

# Gradient flow and the renormalization group

Yoshihiko Abe\* and Masafumi Fukuma\*

*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*

\*E-mail: y.abe@gauge.scphys.kyoto-u.ac.jp, fukuma@gauge.scphys.kyoto-u.ac.jp

Received June 6, 2018; Revised June 28, 2018; Accepted June 28, 2018; Published August 14, 2018

.....  
 We investigate the renormalization group (RG) structure of the gradient flow. Instead of using the original bare action to generate the flow, we propose to use the effective action at each flow time. We write down the basic equation for scalar field theory that determines the evolution of the action, and argue that the equation can be regarded as an RG equation if one makes a field-variable transformation at every step such that the kinetic term is kept in the canonical form. We consider a local potential approximation (LPA) to our equation, and show that the result has a natural interpretation with Feynman diagrams. We make an  $\varepsilon$  expansion of the LPA and show that it reproduces the eigenvalues of the linearized RG transformation around both the Gaussian and the Wilson–Fisher fixed points to the order of  $\varepsilon$ .  
 .....

Subject Index    B32, B37, B38

## 1. Introduction

In recent years the gradient flow has attracted much attention for practical and conceptual reasons [1–7]. Practically, as shown by Lüscher and Weisz [2,3], the gradient flow in non-Abelian gauge theory does not induce extra UV divergences in the bulk, so that the bulk theory is finite once the boundary theory is properly renormalized. Hence the ultralocal products of bulk operators automatically give renormalized composite operators, and this fact yields a lot of applications, including a construction of the energy–momentum tensor on the lattice [5,6].

On the other hand, there has been an expectation that the gradient flow may be interpreted as a renormalization group (RG) flow (see, e.g., Refs. [8–12]). This expectation is based on the observation made in Ref. [2]. To see this, let us consider a Euclidean scalar field theory in  $d$  dimensions with the bare action  $S_0[\phi]$ . We assume that the theory is implemented with some UV cutoff  $\Lambda_0$ . The gradient flow is then given by

$$\partial_\tau \phi_\tau(x) = - \frac{\delta S_0}{\delta \phi(x)}[\phi_\tau], \quad \phi_{\tau=0}(x) = \phi_0(x). \tag{1.1}$$

If the field is canonically normalized as  $\int_x [(1/2)(\partial_\mu \phi)^2 + \dots]$ , then the flow equation gives a heat equation with perturbation:

$$\partial_\tau \phi_\tau(x) = \partial_\mu^2 \phi_\tau(x) + \dots, \tag{1.2}$$

which can be solved as<sup>1</sup>

$$\phi_\tau(x) = \int_y K_\tau(x-y) \phi_0(y) + \dots, \tag{1.3}$$

<sup>1</sup> In this paper we only consider scalar field theory, but our discussion should be easily extended to other field theories. We use a standard polymorphic notation;  $\int_x$  represents  $\int d^d x$  when  $x$  are spacetime coordinates while  $\int_p$  stands for  $\int d^d p / (2\pi)^d$  when  $p$  are momenta. We often denote  $\phi(x)$  by  $\phi_x$ .

where  $K_\tau(x - y)$  is the heat kernel:

$$K_\tau(x - y) = \int_p e^{ip(x-y) - \tau p^2} = \frac{1}{(4\pi\tau)^{d/2}} e^{-(x-y)^2/4\pi\tau}. \quad (1.4)$$

Thus,  $\phi_\tau(x)$  can be interpreted as an effective field that is coarse-grained from  $\phi_0(y)$  within the radius  $r \propto \sqrt{\tau}$ .

However, this interpretation is not perfectly matched with the philosophy of the renormalization group. In fact, if we denote the solution to Eq. (1.1) by  $\phi_\tau(\phi_0) = (\phi_\tau(x; \phi_0))$  so as to specify its initial value, the distribution function of  $\phi$  at time  $\tau$  will be given by

$$p_\tau[\phi] = \frac{1}{Z_0} \int [d\phi_0] \delta[\phi - \phi_\tau(\phi_0)] e^{-S_0[\phi_0]} \quad \left( Z_0 \equiv \int [d\phi_0] e^{-S_0[\phi_0]} \right). \quad (1.5)$$

The flow equation gives the field  $\phi$  a tendency to approach the classical solution of the original bare action  $S_0[\phi]$ , and thus  $p_\tau[\phi]$  will take a sharp,  $\delta$ -function-like peak at the classical solution in the large- $\tau$  limit, but this is not what we expect in the renormalization group;  $\phi_\tau$  at large  $\tau$  should be regarded as a low-energy effective field, which can be well treated as the classical solution to the low-energy effective action at scale  $\Lambda = 1/\sqrt{\tau}$ , not to the bare action, which itself can be regarded as giving an effective theory at the original cutoff  $\Lambda_0 (\gg \Lambda)$ .

