha)\right| - \log|1-(1-\alpha)| + 4\log\left|1 + \sqrt{\frac{1-\alpha}{3\left(\alpha - \frac{1}{9}\right)}}\right| \\ &= O\left(\frac{9}{4}(1-\alpha) - (1-\alpha) + 4\sqrt{\frac{1-\alpha}{3\left(\alpha - \frac{1}{9}\right)}}\right) \\ &\sim 2\sqrt{\frac{3}{2}}\sqrt{1-\alpha}. \end{aligned} \tag{D.33}$$

Then, at $\alpha = 1$ one has $\nu_1 = \frac{1}{2}$.

For $K = 3$ and for $\frac{1}{9} < \alpha < 1$, the fixed point is only

$$x^* = 0. \tag{D.34}$$

In the range of $0 < \alpha < \frac{1}{9}$ and $1 < \alpha$, there are other fixed points $q_\pm(\alpha)$ denoted as $q_\pm^2(\alpha) = -\dfrac{9\alpha - 1}{3(1-\alpha)}$. Then, it holds that

$$\begin{aligned} &\lim_{\alpha \to \frac{1}{9}-0} q_\pm^2(\alpha) = +0, \quad &&\lim_{\alpha \to \frac{1}{9}-0} S'_{3,\alpha}(q_\pm(\alpha)) = 1, \\ &\lim_{\alpha \to 1+0} q_\pm^2(\alpha) = +\infty, \quad &&\lim_{\alpha \to 1+0} S'_{3,\alpha}(q_\pm(\alpha)) = 1. \end{aligned} \tag{D.35}$$

Then, the Floquet multiplier χ at $\alpha = \frac{1}{9}$ and $\alpha = 1$ is

$$\begin{aligned} \chi_{3,\frac{1}{9}} &= S'_{3,\frac{1}{9}}(0) &&= 1, \\ \chi_{3,1} &= \lim_{\alpha \to 1+0} S'_{3,\alpha}(q_\pm(\alpha)) &&= 1. \end{aligned} \tag{D.36}$$

On the basis of these results, we can say that only *Type 1* intermittency occurs for $K = 3$. These results represent new phenomena because, for the Generalized Boole transformation ($K = 2$), one can observe *two* different intermittency types, *Type 1* and *Type 3* [25].

In the case of $K = 4$, the Lyapunov exponent of Eq. (D.31) is

$$\lambda_{4,\alpha} = \log\left|\frac{\alpha}{\gamma_4^2}\right| + 6\log|1+\gamma_4| - 2\log|1+\gamma_4^2|. \tag{D.37}$$

Let us discuss the scaling behavior of $\lambda_{4,\alpha}$ at $\alpha = 0$. Now, the first term of Eq. (D.37) is rewritten as

$$\begin{aligned}\left|\frac{\alpha}{\gamma_4^2}\right| &= \log\left|\frac{2\alpha(1-\alpha)}{6\alpha - 1 + \sqrt{32\alpha^2 - 8\alpha + 1}}\right| \\ &= \left|-\frac{6\alpha - 1 - \sqrt{32\alpha^2 - 8\alpha + 1}}{2}\right|, \end{aligned} \tag{D.38}$$

$$\therefore \lim_{\alpha \to 0}\left|\frac{\alpha}{\gamma_4^2}\right| = 1. \tag{D.39}$$

Then,

$$\begin{aligned} \gamma_4 &\sim \sqrt{\alpha}, \\ \Longrightarrow \log|1+\gamma_4| &\sim \sqrt{\alpha}, \text{ and } \log\left|1+\gamma_4^2\right| \sim \alpha, \\ \therefore \lambda_{4,\alpha} &\sim 6\sqrt{\alpha} = O\left(\sqrt{\alpha}\right). \end{aligned} \tag{D.40}$$

Therefore, the critical exponent of the Lyapunov exponent for $K = 4,\ \alpha = 0$ is

$$\nu_3 = \frac{1}{2}. \tag{D.41}$$

Next, consider the scaling behavior of $\lambda_{4,\alpha}$ at $\alpha = 1-0$. From Eq. (D.38), it holds that, in the vicinity of $\alpha = 1$,

$$\begin{aligned} \lim_{\alpha\to 1-0} \frac{\gamma_4^2(1-\alpha)}{5} &= 1, \\ \therefore \gamma_4 &\sim \sqrt{\frac{5}{1-\alpha}} \text{ as } \alpha \to 1-0. \end{aligned} \tag{D.42}$$