In this paper, we propose a novel gradient flow that gives the field a tendency to approach the classical solution of the effective action at scale  $\Lambda = 1/\sqrt{\tau}$  when the derivative is taken:

$$\partial_\tau \phi_\tau(x) = - \frac{\delta S_\tau}{\delta \phi(x)} [\phi_\tau], \quad \phi_{\tau=0}(x) = \phi_0(x). \quad (1.6)$$

Assuming that the initial value  $\phi_0(x)$  is distributed according to the distribution function  $e^{-S_0[\phi_0]}/Z_0$ , we impose the self-consistency condition that the classical solution  $\phi_\tau(x)$  be distributed with  $e^{-S_\tau[\phi]}/Z_\tau$ :<sup>2</sup>

$$e^{-S_\tau[\phi]} \equiv \int [d\phi_0] \delta[\phi - \phi_\tau(\phi_0)] e^{-S_0[\phi_0]}, \quad (1.7)$$

where  $\phi(x)$  should have only coarse-grained degrees of freedom. We investigate the consequences of this requirement, and argue that the obtained equation for  $S_\tau[\phi]$  may be regarded as an RG equation if one makes a field-variable transformation at every step such that the kinetic term is kept in the canonical form.

This paper is organized as follows. In Sect. 2 we write down the basic equation that determines the evolution of  $S_\tau[\phi]$ . In Sect. 3 we consider a local potential approximation (LPA) to our equation, and show that the result has a nice interpretation with Feynman diagrams. In Sect. 4 we make an  $\epsilon$  expansion of the LPA and show that it reproduces the eigenvalues of the linearized RG transformation around both the Gaussian and the Wilson–Fisher fixed points to the order of  $\epsilon$ . Section 5 is devoted to the conclusion and outlook.

<sup>2</sup> Note that the partition function is constant in time:  $Z_\tau \equiv \int [d\phi] e^{-S_\tau[\phi]} = Z_0$ .

## 2. Formulation

We first rewrite the consistency condition (1.7) in a differential form:<sup>3</sup>

$$\begin{aligned}
\partial_\tau e^{-S_\tau[\phi_\tau]} &= \int [d\phi_0] \int_x \left( \frac{\delta}{\delta\phi(x)} \delta[\phi - \phi_\tau(\phi_0)] \right) (-\partial_\tau \phi_\tau(x)) e^{-S_0[\phi_0]} \\
&= \int [d\phi_0] \int_x \left( \frac{\delta}{\delta\phi(x)} \delta[\phi - \phi_\tau(\phi_0)] \right) \frac{\delta S_\tau}{\delta\phi(x)}[\phi_\tau] e^{-S_0[\phi_0]} \\
&= \int_x \frac{\delta}{\delta\phi(x)} \left[ \frac{\delta S_\tau[\phi]}{\delta\phi(x)} e^{-S_\tau[\phi]} \right],
\end{aligned} \tag{2.1}$$

which in turn gives the following differential equation for  $S_\tau[\phi]$ :

$$\partial_\tau S_\tau[\phi] = \int_x \left[ -\frac{\delta^2 S_\tau[\phi]}{\delta\phi(x)^2} + \frac{\delta S_\tau[\phi]}{\delta\phi(x)} \frac{\delta S_\tau[\phi]}{\delta\phi(x)} \right]. \tag{2.2}$$

However, one can easily see that UV divergences arise from the second-order functional derivative at the same point,  $\delta^2 S/\delta\phi(x)^2$ . The reason why such UV divergences appear in the effective theory is that we have not taken into account the fact that  $\phi(x)$  should have only coarse-grained degrees of freedom with cutoff  $\Lambda = 1/\sqrt{\tau}$ .

To see how to incorporate this fact, it is helpful to consider a sharp cutoff for a while, instead of the smooth smearing with the heat kernel  $K_\tau(x-y)$ . Namely, we assume that the flowed field is cut off as  $\phi_\tau(x) = \int_{|p| \leq 1/\sqrt{\tau}} e^{ipx} \phi_{\tau,p}$ , and accordingly that the action  $S_\tau[\phi]$  depends only on the lower modes  $\phi_p$  ( $|p| \leq 1/\sqrt{\tau}$ ) of the scalar field  $\phi(x) = \int_p e^{ipx} \phi_p$ . Then, the calculation in Eq. (2.1) will be modified as

$$\begin{aligned}
\partial_\tau e^{-S_\tau[\phi]} &= \int [d\phi_0] \int_{|p| \leq 1/\sqrt{\tau}} \left( \frac{\delta}{\delta\phi_p} \delta[\phi - \phi_\tau(\phi_0)] \right) (-\partial_\tau \phi_{\tau,p}) e^{-S_0[\phi_0]} \\
&= \int [d\phi_0] \int_{|p| \leq 1/\sqrt{\tau}} \left( \frac{\delta}{\delta\phi_p} \delta[\phi - \phi_\tau(\phi_0)] \right) \frac{\delta S_\tau}{\delta\phi_{-p}}[\phi_\tau] e^{-S_0[\phi_0]} \\
&= \int_{|p| \leq 1/\sqrt{\tau}} \frac{\delta}{\delta\phi_p} \left[ \frac{\delta S_\tau[\phi]}{\delta\phi_{-p}} e^{-S_\tau[\phi]} \right].
\end{aligned} \tag{2.3}$$

Returning to the smooth cutoff with the heat kernel, Eq. (2.3) will be expressed as

$$\partial_\tau e^{-S_\tau[\phi]} = \int_{x,y} K_\tau(x-y) \frac{\delta}{\delta\phi(x)} \left[ \frac{\delta S_\tau[\phi]}{\delta\phi(y)} e^{-S_\tau[\phi]} \right], \tag{2.4}$$

which is equivalent to the equation

$$\partial_\tau S_\tau[\phi] = \int_{x,y} K_\tau(x-y) \left[ \frac{\delta S_\tau[\phi]}{\delta\phi(x)} \frac{\delta S_\tau[\phi]}{\delta\phi(y)} - \frac{\delta^2 S_\tau[\phi]}{\delta\phi(x)\delta\phi(y)} \right]. \tag{2.5}$$

We see that there no longer exist divergences of the aforementioned type. For the rest of this paper, we treat Eq. (2.5) as the equation that *defines* the flow of  $S_\tau(\phi)$ .

<sup>3</sup> In this paper, in order to simplify discussions, we do not seriously take into account the anomalous dimension  $\gamma = \eta/2$ , which may be incorporated by adding a term  $(\gamma/2\tau) \phi_\tau(x)$  to the right-hand side of the first equation in Eq. (1.6).

We here make an important comment that Eq. (2.4) can be rewritten in the form of a Fokker–Planck equation:

$$\begin{aligned}\partial_\tau e^{-S_\tau[\phi]} &= \int_{x,y} K_\tau(x-y) \left[ \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)} - \frac{\delta S_\tau[\phi]}{\delta\phi(x)} \frac{\delta S_\tau[\phi]}{\delta\phi(y)} \right] e^{-S_\tau[\phi]} \\ &= \int_{x,y} \frac{\delta}{\delta\phi(x)} K_\tau(x-y) \left[ \frac{\delta}{\delta\phi(y)} + 2 \frac{\delta S_\tau[\phi]}{\delta\phi(y)} \right] e^{-S_\tau[\phi]},\end{aligned}\quad (2.6)$$

which corresponds to the Langevin equation

$$\partial_\tau \phi_\tau(x) = v_\tau(x) - 2 \int_y K_\tau(x-y) \frac{\delta S_\tau[\phi]}{\delta\phi(y)} \quad (2.7)$$

with the Gaussian white noise  $v_\tau(x)$  normalized as

$$\langle v_\tau(x) v_{\tau'}(y) \rangle_v = 2 \delta(\tau - \tau') K_\tau(x-y). \quad (2.8)$$