The Lyapunov exponent of Eq. (D.37) is

$$\begin{aligned} \lambda_{4,\alpha} &= \log\left|\frac{\alpha}{\gamma_4^2}\right| + 6\log\gamma_4 + 6\log\left|1+\frac{1}{\gamma_4}\right| - 2\log\left|\gamma_4^2\right| - 2\log\left|1+\frac{1}{\gamma_4^2}\right| \\ &= \log|\alpha| + 6\log\left|1+\frac{1}{\gamma_4}\right| - 2\log\left|1+\frac{1}{\gamma_4^2}\right| \\ &= O\left(-(1-\alpha) + \frac{6}{\gamma_4} - \frac{2}{\gamma_4^2}\right), \\ \therefore \lambda_{4,\alpha} &\sim \frac{6\sqrt{5}}{5}\sqrt{1-\alpha}, \text{ as } \alpha \to 1-0. \end{aligned} \tag{D.43}$$

Therefore, the critical exponent of the Lyapunov exponent for $K = 4,\ \alpha = 1-0$ is

$$\nu_1 = \frac{1}{2}. \tag{D.44}$$

In the case of $K = 4$, the fixed points of $S_{4,\alpha}$ are:

$$x_{4*} = \begin{cases} 0, & \alpha = 0 \\ \pm\sqrt{\dfrac{1-6\alpha+\sqrt{40\alpha^2-16\alpha+1}}{1-\alpha}} & 0 < \alpha < 1, \\ \pm\dfrac{1}{\sqrt{5}} & \alpha = 1. \end{cases} \tag{D.45}$$

It also holds that

$$\begin{aligned} S_{4,\alpha}(x) - x &= \frac{4(\alpha-1)x^4 - 4(6\alpha+1)x^2 + 4\alpha}{4x^3 - 4x}, \\ \lim_{|x|\to\infty}\left(S_{4,1}(x) - x\right) &= 0. \end{aligned} \tag{D.46}$$

In the case of SGB transformations, at $\alpha = 0$, all points are mapped to the original point $x = 0$. As this is not interesting, other fixed points are considered as follows:

$$\begin{aligned} \lim_{\alpha\to+0} S'_{4,\alpha}\left(\pm\sqrt{\frac{1-6\alpha+\sqrt{40\alpha^2-16\alpha+1}}{1-\alpha}}\right) &= 0, \\ S'_{4,1}\left(\pm\frac{1}{\sqrt{5}}\right) &= \frac{27}{2}, \\ \lim_{|x|\to\infty} S'_{4,1}(x) &= 1. \end{aligned} \tag{D.47}$$

For $K = 4$, from Eq. (D.47), one obtains the Floquet multiplier at $\alpha = 1$ as

$$\chi_{4,1} = 1. \tag{D.48}$$

This result indicates that *Type 1* intermittency occurs at $\alpha = 1$.

In the case of $K = 5$, Eq. (D.31) converges to zero in the limit of $\alpha \to \frac{1}{25} + 0$ and $\alpha \to 1 - 0$. In addition, it holds that $\left|\frac{\partial\lambda_{5,\alpha}}{\partial\alpha}\left(\frac{1}{25}\right)\right| = \left|\frac{\partial\lambda_{5,\alpha}}{\partial\alpha}(1)\right| = \infty$. The Lyapunov exponent $\lambda_{5,\alpha}$ for $\frac{1}{25} < \alpha < 1$ can be expanded as

$$\lambda_{5,\alpha} = \log\frac{25}{256} - \log|\alpha| + 4\log|1-\alpha| - 2\log\left|1 - 11\alpha - \sqrt{125\alpha^2 - 30\alpha + 5}\right| + 8\log|1+\gamma_5|. \tag{D.49}$$

Now, the sum from the first term to the fourth term converges to zero such that

$$\lim_{\alpha\to\frac{1}{25}+0}\left[\log\frac{25}{256} - \log|\alpha| + 4\log|1-\alpha| - 2\log\left|1 - 11\alpha - \sqrt{125\alpha^2 - 30\alpha + 5}\right|\right] = 0. \tag{D.50}$$

When the parameter α is close to $\frac{1}{25}$, it holds that

$$\begin{aligned} &4\log|1-\alpha| - 2\log\left|1 - 11\alpha - \sqrt{125\alpha^2 - 30\alpha + 5}\right| \\ &= 2\log\left|\tfrac{1}{4}\left\{1 - 11\alpha + \sqrt{125\alpha^2 - 30\alpha + 5}\right\}\right| \\ &= -4\log 2 + 2\log\tfrac{14}{25} + 2\log\left|1 - \tfrac{275}{14}\left(\alpha - \tfrac{1}{25}\right) + \tfrac{25}{14}\sqrt{\tfrac{20}{5} - 20\left(\alpha - \tfrac{1}{25}\right) + 125\left(\alpha - \tfrac{1}{25}\right)^2}\right|. \end{aligned} \tag{D.51}$$