The solution  $\phi_\tau(x)$  to the Langevin equation now depends on the noise  $v_\tau(x)$  as well as the initial value  $\phi_0(x)$ :

$$\phi_\tau(x) = \phi_\tau(x; \phi_0, v). \quad (2.9)$$

Then, denoting the Gaussian measure of  $v$  by  $[d\rho(v)]$ , the distribution function  $e^{-S_\tau[\phi]}/Z_\tau$  (see Eq. (1.7)) can also be written as

$$\begin{aligned}e^{-S_\tau[\phi]} &= \int [d\phi_0] \langle \delta[\phi - \phi_\tau(\phi_0, v)] \rangle_v e^{-S_0[\phi_0]} \\ &= \int [d\phi_0] [d\rho(v)] \delta[\phi - \phi_\tau(\phi_0, v)] e^{-S_0[\phi_0]}.\end{aligned}\quad (2.10)$$

The Langevin equation (2.7) shows that the field  $\phi_\tau(x)$  makes a random walk due to the noise term, but at the same time it tries to approach the classical solution to  $S_\tau[\phi]$ . We thus find mathematical equivalence between two expressions (1.7) and (2.10) that have different meanings; the former is purely deterministic in the course of evolution while the latter is stochastic. This observation may support the idea that a seemingly deterministic evolution is actually accompanied by an integration over some fluctuating degrees of freedom.

### 3. Local potential approximation

In order to investigate how Eq. (2.5) works as an RG equation, we make a local potential approximation [13–15]:

$$S_\tau[\phi] = \int_x \left[ V_\tau(\phi_x) + \frac{1}{2} (\partial_\mu \phi_x)^2 \right]. \quad (3.1)$$

The canonical form of the kinetic term is particularly important for our purpose to interpret the gradient flow as an RG flow (see discussions around Eq. (1.2)). However, even when we normalize the field  $\phi_x$  in this way at time  $\tau$ , the action may no longer take a canonical form at  $\tau + \epsilon$ . In order for the interpretation  $\Lambda = 1/\sqrt{\tau}$  to hold also at time  $\tau + \epsilon$  (i.e.,  $\Lambda - \delta\Lambda = 1/\sqrt{\tau + \epsilon} = (\tau e^{\epsilon/\tau})^{-1/2}$ ), we then need to make a field-variable transformation at  $\tau + \epsilon$  to retain the kinetic term in the canonical form.

To make the necessary calculations, it is convenient to start from the local potential approximation of the second order:

$$I_\tau[\varphi] \equiv \int_x \left[ U_\tau(\varphi_x) + \frac{1}{2} W_\tau(\varphi_x) (\partial_\mu \varphi_x)^2 \right] \quad (3.2)$$

and to investigate the evolution of  $U_\tau(\varphi)$  and  $W_\tau(\varphi)$  from  $\tau$  to  $\tau + \epsilon$  with the initial values  $U_\tau(\varphi) = V_\tau(\varphi)$  and  $W_\tau(\varphi) = 1$ . One can easily derive the following combined equations:<sup>4</sup>

$$\partial_\tau U_\tau(\varphi) = U'_\tau(\varphi)^2 - \frac{1}{(4\pi\tau)^{d/2}} U''_\tau(\varphi) - \frac{d}{2\tau} \frac{1}{(4\pi\tau)^{d/2}} W(\varphi), \quad (3.3)$$

$$\partial_\tau W_\tau(\varphi) = 2 U'_\tau(\varphi) W'_\tau(\varphi) + 4 U''_\tau(\varphi) W_\tau(\varphi) - 2\tau U''_\tau(\varphi)^2 - \frac{1}{(4\pi\tau)^{d/2}} W''_\tau(\varphi). \quad (3.4)$$

From these, we find that the coefficient of  $(1/2)(\partial_\mu \varphi_x)^2$  changes from the normalized value  $W_\tau(\varphi) \equiv 1$  to

$$\begin{aligned} W_{\tau+\epsilon}(\varphi) &= 1 + \epsilon \partial_\tau W_\tau(\varphi) = 1 + \epsilon [4 U''_\tau(\varphi) - 2\tau U''_\tau(\varphi)^2] \\ &\equiv 1 + 2\epsilon \rho'_\tau(\varphi). \end{aligned} \quad (3.5)$$

Thus, the canonically normalized field  $\phi$  at  $\tau + \epsilon$  is given by integrating the equation  $d\phi/d\varphi = \sqrt{W_{\tau+\epsilon}(\varphi)} = 1 + \epsilon \rho'_\tau(\varphi)$ , and we find the following relation to the order of  $\epsilon$ :

$$\varphi = \phi - \epsilon \rho_\tau(\phi) = \phi - \epsilon \int_0^\phi d\phi [2 U''_\tau(\phi) - \tau U''_\tau(\phi)^2]. \quad (3.6)$$