Then, the Lyapunov exponent increases as follows:

$$\begin{aligned} \lambda_{5,\alpha} = {} & \log\tfrac{25}{256} - \log\tfrac{1}{25} - \log\left|1 + 25\left(\alpha - \tfrac{1}{25}\right)\right| \\ & - 4\log 2 + 2\log\tfrac{14}{25} \\ & + 2\log\left|1 - \tfrac{275}{14}\left(\alpha - \tfrac{1}{25}\right) + \tfrac{25}{14}\sqrt{\tfrac{20}{5} - 20\left(\alpha - \tfrac{1}{25}\right) + 125\left(\alpha - \tfrac{1}{25}\right)^2}\right| \\ & + 8\log|1+\gamma_5|. \end{aligned} \tag{D.52}$$

From Eq. (D.50), the Lyapunov exponent obeys

$$\lambda_{5,\alpha} = O\left(\left(\alpha - \frac{1}{25}\right) + 8\gamma_5\right) \sim 8\gamma_5 \sim 20\sqrt{\frac{5}{6}\left(\alpha - \frac{1}{25}\right)}. \tag{D.53}$$

Therefore, the critical exponent ν_2 is

$$\nu_2 = \frac{1}{2}. \tag{D.54}$$

For $\alpha \lesssim 1$, it holds that

$$\lim_{\alpha\to 1-0}\left\{\log\frac{25}{256}-\log|\alpha|-2\log\left|1-11\alpha-\sqrt{125\alpha^2-30\alpha+5}\right|\right\}=-12\log 2, \tag{D.55}$$

$$4\log|1-\alpha|+8\log|1+\gamma_5|\sim 8\log\left|2\sqrt{2}+\sqrt{1-\alpha}\right|\sim 12\log 2+8\log\left|1+\frac{1}{2}\sqrt{\frac{1-\alpha}{2}}\right|. \tag{D.56}$$

Then, the Lyapunov exponent increases as follows:

$$\begin{aligned}\lambda_{5,\alpha}=&\log\tfrac{25}{256}-\log|1-(1-\alpha)|\\&-2\log 10-2\log\left|1-\tfrac{11}{10}(1-\alpha)+\sqrt{1-22(1-\alpha)+\tfrac{125}{100}(1-\alpha)^2}\right|\\&+4\log|1-\alpha|+8\log|1+\gamma_5|\,.\end{aligned} \tag{D.57}$$

Thus, one has that, in the limit of $\alpha\to 1-0$,

$$\lambda_{5,\alpha}=O\left((1-\alpha)+4\sqrt{\frac{1-\alpha}{2}}\right)\sim 4\sqrt{\frac{1-\alpha}{2}}. \tag{D.58}$$

Therefore, the critical exponent ν_1 is

$$\nu_1=\frac{1}{2}. \tag{D.59}$$

In the case of $K=5$, the fixed points of $S_{5,\alpha}$ are as follows:

$$x_*=\begin{cases}0 & 0<\alpha\le 1,\\ \pm\sqrt{\frac{5}{3}} & \alpha=\frac{1}{25},\\ \pm\sqrt{\frac{5(1-5\alpha)+2\sqrt{5(25\alpha^2-6\alpha+1)}}{5(1-\alpha)}} & \frac{1}{25}<\alpha<1,\\ \pm\sqrt{\frac{3}{5}} & \alpha=1.\end{cases} \tag{D.60}$$

It also holds that

$$\lim_{|x|\to\infty}\left(S_{5,1}(x)-x\right)=0. \tag{D.61}$$

At the fixed points $x_*=0,\pm\sqrt{\frac{5}{3}},\pm\sqrt{\frac{3}{5}}$, and $\pm\infty$, the derivatives $S'_{5,\alpha}(x)=\frac{25\alpha(1+x^2)^4}{(5x^4-10x^2+1)^2}$ are

$$\begin{aligned}S'_{5,\frac{1}{25}}(0)&=1,\\ S'_{5,1}(0)&=25,\\ S'_{5,\frac{1}{25}}\left(\pm\sqrt{\tfrac{5}{3}}\right)&=16,\\ S'_{5,1}\left(\pm\sqrt{\tfrac{3}{5}}\right)&=16,\\ \lim_{|x|\to\infty}S'_{5,1}(x)&=1.\end{aligned} \tag{D.62}$$

From Eq. (D.62), for $K = 5$ and at $\alpha = \frac{1}{25}$ and $\alpha = 1$, the Floquet multipliers are obtained as

$$\begin{aligned} \chi_{5,\frac{1}{25}} &= S'_{5,\frac{1}{25}}(0) &&= 1, \\ \chi_{5,1} &= \lim_{|x|\to\infty} S'_{5,1}(x) &&= 1. \end{aligned} \tag{D.63}$$

Therefore, similar to the case of $K = 3$, only *Type 1* intermittency occurs, which is *different* from the case with the Generalized Boole transformation (K=2).

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