The Jacobian<sup>5</sup>  $\text{Det}'(\delta\varphi/\delta\phi) = e^{\text{Tr}' \log(\delta\varphi/\delta\phi)}$  is calculated with

$$\text{Tr}' \log(\delta\varphi/\delta\phi) = \int_{x,y} K_\tau(x-y) \log[1 - \epsilon \rho'_\tau(\phi_x)] \delta^d(x-y) = -\epsilon \int_x \frac{1}{(4\pi\tau)^{d/2}} \rho'_\tau(\phi_x). \quad (3.7)$$

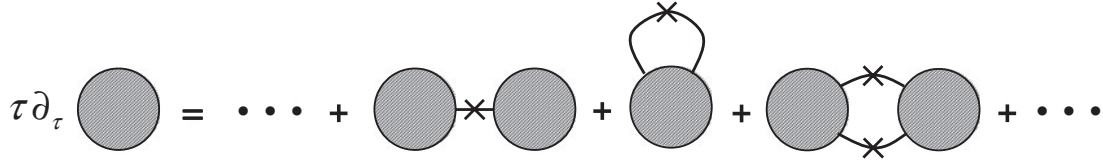
By putting everything together, the change of the local potential for the canonically normalized field  $\phi$  is given as follows (recall the initial condition  $U_\tau(\phi) = V_\tau(\phi)$ ):

$$\begin{aligned} V_{\tau+\epsilon}(\phi) &= [U_\tau(\varphi) + \epsilon \partial_\tau U_\tau(\varphi)]|_{\varphi=\phi-\epsilon\rho_\tau(\phi)} + \epsilon \frac{1}{(4\pi\tau)^{d/2}} \rho'_\tau(\phi) \\ &= V_\tau(\phi) + \epsilon \left[ -V'_\tau(\phi)^2 + \frac{1}{(4\pi\tau)^{d/2}} V''_\tau(\phi) + \tau V'_\tau(\phi) \int_0^\phi d\phi V''_\tau(\phi)^2 \right. \\ &\quad \left. - \frac{\tau}{(4\pi\tau)^{d/2}} V''_\tau(\phi)^2 - \frac{d}{2\tau(4\pi\tau)^{d/2}} \right]. \end{aligned} \quad (3.8)$$

<sup>4</sup> Among the formulas that may be useful in deriving the equations are

$$\begin{aligned} \partial_x^2 K_\tau(x-y) &= \partial_\tau K_\tau(x-y), \quad \int_{x-y} K_\tau(x-y) (x-y)_\mu (x-y)_\nu = 2\tau \delta_{\mu\nu}, \\ \int_{x,y} K_\tau(x-y) f(\phi_x) g(\phi_y) &= \int_x [f(\phi_x) g(\phi_x) - \tau (\partial_\mu \phi_x)^2 f'(\phi_x) g'(\phi_x) + O(\tau^2)]. \end{aligned}$$

<sup>5</sup> The prime means that the determinant or the trace should be taken on the partial functional space under the projection of  $K_\tau(x-y)$ .



**Fig. 1.** A Feynman diagrammatic interpretation of Eq. (3.13). The shaded circle represents minus the potential,  $-V_\tau(\phi)$ .

Note that the terms  $V'_\tau(\phi)^2$  and  $V''_\tau(\phi)$  appear in Eq. (3.8) as  $-V'_\tau(\phi)^2 + \text{const. } V''_\tau(\phi)$ , which have the same signs as those in the Polchinski equation [17], although the signs of the terms  $U'_\tau(\phi)^2$  and  $U''_\tau(\phi)$  are opposite in Eq. (3.3).

To get dimensionless expressions, we use the cutoff  $\Lambda = 1/\sqrt{\tau} = \tau^{-1/2}$  at time  $\tau$  as

$$x_\mu = \tau^{1/2} \bar{x}_\mu, \quad \partial_\mu = \tau^{-1/2} \bar{\partial}_\mu, \quad \phi_x = \tau^{-(d-2)/4} \bar{\phi}_x, \tag{3.9}$$

which gives the relation

$$V_\tau(\phi) = \tau^{-d/2} \bar{V}_\tau(\bar{\phi}) \quad \text{with} \quad \phi = \tau^{-(d-2)/4} \bar{\phi}. \tag{3.10}$$

Here we have placed bars on quantities to indicate that they are dimensionless. On the other hand, we use the cutoff  $\Lambda - \delta\Lambda = 1/\sqrt{\tau + \epsilon} = (\tau e^{\epsilon/\tau})^{-1/2}$  at time  $\tau + \epsilon$  as

$$x_\mu = (\tau e^{\epsilon/\tau})^{1/2} \bar{x}_\mu, \quad \partial_\mu = (\tau e^{\epsilon/\tau})^{-1/2} \bar{\partial}_\mu, \quad \phi_x = (\tau e^{\epsilon/\tau})^{-(d-2)/4} \bar{\phi}_x, \tag{3.11}$$

which leads to the relation

$$V_{\tau+\epsilon}(\phi) = (\tau e^{\epsilon/\tau})^{-d/2} \bar{V}_{\tau+\epsilon}(\bar{\phi}) \quad \text{with} \quad \phi = (\tau e^{\epsilon/\tau})^{-(d-2)/4} \bar{\phi}. \tag{3.12}$$

Substituting Eqs. (3.10) and (3.12) into Eq. (3.8), we finally obtain the following equation for the dimensionless local potential (we remove the bars from the expression for notational simplicity):

$$\begin{aligned} \tau \partial_\tau V_\tau(\phi) &= \frac{d}{2} V_\tau(\phi) - \frac{d-2}{4} \phi V'_\tau(\phi) - V'_\tau(\phi)^2 + B_d V''_\tau(\phi) - B_d V''_\tau(\phi)^2 \\ &+ V'_\tau(\phi) \int_0^\phi d\phi V''_\tau(\phi)^2 - \frac{d}{2} B_d \quad \left( B_d \equiv \frac{1}{(4\pi)^{d/2}} \right). \end{aligned} \tag{3.13}$$

Note that the first two terms in Eq. (3.13) reflect the simple rescalings of the potential and the field variable. The next three terms have a natural interpretation with Feynman diagrams (see Fig. 1). In fact, the third term in Eq. (3.13) represents the contraction of a propagator in a one-particle reducible diagram, while the fourth term stands for that of a propagator in a one-particle irreducible diagram. The fifth term represents the contraction of propagators in a two-particle reducible diagram.

#### 4. $\epsilon$ expansion

Equation (3.13) can be solved iteratively in dimension  $d = 4 - \epsilon$  with  $0 < \epsilon \ll 1$ . Expanding the potential as

$$V(\phi) = v_0 + \frac{v_2}{2!} \phi^2 + \frac{v_4}{4!} \phi^4 + \dots, \tag{4.1}$$

the first few terms in Eq. (3.13) are given by

$$\tau \partial_\tau v_2 = v_2 - 2 v_2^2 + 2 v_2^3 + B_d v_4 - 2 B_d v_2 v_4, \tag{4.2}$$

$$\tau \partial_\tau v_4 = \frac{\varepsilon}{2} v_4 - 8 v_2 v_4 + 12 v_2^2 v_4 - 6 B_d v_4^2 - 2 B_d v_2 v_6 + B_d v_6, \quad (4.3)$$

$$\begin{aligned} \tau \partial_\tau v_6 = & (-1 + \varepsilon) v_6 - 20 v_4^2 + 76 v_2 v_4^2 - 12 v_2 v_6 + 18 v_2^2 v_6 \\ & - 30 B_d v_4 v_6 + B_d v_8 - 2 B_d v_2 v_8, \end{aligned} \quad (4.4)$$

$$\begin{aligned} \tau \partial_\tau v_8 = & \left(-2 + \frac{3\varepsilon}{2}\right) v_8 - 16 v_2 v_8 - 112 v_4 v_6 + 24 v_2^2 v_8 + 336 v_4^3 \\ & + 464 v_2 v_4 v_6 - 56 B_d v_4 v_8 - 70 B_d v_6^2. \end{aligned} \quad (4.5)$$

In addition to the Gaussian fixed point ( $v_n^* = 0$ ), a nontrivial fixed point  $v_n^*$  can be found with the ansatz  $v_2^* = O(\varepsilon)$ ,  $v_4^* = O(\varepsilon)$ ,  $v_6^* = O(\varepsilon^2)$ , and  $v_n^* = O(\varepsilon^3)$  ( $n \geq 8$ ):

$$v_2^* = -\frac{1}{36} \varepsilon + O(\varepsilon^2), \quad v_4^* = \frac{1}{36B_4} \varepsilon + O(\varepsilon^2), \quad v_6^* = -\frac{20}{(36B_4)^2} \varepsilon^2 + O(\varepsilon^3), \quad v_8^* = O(\varepsilon^3). \quad (4.6)$$

By linearizing Eqs. (4.2)–(4.5) around these values, the first two eigenvalues are found to be  $1 - \varepsilon/6 + O(\varepsilon^2)$  and  $-\varepsilon/2 + O(\varepsilon^2)$ , which agree with those of the linearized RG transformation at the Wilson–Fisher fixed point (note that  $-\Lambda \partial_\Lambda = 2 \tau \partial_\tau$ ).

## 5. Conclusion and outlook

In this paper, we have investigated the RG structure of the gradient flow. To generate the flow, instead of using the original bare action, we proposed to use the action  $S_\tau[\phi]$  at flow time  $\tau$ . We wrote down the basic equation that determines the evolution of the action and considered an LPA to our equation, and showed that the result has a nice interpretation with Feynman diagrams. We also made an  $\varepsilon$  expansion of the LPA and showed that it reproduces the eigenvalues of the linearized RG transformation around both the Gaussian and the Wilson–Fisher fixed points to the order of  $\varepsilon$ .

In order to simplify the argument, we have not seriously taken into account the anomalous dimension, which actually could be neglected to the order of the approximation that we made in the  $\varepsilon$  expansion. A careful treatment of the anomalous dimension will be given in a forthcoming paper. In addition to higher-order calculations of  $\varepsilon$  expansion, it should be interesting to investigate the LPA of the  $O(N)$  vector model.

It is tempting to regard Eq. (2.5) as a sort of exact renormalization group [13, 16–18] (see Refs. [19–21] for a nice review on this subject). However, one must be careful in establishing this relationship, because the RG interpretation of Eq. (2.5) is possible only when we make a field-variable transformation at every step such that the kinetic term is kept in the canonical form (see discussions below Eq. (3.1)). It thus should be interesting to write down an equation that incorporates the effect of the change of variable in a form of differential equation.

In developing the present work further, it will be important to investigate whether the gradient flow of the present paper (Eq. (1.6)) also has a nice property in the renormalization of the flowed fields and their composite operators. In fact, a prominent feature of the conventional gradient flow (1.1) is, as was mentioned in the introduction, that there appear no extra divergences in the  $(d+1)$ -dimensional bulk theory. For example, let us consider the expectation value of an operator constructed from the flowed field,  $\mathcal{O}[\phi_\tau]$ :

$$\langle \mathcal{O}[\phi_\tau] \rangle_{S_0} \equiv \frac{1}{Z_0} \int [d\phi_0] e^{-S_0[\phi_0]} \mathcal{O}[\phi_\tau(\phi_0)], \quad (5.1)$$

where  $\phi_\tau(\phi_0)$  is the solution to Eq. (1.1). This gives a finite quantity once a proper regularization is implemented at the initial cutoff  $\Lambda_0$ , and this absence of extra divergences is attributed to the fact that  $\phi_\tau(x; \phi_0)$  takes the form  $\phi_\tau(x; \phi_0) = \int_y K_\tau(x-y) \phi_0(y) + \dots$ . Now let us consider the expectation value of the same operator  $\mathcal{O}[\phi]$  with respect to our effective action  $S_\tau[\phi]$ :

$$\begin{aligned} \langle \mathcal{O}[\phi] \rangle_{S_\tau} &\equiv \frac{1}{Z_\tau} \int [d\phi] e^{-S_\tau[\phi]} \mathcal{O}[\phi] \\ &= \frac{1}{Z_\tau} \int [d\phi][d\phi_0] e^{-S_0[\phi_0]} \delta[\phi - \phi_\tau(\phi_0)] \mathcal{O}(\phi) \\ &= \frac{1}{Z_\tau} \int [d\phi_0] e^{-S_0[\phi_0]} \mathcal{O}[\phi_\tau(\phi_0)], \end{aligned} \quad (5.2)$$

where  $\phi_\tau(x; \phi_0)$  is now the solution to our flow equation (1.6). Note that this solution also has the form  $\phi_\tau(x; \phi_0) = \int_y K_\tau(x-y) \phi_0(y) + \dots$  because we make a field-variable transformation at every step such that  $S_\tau[\phi]$  takes the canonical form,  $S_\tau[\phi] = \int_x [(1/2) (\partial_\mu \phi(x))^2 + \dots]$ . We thus expect that the two expectation values (5.1) and (5.2) share the same properties for finiteness at short distances. We leave the confirmation of this expectation for future work.

Although the present paper only discusses scalar field theory, the extension to other field theories should be straightforward. The generalization to field theories in curved spacetime will also be interesting.

A study along these lines is now in progress and will be reported elsewhere.

## Acknowledgements

The authors thank Daisuke Kadoh, Yoshio Kikukawa, Nobuyuki Matsumoto, Tetsuya Onogi, Hidenori Sonoda, and Hiroshi Suzuki for useful discussions. This work was partially supported by Japan Society for the Promotion of Science (JSPS) KAKENHI (grant number 16K05321).

## Funding

Open Access funding: SCOAP<sup>3</sup>.

## References

- [1] M. Lüscher, Commun. Math. Phys. **293**, 899 (2010) [arXiv:0907.5491 [hep-lat]] [Search INSPIRE].
- [2] M. Lüscher, J. High Energy Phys. **1008**, 071 (2010); **1403**, 092 (2014) [erratum] [arXiv:1006.4518 [hep-lat]] [Search INSPIRE].
- [3] M. Lüscher and P. Weisz, J. High Energy Phys. **1102**, 051 (2011) [arXiv:1101.0963 [hep-th]] [Search INSPIRE].
- [4] M. Lüscher, J. High Energy Phys. **1304**, 123 (2013) [arXiv:1302.5246 [hep-lat]] [Search INSPIRE].
- [5] H. Suzuki, Prog. Theor. Exp. Phys. **2013**, 083B03 (2013); **2015**, 079201 (2015) [erratum] [arXiv:1304.0533 [hep-lat]] [Search INSPIRE].
- [6] H. Makino and H. Suzuki, Prog. Theor. Exp. Phys. **2014**, 063B02 (2014); **2015**, 079202 (2015) [erratum] [arXiv:1403.4772 [hep-lat]] [Search INSPIRE].
- [7] K. Kikuchi and T. Onogi, J. High Energy Phys. **1411**, 094 (2014) [arXiv:1408.2185 [hep-th]] [Search INSPIRE].
- [8] M. Lüscher, PoS LATTICE **2013**, 016 (2014) [arXiv:1308.5598 [hep-lat]] [Search INSPIRE].
- [9] A. Kagimura, A. Tomiya, and R. Yamamura, arXiv:1508.04986 [hep-lat] [Search INSPIRE].
- [10] R. Yamamura, Prog. Theor. Exp. Phys. **2016**, 073B10 (2016) [arXiv:1510.08208 [hep-lat]] [Search INSPIRE].
- [11] S. Aoki, J. Balog, T. Onogi, and P. Weisz, Prog. Theor. Exp. Phys. **2016**, 083B04 (2016) [arXiv:1605.02413 [hep-th]] [Search INSPIRE].



- [12] H. Makino, O. Morikawa, and H. Suzuki, Prog. Theor. Exp. Phys. **2018**, 053B02 (2018) [[arXiv:1802.07897](#) [hep-th]] [[Search INSPIRE](#)].
- [13] K. G. Wilson and J. Kogut, Phys. Rept. **12**, 75 (1974).
- [14] A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B **270**, 687 (1986).
- [15] T. R. Morris, Phys. Lett. B **329**, 241 (1994) [[arXiv:hep-ph/9403340](#)] [[Search INSPIRE](#)].
- [16] F. J. Wegner and A. Houghton, Phys. Rev. A **8**, 401 (1973).
- [17] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).
- [18] C. Wetterich, Phys. Lett. B **301**, 90 (1993) [[arXiv:1710.05815](#) [hep-th]] [[Search INSPIRE](#)].
- [19] C. Bagnuls and C. Bervillier, Int. J. Mod. Phys. A **16**, 1825 (2001) [[arXiv:hep-th/0101110](#)] [[Search INSPIRE](#)].
- [20] J. Polonyi, Central Eur. J. Phys. **1**, 1 (2003) [[arXiv:hep-th/0110026](#)] [[Search INSPIRE](#)].
- [21] O. J. Rosten, Phys. Rept. **511**, 177 (2012) [[arXiv:1003.1366](#) [hep-th]] [[Search INSPIRE](#)].